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Equivalence transformations of rational matrices and applications

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The known theories of transformations between polynomial matrices are extended to the case of rational matrices. Specifically, Ω-equivalence between rational matrices having possibly different dimensions is defined, and this has the property of preserving the zero structure of rational matrices in the region $\Omega \subseteq \boldsymbol{C} \cup \{\infty\}$. Some implications of this new equivalence transformation for linear system theory are also provided.

1. Introduction

Some open questions surround the existence of transformations which preserve both the finite and infinite zero structure of polynomial matrices and, more generally, rational matrices of different dimensions. Hayton *et al.* (1988) proposed full equivalence (f.e.) for the special case of polynomial matrices, which includes both extended unimodular equivalence (e.u.e.) (Pugh and Shelton 1978) and extended causal equivalence (e.c.e.) (Anderson *et al.* 1985, Cullen 1987, Walker 1988) within one single transformation. It thus preserves both the finite and infinite zero structure of polynomial matrices with different dimensions.

In this paper an extension, in two separate senses, of the above ideas is presented. The first extension is the development of a general equivalence transformation for rational matrices which are not necessarily of the same dimension. The second extension concerns the reference of the transformation to a specific region $\Omega \subseteq \boldsymbol{C} \cup \{\infty\}$, where $\boldsymbol{C}$ is the complex plane, and the confinement of its invariants to the given region.

The study of the zero structure in a region $\Omega \subseteq \boldsymbol{C} \cup \{\infty\}$ of a given transfer function matrix, under Ω-unimodular output feedback, provides an application of the notion of Ω-equivalence. As a consequence, some interesting results for special cases of the region $\Omega \subseteq \boldsymbol{C} \cup \{\infty\}$ are obtained. In addition, an extension of the known conditions for the absence of infinite zeros of polynomial matrices (Pugh *et al.* 1992, Zhang 1989 a, b), to the case of rational matrices is also provided.

2. Structure of rational matrices in $\Omega \subseteq C$

Let $\Omega \subseteq \boldsymbol{C}$ be symmetrically located with respect to the real axis $\boldsymbol{R}$ of the complex plane $\boldsymbol{C}$. Let the rational function $t(s) \in \boldsymbol{R}(s)$ be written as

$$t(s) = t_{\Omega}(s)\,\hat{t}(s) \tag{2.1}$$

where $t_{\Omega}(s) = n_{\Omega}(s)/d_{\Omega}(s)$ and $n_{\Omega}(s)$, $d_{\Omega}(s) \in \boldsymbol{R}[s]$ are coprime with all their zeros within Ω and $\hat{t}(s) = \hat{n}(s)/\hat{d}(s)$ and $\hat{n}(s)$, $\hat{d}(s) \in \boldsymbol{R}[s]$ are coprime with all

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their zeros outside Ω. Let δ_Ω: $\boldsymbol{R}(s) \to \boldsymbol{Z} \cup \{-\infty\}$ be defined as

$$\delta_\Omega(t(s)) = \begin{cases} \deg n_\Omega(s) - \deg d_\Omega(s) & t_\Omega \neq 0 \\ -\infty & t_\Omega \equiv 0 \end{cases} \tag{2.2}$$

where $\deg(\cdot)$ denotes the usual degree of the indicated polynomial. Then, $\delta_\Omega(\cdot)$ is a discrete valuation on $\boldsymbol{R}(s)$. In the case where $t(s)$ is a conventional polynomial note that $\delta_\Omega(t(s)) \leqslant \deg(t(s))$ with equality holding if and only if all the zeros of $t(s)$ lie in Ω. The zeros of $n_\Omega(s)$ (respectively $d_\Omega(s)$) are termed the *zeros* (respectively *poles*) of $t(s)$ in Ω. Let $\boldsymbol{R}_\Omega(s)$ be the ring of rational functions with no poles in Ω. These will be called Ω-polynomials (Pernebo 1981) and note that if $t(s) \in \boldsymbol{R}_\Omega(s)$ then $\lim_{s\to s_0} t(s)$ exists $\forall s_0 \in \Omega$. The following result is an extension of that of Pernebo (1981) to the case of rational functions.

Lemma 1: *Let* $t_1(s)$, $t_2(s) \in \boldsymbol{R}(s)$, $t_2(s) \neq 0$ *and* $\delta_\Omega(t_1(s)) > \delta_\Omega(t_2(s))$. *Then there exist* $\hat{q}(s) \in \boldsymbol{R}_\Omega(s)$, $r(s) \in \boldsymbol{R}(s)$ *such that*

$$t_1(s) = t_2(s)\hat{q}(s) + r(s) \tag{2.3}$$

where $\delta_\Omega(r(s)) < \delta_\Omega(t_2(s))$ *or* $r(s) = 0$.

Proof: Let $t_i(s) = (\hat{n}_i(s)/\hat{d}_i(s))(n_{\Omega i}(s)/d_{\Omega i}(s))$, with $\hat{n}_i(s)$, $\hat{d}_i(s)$, $n_{\Omega i}(s)$, $d_{\Omega i}(s) \in \boldsymbol{R}[s]$, $i = 1, 2$. From the usual euclidean division algorithm of polynomials and the assumption that $\delta_\Omega(t_1(s)) > \delta_\Omega(t_2(s))$ we have that there exist two polynomials $q(s)$ and $p(s)$ such that

$$n_{\Omega 1}(s)d_{\Omega 2}(s) = q(s)[n_{\Omega 2}(s)d_{\Omega 1}(s)] + p(s) \tag{2.4}$$

where

$$\deg[p(s)] < \deg[n_{\Omega 2}(s)d_{\Omega 1}(s)] \tag{2.5}$$

or $p(s) = 0$. Dividing both sides of (2.4) by $d_{\Omega 2}(s)d_{\Omega 1}(s)$ gives

$$\frac{n_{\Omega 1}(s)}{d_{\Omega 1}(s)} = q(s)\frac{n_{\Omega 2}(s)}{d_{\Omega 2}(s)} + \frac{p(s)}{d_{\Omega 1}(s)d_{\Omega 2}(s)} \tag{2.6}$$

where

$$\begin{aligned} \deg[p(s)] &< \deg[n_{\Omega 2}(s)] + \deg[d_{\Omega 1}(s)] \\ \Rightarrow \deg[p(s)] - \deg[d_{\Omega 1}(s)] &< \deg[n_{\Omega 2}(s)] \\ \Rightarrow \deg[p(s)] - \deg[d_{\Omega 1}(s)] - \deg[d_{\Omega 2}(s)] &< \deg[n_{\Omega 2}(s)] - \deg[d_{\Omega 2}(s)] \\ \Rightarrow \deg[p(s)] - \deg[d_{\Omega 1}(s)] - \deg[d_{\Omega 2}(s)] &< \delta_\Omega(t_2(s)) \end{aligned} \tag{2.7}$$

(2.6) may be rewritten as

$$\begin{aligned} \frac{\hat{d}_1(s)}{\hat{n}_1(s)}\left[\frac{\hat{n}_1(s)}{\hat{d}_1(s)}\frac{n_{\Omega 1}(s)}{d_{\Omega 1}(s)}\right] &= q(s)\frac{\hat{d}_2(s)}{\hat{n}_2(s)}\left[\frac{\hat{n}_2(s)}{\hat{d}_2(s)}\frac{n_{\Omega 2}(s)}{d_{\Omega 2}(s)}\right] + \frac{p(s)}{d_{\Omega 1}(s)d_{\Omega 2}(s)} \\ \Rightarrow t_1(s) &= \left[q(s)\frac{\hat{d}_2(s)}{\hat{n}_2(s)}\frac{\hat{n}_1(s)}{\hat{d}_1(s)}\right]t_2(s) + \frac{\hat{n}_1(s)}{\hat{d}_1(s)}\frac{p(s)}{d_{\Omega 1}(s)d_{\Omega 2}(s)} \end{aligned} \tag{2.8}$$

where, according to (2.7), we have that

$$\left.\begin{aligned}&\delta_\Omega\left[\frac{\hat{n}_1(s)}{\hat{d}_1(s)}\,\frac{p(s)}{d_{\Omega 1}(s)d_{\Omega 2}(s)}\right]\\&= \delta_\Omega\left[\frac{p(s)}{d_{\Omega 1}(s)d_{\Omega 2}(s)}\right] = \delta_\Omega(p(s)) - \delta_\Omega(d_{\Omega 1}(s)d_{\Omega 2}(s))\\&\leqslant \deg p(s) - \deg d_{\Omega 1}(s) - \deg d_{\Omega 2}(s)\\&\leqslant \deg n_{\Omega 2}(s) + \deg d_{\Omega 1}(s) - \deg d_{\Omega 1}(s) - \deg d_{\Omega 2}(s)\\&= \delta_\Omega(t_2(s))\end{aligned}\right\} \tag{2.9}$$

and the rational function

$$\hat{q}(s) = q(s)\,\frac{\hat{d}_2(s)}{\hat{n}_2(s)}\,\frac{\hat{n}_1(s)}{\hat{d}_1(s)} \tag{2.10}$$

has no poles in Ω, which proves the lemma. □

For $t(s)(\neq 0) \in \boldsymbol{R}_\Omega(s)$, $\delta_\Omega(t(s)) \geqslant 0$, and so $\delta_\Omega(t)$ serves as a degree on $\boldsymbol{R}_\Omega(s)$. Thus, by Lemma 1, $\boldsymbol{R}_\Omega(s)$ is a euclidean ring and hence a principal ideal domain.

$T(s) \in \boldsymbol{R}(s)^{p\times m}$ is said to be *Ω-polynomial* if $\lim_{s\to s_0} T(s)$ exists $\forall s_0 \in \Omega$. The set of such matrices is denoted $\boldsymbol{R}_\Omega(s)^{p\times m}$. $T(s) \in \boldsymbol{R}_\Omega(s)^{p\times p}$ is said to be *Ω-unimodular* in case there exist $\hat{T}(s) \in \boldsymbol{R}_\Omega(s)^{p\times p}$ such that: $T(s)\hat{T}(s) = I$ or, equivalently, if and only if $\lim_{s\to s_0} T(s)$ exists, and is invertible $\forall s_0 \in \Omega$. Obvious row/column operations correspond to pre/post-multiplication by an appropriate Ω-unimodular matrix.

Definition 1: $T_1(s)$, $T_2(s) \in \boldsymbol{R}(s)^{p\times m}$ are said to be *Ω-unimodular equivalent* (Ω-u.e.) in case there exist Ω-unimodular matrices $U_L(s) \in \boldsymbol{R}_\Omega(s)^{p\times p}$ and $U_R(s) \in \boldsymbol{R}_\Omega(s)^{m\times m}$, such that

$$U_L(s)T_1(s)U_R(s) = T_2(s) \tag{2.11}$$

□

The following result is fairly immediate from Verghese (1979), Vardulakis and Karcanias (1983), Vardulakis (1991).

Theorem 1: *$T(s) \in \boldsymbol{R}(s)^{p\times m}$ is (Ω−u.e.) to*

$$S^\Omega_{T(s)} = \text{blockdiag}\left(\frac{\varepsilon_1(s)}{\psi_1(s)}, \frac{\varepsilon_2(s)}{\psi_2(s)}, \ldots, \frac{\varepsilon_r(s)}{\psi_r(s)}, \boldsymbol{O}_{p-r,m-r}\right) \tag{2.12}$$

where $\varepsilon_i(s)$, $\psi_i(s) \in \boldsymbol{R}[s]$ are monic, have no zeros outside Ω, are pairwise coprime and $\varepsilon_i(s)\backslash\varepsilon_{i+1}(s)$, $\psi_{i+1}(s)\backslash\psi_i(s)$ for each $i = 1, 2, \ldots, r-1$. $S^\Omega_{T(s)}(s)$ is the Smith-McMillan form of $T(s)$ in Ω and r is the rank of $T(s)$.

$\varepsilon_i(s)/\psi_i(s) := f_i(s)$, called the Ω-invariant rational functions of $T(s)$, constitute a complete set of invariants for rational matrices under (Ω−u.e.). Further the zeros (respectively poles) in Ω of $T(s)$ are the zeros of $\varepsilon_i(s)$ (respectively $\psi_i(s)$), $i = 1, 2, \ldots, r$.

Definition 2 (Pernebo 1981): $A(s) \in \boldsymbol{R}_\Omega(s)^{p\times p}$ and $B(s) \in \boldsymbol{R}_\Omega(s)^{p\times m}$ are said to be *Ω-left coprime* if and only if

$$\mathrm{rank}_{\boldsymbol{R}}\,(A(s_0)\;\; B(s_0)) = p \;\; \forall s_0 \in \Omega \tag{2.13}$$

□

For $\Omega \subseteq \boldsymbol{C}$ the following holds.

Proposition 1 (Pernebo 1981): *Let $T_1(s) \in \boldsymbol{R}_\Omega(s)^{p\times l}$, $T_2(s) \in \boldsymbol{R}_\Omega(s)^{p\times t}$ be two Ω-polynomial matrices with $l + t := m \geqslant p = \mathrm{rank}[T_1(s)\;\; T_2(s)]$. Then the following statements are equivalent.*

(1) *$T_1(s)$ and $T_2(s)$ are Ω-left coprime.*

(2) *$T(s) = [T_1(s)\;\; T_2(s)]$ has no zeros in Ω.*

(3) *There exists $X(s) \in \boldsymbol{R}_\Omega(s)^{l\times p}$, $Y(s) \in \boldsymbol{R}_\Omega(s)^{t\times p}$ such that*

$$T_1(s)X(s) + T_2(s)Y(s) = I_p \tag{2.14}$$

Lemma 2: *Let $T(s) \in \boldsymbol{R}(s)^{p\times m}$ be of rank r. Then there exist (non-unique) Ω-left coprime matrices $A_1(s) \in \boldsymbol{R}_\Omega(s)^{p\times p}$, $B_1(s) \in \boldsymbol{R}_\Omega(s)^{p\times m}$ such that*

$$T(s) = A_1^{-1}(s)B_1(s) \tag{2.15}$$

Every other Ω-left coprime factorization $\widetilde{A}_1^{-1}(s)\widetilde{B}_1(s)$ is such that

$$\widetilde{A}_1(s) = U_{\mathrm{L}}(s)A_1(s):\quad \widetilde{B}_1(s) = U_{\mathrm{L}}(s)B_1(s) \tag{2.16}$$

where $U_{\mathrm{L}}(s) \in \boldsymbol{R}_\Omega(s)^{p\times p}$ is Ω-unimodular.

Proof: The proof follows in a similar way to the result concerning conventional polynomial matrices (e.g. see Rosenbrock 1970 p 139). □

Equation (2.15) is called a *Ω-left coprime matrix fraction description* (Ω−MFD) of $T(s)$. Definition 2 may now be extended to include general rational matrices and not simply Ω-polynomial matrices.

Definition 3: Let $A(s) \in \boldsymbol{R}(s)^{p\times m}$ and $B(s) \in \boldsymbol{R}(s)^{p\times l}$ be general rational matrices and consider the left Ω-coprime MFD

$$(A(s)\;\; B(s)) = D_1^{-1}(s)(\bar{A}(s)\;\; \bar{B}(s)) \tag{2.17}$$

$A(s)$, $B(s)$ are *Ω-left coprime* if and only if the Ω-polynomial matrices $\bar{A}(s)$, $\bar{B}(s)$ are Ω-left coprime. □

Similar definitions to those above may be given in respect of Ω-right coprimeness. We call a $p \times m$ Ω-polynomial matrix such as $B_1(s)$ a *numerator* of $T(s)$ and a $p \times p$ Ω-polynomial matrix such as $A_1(s)$ a *denominator of* $T(s)$. In the usual way we have

$$\text{zeros in } \Omega \text{ of } T(s) \equiv \text{zeros in } \Omega \text{ of } B_1(s) \tag{2.18}$$

$$\text{poles in } \Omega \text{ of } T(s) \equiv \text{zeros in } \Omega \text{ of } A_1(s) \tag{2.19}$$

Definition 4: If $\Omega \subseteq \boldsymbol{C} \cup \{\infty\}$ then the *Ω-least order of* $T(s)$, denoted $v_\Omega(T(s))$, is the number of poles of $T(s)$ occurring in Ω counted according to degree and multiplicity. □

Corollary 1: *When $\Omega \subseteq C$ then the MFD (2.15) is Ω-left coprime if and only if*

$$\delta_\Omega(|A_1(s)|) = \nu_\Omega(T(s)) \tag{2.20}$$

3. Equivalence transformations of rational matrices in $\Omega \subseteq C \cup \{\infty\}$

It is of interest to know if there exist transformations which preserve, in a given region, the zero structure of rational matrices with possibly different dimensions. Initially we consider merely the case $\Omega \subseteq C$ since there are some technical problems in treating the point at infinity and C together (i.e. $\Omega = C \cup \{\infty\}$) as noted by Pernebo (1981). It should be noted however that in case we wish to consider $\Omega \subset C \cup \{\infty\}$ a bilinear transformation may be employed to reduce the problem to $\Omega \subseteq C$. Let $P(p, m)$ and $P_\Omega(p, m)$ be, respectively, the sets of $(r+p) \times (r+m)$ rational and Ω-polynomial matrices, where the integer $r \geqslant \max(-p, -m)$.

Definition 5: $P_1(s)$, $P_2(s) \in P(p, m)$ are said to be *Ω-equivalent* in case there exist rational matrices $M(s)$, $N(s)$ such that

$$[M(s) \quad P_2(s)]\begin{bmatrix} P_1(s) \\ -N(s) \end{bmatrix} = 0 \tag{3.1}$$

where the elements of the compound matrices

$$[M(s) \quad P_2(s)]; \quad \begin{bmatrix} P_1(s) \\ -N(s) \end{bmatrix} \tag{3.2}$$

are Ω-left and Ω-right coprime respectively and satisfy the following Ω-least order conditions

$$\nu_\Omega[M(s) \quad P_2(s)] = \nu_\Omega(P_2(s)); \quad \nu_\Omega\begin{bmatrix} P_1(s) \\ -N(s) \end{bmatrix} = \nu_\Omega(P_1(s)) \tag{3.3}$$

□

Theorem 2:

(i) *If $\Omega \subset C \cup \{\infty\}$ then Ω-u.e. implies Ω-equivalence.*

(ii) *If $\Omega \subseteq C \cup \{\infty\}$ and $\Omega_1 \subset \Omega$ then Ω-equivalence implies Ω_1-equivalence.*

Proof:

(i) Let $P_1(s)$, $P_2(s) \in P(p, m)$ be Ω-u.e. where $\Omega \subseteq C$ under the following Ω-u.e. transformation

$$P_2(s) = U_L(s)P_1(s)U_R(s) \tag{3.4}$$

where $U_L(s)$, $U_R(s)$ are Ω-unimodular matrices. Equivalently, (3.4) may be written as

$$U_L^{-1}(s)P_2(s) = P_1(s)U_R(s) \Leftrightarrow \tilde{U}_L(s)P_2(s) = P_1(s)U_R(s) \tag{3.5}$$

where $\tilde{U}_L(s) = U_L^{-1}(s)$ is also a Ω-unimodular matrix, and so has no poles in the region $\Omega \subseteq C$. Thus, the pole structure in Ω of the compound matrix

$$[\tilde{U}_L(s) \quad P_1(s)] \tag{3.6}$$

coincides exactly with that of the rational matrix $P_1(s)$, and so (3.6) satisfies the least-order conditions. Let now $P_1(s) = D_\Omega^{-1}(s)N_\Omega(s)$ be an Ω-right coprime MFD of $P_1(s)$. Then

$$[\widetilde{U}_L(s) \quad P_1(s)] = D_\Omega^{-1}(s)[D_\Omega(s)\cdot \widetilde{U}_L(s) \quad N_\Omega(s)] \tag{3.7}$$

is an Ω-MFD of the compound matrix in (3.6). Notice that the Ω-polynomial matrix

$$[D_\Omega(s) \quad D_\Omega(s)\cdot \widetilde{U}_L(s) \quad N_\Omega(s)] \tag{3.8}$$

is Ω-u.e. to the Ω-polynomial matrix

$$[D_\Omega(s) \quad 0 \quad N_\Omega(s)] = [D_\Omega(s) \quad D_\Omega(s)\cdot \widetilde{U}_L(s) \quad N_\Omega(s)]\begin{bmatrix} I & -\widetilde{U}_L(s) & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \tag{3.9}$$

which has no zeros in Ω. Thus, the Ω-MFD in (3.7) is left coprime and therefore the zeros of the compound matrix $[D_\Omega(s)\cdot \widetilde{U}_L(s) \ N_\Omega(s)]$ coincide with the zeros of the compound matrix (3.6). However

$$(D_\Omega(s)\cdot \widetilde{U}_L(s) \ N_\Omega(s)) = (D_\Omega(s) \ N_\Omega(s))\begin{bmatrix} \widetilde{U}_L(s) & 0 \\ 0 & I \end{bmatrix} \tag{3.10}$$

is a Ω-u.e. transformation and so the zeros in Ω of the compound matrix in (3.6), or equivalently the zeros of $(D_\Omega(s)\cdot \widetilde{U}_L(s) \ N_\Omega(s))$, coincide with those of $(D_\Omega(s) \ N_\Omega(s))$ which of course has none. Thus, the compound matrix in (3.6) has no zeros in Ω. In the same way we can show that the compound matrix $[P_2(s)^T \ U_R(s)^T]^T$ has no zeros in Ω and so (3.5) is an Ω-equivalence transformation in the case where $\Omega \subseteq \boldsymbol{C}$. In the case where $\Omega \subset \boldsymbol{C} \cup \{\infty\}$ then a bilinear transformation may be applied to reduce the problem to $\Omega \subseteq \boldsymbol{C}$ and the result follows.

(ii) Let $P_1(s)$, $P_2(s)$ be Ω-equivalent, under the Ω-equivalence transformation (3.1), with $\Omega \subseteq \boldsymbol{C} \cup \{\infty\}$. If $s_0 \in \Omega$ is a pole of $P_2(s)$ then s_0 is also a pole of the compound matrix $(M(s) \ P_2(s))$ whose multiplicities can only be greater than its corresponding multiplicities in $P_2(s)$, since $M(s)$ is a rational matrix. If $M(s)$ is such as to increase the multiplicities of $P_2(s)$ then the Ω-least order conditions cannot be satisfied. It follows therefore that the Ω least-order conditions will be satisfied with respect to each point $s_0 \in \Omega$. Hence, if these least-order conditions are satisfied with respect to a given set Ω then they will be satisfied with respect to any subset $\Omega_1 \subseteq \Omega$. It can also be seen that the absence of the zeros in Ω of the compound matrix $(M(s) \ P_2(s))$ directly implies the absence of the zeros in Ω_1 of the same compound matrix. The same observations also apply to the compound matrix $(P_1(s)^T \ -N(s)^T)^T$ which establishes the theorem. □

Corollary 2:

(i) *If the Ω-least order conditions of Ω-equivalence are satisfied in $\Omega \subseteq \boldsymbol{C} \cup \{\infty\}$ then they are satisfied at each point of Ω.*

(ii) *If $P_1(s), P_2(s) \in P(p, m)$ are Ω-equivalent for $\Omega \subseteq \boldsymbol{C} \cup \{\infty\}$ then $P_1(s)$, $P_2(s)$ are s_0-equivalent for each $s_0 \in \Omega$ including $s_0 = \infty$.*

Note that conditions (i) and (ii) of Theorem 2 are only necessary conditions. We can see this in the following.

Example 1: Let $P_1(s) = (s+1)/(s+2)^2$ and $P_2(s) = s+1$. With $\Omega = \boldsymbol{C}$ consider the transformation

$$[(s+2)^2 \quad s+1]\begin{bmatrix} \dfrac{s+1}{(s+2)^2} \\ -1 \end{bmatrix} = 0 \tag{3.11}$$

Equation (3.11) is an Ω-equivalence transformation. It is easily seen however that $P_1(s)$ and $P_2(s)$ have the same finite zero structure but different finite pole structure and so $P_1(s)$ and $P_2(s)$ are not $\boldsymbol{C}$-unimodular equivalent. Thus, the converse of statement (i) of Theorem 2 is not true in this case. It can also be seen that

$$\nu_{C\cup\{\infty\}}[(s+2)^2 \quad s+1] = 2 \neq 1 = \nu_{C\cup\{\infty\}}(s+1) \tag{3.12}$$

and so $P_1(s)$ and $P_2(s)$ are not $\boldsymbol{C}\cup\{\infty\}$-equivalent which contradicts the converse of statement (ii) in Theorem 2.

The importance of Ω-equivalence is seen in the following theorem.

Theorem 3: *Ω-equivalence on $P(p, m)$ preserves the zero structure in Ω and the rank defect.*

Proof: Let $P_1(s)$, $P_2(s) \in P(p, m)$ and let

$$[M(s) \quad P_2(s)] = D_2^{-1}(s)[\widetilde{M}(s) \quad \widetilde{P}_2(s)]; \quad \begin{bmatrix} P_1(s) \\ -N(s) \end{bmatrix} = \begin{bmatrix} \widetilde{P}_1(s) \\ -\widetilde{N}(s) \end{bmatrix} D_1^{-1}(s) \tag{3.13}$$

be Ω-coprime MFDs. Then

$$\delta_\Omega(|D_2(s)|) = \nu_\Omega(M(s) \;\; P_2(s)) \overset{(3.3)}{=} \nu_\Omega(P_2(s)) \tag{3.14}$$

which means, from Corollary 1, that the Ω-MFD

$$P_2(s) = D_2^{-1}(s)\widetilde{P}_2(s) \tag{3.15}$$

is Ω-left coprime and so the zero structure in Ω and the rank defect of $P_2(s)$ and $\widetilde{P}_2(s)$ coincide. Similarly, $P_1(s) = \widetilde{P}_1(s)D_1^{-1}(s)$ is Ω-right coprime MFD and so the zero structure in Ω and the rank defect of $P_1(s)$ and $\widetilde{P}_1(s)$ coincide. Thus, from (3.1) and (3.13)

$$[\widetilde{M}(s) \quad \widetilde{P}_2(s)]\begin{bmatrix} \widetilde{P}_1(s) \\ -\widetilde{N}(s) \end{bmatrix} = 0 \tag{3.16}$$

where $[\widetilde{M}(s) \;\; \widetilde{P}_2(s)]$ and $[\widetilde{P}_1(s)^{\mathrm{T}} \;\; -\widetilde{N}(s)^{\mathrm{T}}]^{\mathrm{T}}$ have full rank $\forall s_0 \in \Omega$. From (2.14) it follows that there exist Ω-polynomial matrices $A_1(s)$, $A_2(s)$, $B_1(s)$, $B_2(s)$ such that

$$\widetilde{M}(s)A_1(s) + \widetilde{P}_2(s)B_1(s) = I_1 \tag{3.17}$$

$$A_2(s)\widetilde{N}(s) + B_2(s)\widetilde{P}_1(s) = I_2 \tag{3.18}$$

From (3.16), (3.17) and (3.18) we obtain

$$\begin{bmatrix} \widetilde{P}_2(s) & \widetilde{M}(s) \\ -A_3(s) & B_3(s) \end{bmatrix} \begin{bmatrix} B_1(s) & -\widetilde{N}(s) \\ A_1(s) & \widetilde{P}_1(s) \end{bmatrix} = \begin{bmatrix} I_1 & 0 \\ 0 & I_2 \end{bmatrix} \tag{3.19}$$

where

$$A_3(s) = -(A_2(s)B_1(s) - B_2(s)A_1(s))\widetilde{P}_2(s) + A_2(s) \tag{3.20 a}$$

and

$$B_3(s) = (A_2(s)B_1(s) - B_2(s)A_1(s))\widetilde{M}(s) + B_2(s) \tag{3.20 b}$$

From (3.19) the two Ω-polynomial matrices on the left-hand side are Ω-unimodular. Thus

$$\begin{bmatrix} B_1(s) & -\widetilde{N}(s) \\ A_1(s) & \widetilde{P}_1(s) \end{bmatrix} \begin{bmatrix} \widetilde{P}_2(s) & \widetilde{M}(s) \\ -A_3(s) & B_3(s) \end{bmatrix} = \begin{bmatrix} I_3 & 0 \\ 0 & I_4 \end{bmatrix} \tag{3.21}$$

and it follows immediately that

$$\begin{bmatrix} B_1(s) & -\widetilde{N}(s) \\ A_1(s) & \widetilde{P}_1(s) \end{bmatrix} \begin{bmatrix} \widetilde{P}_2(s) & 0 \\ -A_3(s) & I_2 \end{bmatrix} = \begin{bmatrix} I_3 & -\widetilde{N}(s) \\ 0 & \widetilde{P}_1(s) \end{bmatrix} \tag{3.22}$$

Thus, since $\widetilde{N}(s)$ and $A_3(s)$ are Ω-polynomials

$$\begin{bmatrix} I_2 & 0 \\ 0 & \widetilde{P}_2(s) \end{bmatrix} \text{ and } \begin{bmatrix} I_3 & 0 \\ 0 & \widetilde{P}_1(s) \end{bmatrix} \tag{3.23}$$

are Ω-u.e., and so $\widetilde{P}_1(s)$ and $\widetilde{P}_2(s)$, or equivalently $P_1(s)$ and $P_2(s)$, have the same zero structure and rank defect in Ω. □

Ω-equivalence therefore has the property of preserving the zero structure in Ω. The proof also indicates that the Ω-polynomial matrices $D_1(s)$ and $D_2(s)$ which define the pole structure in Ω of $P_1(s)$, $P_2(s)$ may be cancelled in (3.1) in the manner of (3.16). Thus, the pole structure in Ω is not invariant under Ω-equivalence.

Example 2: Consider again the transformation in Example 1

$$[(s+2)^2 \quad s+1] \begin{bmatrix} \dfrac{s+1}{(s+2)^2} \\ -1 \end{bmatrix} = 0 \tag{3.24}$$

Equation (3.24) is a $\boldsymbol{C}$-equivalence transformation and so $P_1(s)$ and $P_2(s)$ have the same zero structure in $\boldsymbol{C}$. Note, however, that $P_1(s)$ has one pole at $s = -2$ while $P_2(s)$ has no poles in $\boldsymbol{C}$. □

Theorem 4: *If $P_1(s)$, $P_2(s) \in P(p, m)$ have the same zero structure in Ω and the same rank defect they they are Ω-equivalent.*

Proof: Let $S^{\Omega}_{P_1(s)}$, $S^{\Omega}_{P_2(s)}$, be the Smith–McMillan forms in Ω of $P_1(s)$, $P_2(s)$. Then there exist Ω-unimodular matrices $U_1(s)$, $U_2(s)$, $U_3(s)$, $U_4(s)$ such that

$$U_1(s)P_1(s)U_2(s) = S^{\Omega}_{P_1(s)} = \text{blockdiag}\left(\frac{\varepsilon_1(s)}{\psi_1(s)}, \frac{\varepsilon_2(s)}{\psi_2(s)}, \ldots, \frac{\varepsilon_r(s)}{\psi_r(s)}, 0\right) \tag{3.25 a}$$

and

$$U_3(s)P_2(s)U_4(s) = S^{\Omega}_{P_2(s)}(s)$$
$$= \text{blockdiag}\left(\frac{1}{\hat{\psi}_1(s)}, \dots, \frac{1}{\hat{\psi}_{k-1}(s)}, \frac{\varepsilon_1(s)}{\hat{\psi}_k(s)}, \dots, \frac{\varepsilon_r(s)}{\hat{\psi}_{r+k-1}(s)}, 0\right) \tag{3.25 b}$$

where $S^{\Omega}_{P_1(s)}$, $S^{\Omega}_{P_2(s)}$ have the same zero structure and rank defect. Now

$$\left[\begin{array}{c|c} \Psi' & 0 \\ \hline 0 & I \end{array}\right] S^{\Omega}_{P_2(s)} = S^{\Omega}_{P_1(s)} \left[\begin{array}{c|c} \Psi & 0 \\ \hline 0 & I \end{array}\right] \tag{3.26}$$
$$\Psi' = (0_{r,k-1}, \text{diag}[\hat{\psi}_k(s), \dots, \hat{\psi}_{r+k-1}(s)])$$
$$\Psi = (0_{r,k-1}, \text{diag}[\psi_1(s), \dots, \psi_r(s)])$$

is an Ω-equivalence transformation since the compound matrices (3.2) arising from (3.26) satisfy all the conditions of Definition 5. From (3.26)

$$\left(U_1^{-1}(s)\left[\begin{array}{c|c} \Psi' & 0 \\ \hline 0 & I \end{array}\right] U_3(s)\right)P_2(s) = P_1(s)\left(U_2(s)\left[\begin{array}{c|c} \Psi & 0 \\ \hline 0 & I \end{array}\right] U_4^{-1}(s)\right) \tag{3.27}$$

Equation (3.26) is still an Ω-equivalence transformation since multiplication by Ω-unimodular matrices does not alter the conditions on (3.26). □

From Theorems 3 and 4 we see that Ω-equivalence is a necessary and sufficient condition for two rational matrices to have the same zero structure and rank defect in $\Omega \subset \boldsymbol{C} \cup \{\infty\}$. Also, note from Theorem 4 the following corollary.

Corollary 3: *The transforming matrices under Ω-equivalence may always be taken to be Ω-polynomial.*

Proof: The proof follows immediately from relation (3.27). □

Corollary 4: *Let $P_1(s)$, $P_2(s)$ be two Ω-polynomial matrices of the same dimension with $\Omega \subseteq \boldsymbol{C}$. Then $P_1(s)$ and $P_2(s)$ are Ω-*u.e. *if and only if they are Ω-equivalent.*

Proof: The necessity has been proved in (i) of Theorem 2. The sufficiency is because in this special case we always have the ability to take the transforming matrices to be Ω-unimodular (see (3.27) and Corollary 3). □

A consequence of the above remark is that (e.u.e.) defines the same equivalence classes as (u.e.) in the case of polynomial matrices with the same dimension. The above remark may also be applied in the case where $\Omega \subset \boldsymbol{C} \cup \{\infty\}$, under the usual bilinear transformation, which proves that (e.c.e.) defines the same equivalence classes as bicausal equivalence (Vardulakis *et al.* 1982, Vardulakis 1991) in the case of causal matrices with the same dimensions.

Theorem 5: *Ω-equivalence is an equivalence relation on $P(p, m)$.*

Proof:

(i) *Reflexivity property*. Clearly

$$I \cdot P(s) = P(s) \cdot I \tag{3.28}$$

is an Ω-equivalence transformation.

(ii) *Symmetry property*. Let $P_1(s)$, $P_2(s) \in P(p, m)$ be Ω-equivalent, i.e.

$$M(s)P_1(s) = P_2(s)N(s) \tag{3.29}$$

Then $P_1(s)$, $P_2(s)$ will have the same zero structure and rank defect in Ω by Theorem 3. Thus, there exists a transformation of the form (3.27) between $P_2(s)$ and $P_1(s)$, which proves the symmetry property.

(iii) *Transitivity property*. Let $P_1(s)$, $P_2(s) \in P(p, m)$ and $P_2(s)$, $P_3(s) \in P(p, m)$ be Ω-equivalent respectively. Then, $P_1(s)$, $P_2(s)$ and $P_2(s)$, $P_3(s)$ have the same zero structure and rank defect in Ω. Thus, $P_1(s)$, $P_3(s)$ have the same zero structure and rank defect in Ω and so, from Theorem 4, $P_1(s)$, $P_3(s)$ are Ω-equivalent. □

Example 3: In the special case where $\Omega \equiv \boldsymbol{C}$ and $P_1(s)$, $P_2(s)$ are polynomial matrices then Ω-equivalence coincides with (e.u.e.), (Pugh and Shelton 1978). This is because the Ω-least order conditions require the transforming matrices to be polynomial while the conditions that the compound matrices of (3.2) possess no zeros in Ω reduce to the usual relative primeness conditions of (e.u.e.). □

Example 4: In the special case that $\Omega \equiv s_0$ where $s_0 \in \boldsymbol{C}$, and $P_1(s)$, $P_2(s)$ are polynomial matrices then Ω-equivalence reduces to local equivalence (Cullen 1987). If $\Omega = \{\infty\}$, then from Corollary 3 the transforming matrices of Ω-equivalence are rational but by Theorem 4 these may be taken to be proper rational matrices. If, however, $P_1(s)$, $P_2(s)$ are polynomial then the Ω-least order conditions are redundant and Ω-equivalence reduces immediately to (e.c.e.) (Walker 1988). □

An interesting question still remains as to what transformations will preserve the zero structure of a rational matrix $P(s)$ in the region $\Omega = \boldsymbol{C} \cup \{\infty\}$. The following result shows that Ω-equivalence under the same conditions of Definition 5, even with $\Omega = \boldsymbol{C} \cup \{\infty\}$ still provides an answer.

Theorem 6: *If $P_1(s)$, $P_2(s) \in P(p, m)$ are $\boldsymbol{C} \cup \{\infty\}$-equivalent then they have the same finite and infinite zero structure.*

Proof: Let $\mathscr{A}$ denote the set of all locations of poles and zeros of $P_1(s)$, $P_2(s)$ in $\boldsymbol{C} \cup \{\infty\}$. Since $P_1(s)$, $P_2(s)$ are rational matrices over $\boldsymbol{R}$ it follows that $\mathscr{A}$ is finite and symmetric with respect to the real axis, and hence that $\mathscr{A} \subset \boldsymbol{C} \cup \{\infty\}$. Thus, there exist a real number $\alpha \notin \mathscr{A}$. From Theorem 2, $P_1(s)$, $P_2(s)$ are $\boldsymbol{C} \cup \{\infty\} \backslash \{\alpha\}$-equivalent since they are $\boldsymbol{C} \cup \{\infty\}$-equivalent. Thus, from Theorem 3 $P_1(s)$, $P_2(s)$ have identical zero structures in $\boldsymbol{C} \cup \{\infty\} \backslash \{\alpha\}$. From the definition of α, the matrices $P_1(s)$, $P_2(s)$ have no zeros at α, which completes the proof. □

Corollary 5: *In the case where $P_1(s)$, $P_2(s)$ are polynomial matrices, then $\boldsymbol{C} \cup \{\infty\}$-equivalence coincides with (f.e.) (Hayton et al. 1988).*

Proof: Note that if $P_1(s)$, $P_2(s)$ are polynomial then the $\boldsymbol{C} \cup \{\infty\}$-least order conditions (3.3) (and therefore from Corollary 2 the $\boldsymbol{C}$-least order conditions) imply that the transforming matrices can have no finite poles and are therefore polynomial. These particular conditions then reduce immediately to the McMillan degree conditions of (f.e.). The other conditions of $C \cup \{\infty\}$-equivalence coincide directly with those of (f.e.), as required. □

Corollary 6: *If $P_1(s)$, $P_2(s) \in P(p, m)$ are Ω-equivalent for any $\Omega \subseteq \boldsymbol{C} \cup \{\infty\}$ then $P_1(s)$, $P_2(s)$ have identical zero structure within Ω.*

Proof: If $\Omega \subseteq \boldsymbol{C}$ then Theorem 3 gives the result. If $\Omega = \boldsymbol{C} \cup \{\infty\}$ then Theorem 6 provides the result. If, however, $\Omega \subset \boldsymbol{C} \cup \{\infty\}$ then employing the usual bilinear transformation takes the region Ω to $\Omega' \subseteq \boldsymbol{C}$ and Theorem 3 then applies. □

Corollary 7: *Full equivalence implies (e.c.e.) and (e.u.e.).*

Proof: Let $P_1(s)$, $P_2(s)$ be two polynomial matrices which are (f.e.), or equivalently from Corollary 5, $\boldsymbol{C} \cup \{\infty\}$-equivalent. Obviously, $\{\infty\} \subset \boldsymbol{C} \cup \{\infty\}$ and $\boldsymbol{C} \subset \boldsymbol{C} \cup \{\infty\}$, and so from Theorem 2, $P_1(s)$ and $P_2(s)$ are both $\{\infty\}$-equivalent and $\boldsymbol{C}$-equivalent. However, from Examples 3 and 4, $\{\infty\}$-equivalence and $\boldsymbol{C}$-equivalence reduce to (e.c.e.) and (e.u.e.) respectively which completes the proof. □

4. Zero structure in $\Omega \subseteq \mathbf{C} \cup \{\infty\}$ under output feedback

Let $G(s) \in \boldsymbol{R}(s)^{p \times m}$ be the transfer function matrix of an open-loop system. If $G_F(s)$ denotes the transfer function matrix of the closed-loop system (see the Figure) formed under output feedback of the form

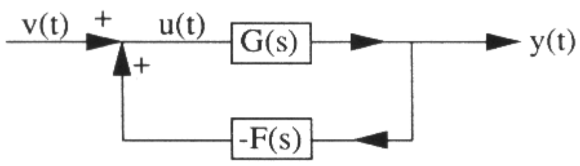
$$u(t) = -F(\rho)y(t) + v(t) \tag{4.1}$$

then

$$G_F(s) = G(s)(I + F(s)G(s))^{-1} = (I + G(s)F(s))^{-1}G(s) \tag{4.2}$$

provided, as will always be assumed, that

$$|I + G(s)F(s)| \not\equiv 0 \quad \text{or} \quad |I + F(s)G(s)| \not\equiv 0 \tag{4.3}$$



The study of the feedback invariants of (4.2) provides a ready application of the theory of Ω-equivalence. Consider, therefore, the transformation

$$(I + G(s)F(s)) \cdot G_F(s) = G(s) \cdot I \tag{4.4}$$

Theorem 7: *The zero structure in $\Omega \subseteq \boldsymbol{C} \cup \{\infty\}$ of the transfer function matrix $G(s)$, of the open-loop system remains invariant under Ω-polynomial output feedback, i.e. $F(s) \in \boldsymbol{R}_\Omega(s)^{m \times p}$, or equivalently $F(s)$ has no poles in $\Omega \subseteq \boldsymbol{C} \cup \{\infty\}$.*

Proof: We shall show that the transformation (4.4) is an Ω-equivalence transformation and so from Corollary 6, $G(s)$ and $G_F(s)$ will have the same zero structure in $\Omega \subseteq \boldsymbol{C} \cup \{\infty\}$.

Consider therefore the compound matrices, which arise from the transformation (4.4)

$$[I + G(s)F(s) \quad G(s)]; \quad \begin{bmatrix} G_F(s) \\ -I \end{bmatrix} \tag{4.5}$$

Let

$$G_F(s) = N_F(s)D_F^{-1}(s) \tag{4.6}$$

be a right coprime $\Omega \backslash \{\infty\}$-MFD of $G_F(s)$. Then

$$\begin{bmatrix} G_F(s) \\ -I \end{bmatrix} = \begin{bmatrix} N_F(s) \\ -D_F(s) \end{bmatrix} D_F^{-1}(s) \tag{4.7}$$

is clearly a right coprime $\Omega \backslash \{\infty\}$-MFD. Thus, the zero structure of $(G_F(s)^T \ -I)^T$ in $\Omega \backslash \{\infty\}$ coincides with that of the $(N_F(s)^T \ -D_F(s)^T)^T$ and so it has no zeros in $\Omega \backslash \{\infty\}$. In a similar way, and in the case $\{\infty\} \subseteq \Omega$, then $(G_F(\frac{1}{w})^T \ -I)^T$ can be seen to have no zeros at $w = 0$ and so $(G_F(s)^T \ -I)^T$ has no infinite zeros. Thus, the compound matrix $(G_F(s)^T \ -I)^T$ has no zeros in any region $\Omega \subseteq \boldsymbol{C} \cup \{\infty\}$. Clearly

$$\nu_\Omega \begin{bmatrix} G_F(s) \\ -I \end{bmatrix} = \nu_\Omega(G_F(s)) \tag{4.8}$$

and so the compound matrix $(G_F(s)^T \ -I)^T$ satisfies the conditions of Ω-equivalence.

For the other compound matrix, note that since $F(s)$ is Ω-polynomial, then

$$[I + G(s)F(s) \quad G(s)] = [I \quad G(s)] \begin{bmatrix} I & 0 \\ F(s) & I \end{bmatrix} \tag{4.9}$$

is a Ω-unimodular equivalence transformation of $(I \ G(s))$ which therefore preserves the pole–zero structure in a region $\Omega \subseteq \boldsymbol{C}$. As before, if $\Omega \subset \boldsymbol{C} \cup \{\infty\}$ then a bilinear transformation may be employed to transform Ω into a subset of $\boldsymbol{C}$. Thus, the pole–zero structure is also preserved in this case. Finally, if $\Omega \equiv \boldsymbol{C} \cup \{\infty\}$ then the transformation (4.9) is a strict equivalence transformation since, under the conditions $F(s)$, is $\boldsymbol{C} \cup \{\infty\}$-polynomial, $F(s)$ is then constant. Thus, in this case the finite and infinite pole-zero structure of $(I \ G(s))$ and the matrix on the left-hand side of (4.3) are identical. Thus, in all cases of the region $\Omega \subseteq \boldsymbol{C} \cup \{\infty\}$ the pole–zero structure in Ω of the compound matrix $(I + G(s)F(s) \ G(s))$ will coincide with that of the compound matrix $(I \ G(s))$. Hence, the compound matrix $(I + G(s)F(s) \ G(s))$ satisfies the conditions of Ω-equivalence since $(I \ G(s))$ satisfies these conditions. Consequently, (4.4) is a transformation of Ω-equivalence and so $G(s)$ and $G_F(s)$ have the same zero structure in the given region $\Omega \subseteq \boldsymbol{C} \cup \{\infty\}$. □

The above theorem gives an interesting enough result for a general region $\Omega \subseteq \boldsymbol{C} \cup \{\infty\}$. There are, however, several notable special cases.

Case 1: $\Omega \equiv \boldsymbol{C}$, then $F(s)$ is $\boldsymbol{C}$-polynomial, means that $F(s)$ is a conventional polynomial matrix. In this case Theorem 7 indicates that the finite zero structure of any open-loop system, remains invariant under pure derivative output feedback, while the finite pole structure and the infinite pole and zero structure does not remain invariant. □

Example 5: Let

$$G(s) = \begin{bmatrix} s+1 & 0 \\ \dfrac{s(s+2)}{s+1} & \dfrac{1}{s+1} \end{bmatrix} \quad \text{and} \quad F(s) = \begin{bmatrix} s-1 & 0 \\ -4 & s+3 \end{bmatrix} \tag{4.10}$$

Then

$$G_F(s) = \begin{bmatrix} \dfrac{s+1}{s^2} & 0 \\ \dfrac{s^2+6s+4}{2s^2(s+2)} & \dfrac{1}{2(s+2)} \end{bmatrix} \tag{4.11}$$

and it follows that

$$S^{C}_{G(s)}(s) = \begin{bmatrix} \dfrac{1}{s+1} & 0 \\ 0 & s+1 \end{bmatrix}; \quad S^{C}_{G_F(s)}(s) = \begin{bmatrix} \dfrac{1}{s^2(s+2)} & 0 \\ 0 & s+1 \end{bmatrix} \tag{4.12}$$

and

$$S^{\infty}_{G(s)}(s) = \begin{bmatrix} s & 0 \\ 0 & \dfrac{1}{s} \end{bmatrix}; \quad S^{\infty}_{G_F(s)}(s) = \begin{bmatrix} \dfrac{1}{s} & 0 \\ 0 & \dfrac{1}{s} \end{bmatrix} \tag{4.13}$$

which indicates that the finite zero structure remains invariant under polynomial output feedback while the finite pole structure and the infinite pole/zero structure does not. □

Case 2: $\Omega \equiv \{\infty\}$, then $F(s)$ is $\{\infty\}$-polynomial, means that $F(s)$ is causal. Theorem 7 then establishes that the infinite zero structure of any open loop system remains invariant under causal output feedback, i.e. $F(s)$ is a causal matrix, while the infinite pole structure together with the finite pole/zero structure may change. □

Example 6: Let

$$G(s) = \begin{bmatrix} s+1 & 0 \\ \dfrac{s(s+2)}{s+1} & \dfrac{1}{s+1} \end{bmatrix} \quad \text{and} \quad F(s) = \begin{bmatrix} \dfrac{s}{s+1} & 0 \\ \dfrac{1}{s(s+1)} & 1 \end{bmatrix} \tag{4.14}$$

Then

$$G_F(s) = \begin{bmatrix} 1 & 0 \\ \dfrac{s^2+s-1}{s(s+2)} & \dfrac{1}{s+2} \end{bmatrix} \tag{4.15}$$

If we compare the Smith–McMillan form at $s \in \boldsymbol{C}$ and at $s = \infty$ of $G(s)$ and $G_F(s)$ we have that

$$S^{\mathbf{C}}_{G(s)}(s) = \begin{bmatrix} \dfrac{1}{s+1} & 0 \\ 0 & s+1 \end{bmatrix}; \quad S^{\mathbf{C}}_{G_F(s)}(s) = \begin{bmatrix} \dfrac{1}{s(s+2)} & 0 \\ 0 & s \end{bmatrix} \tag{4.16}$$

and

$$S^{\infty}_{G(s)}(s) = \begin{bmatrix} s & 0 \\ 0 & \dfrac{1}{s} \end{bmatrix}; \quad S^{\infty}_{G_F(s)}(s) = \begin{bmatrix} 1 & 0 \\ 0 & \dfrac{1}{s} \end{bmatrix} \tag{4.17}$$

It is obvious that the infinite zero structure of $G(s)$ remains invariant under causal output feedback whereas the infinite pole structure and the finite pole/zero structure are different. □

Case 3: $\Omega \equiv \boldsymbol{C} \cup \{\infty\}$ (Pugh and Ratcliffe 1980), then $F(s)$ is $\boldsymbol{C} \cup \{\infty\}$-polynomial, means that F is constant. Theorem 7 then reduces to the well-known result that the finite and infinite zero structure of the open loop system remains invariant under constant output feedback, i.e. $F(s)$ is a constant matrix. □

Example 7: Consider the open loop system

$$G(s) = \begin{bmatrix} s+1 & 0 \\ \dfrac{s(s+2)}{s+1} & \dfrac{1}{s+1} \end{bmatrix} \tag{4.18}$$

under the constant (unity) output feedback given by

$$F(s) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I \tag{4.19}$$

Then

$$G_F(s) = \begin{bmatrix} \dfrac{s+1}{s+2} & 0 \\ \dfrac{s}{s+2} & \dfrac{1}{s+2} \end{bmatrix} \tag{4.20}$$

Note that

$$S^{\mathbf{C}}_{G(s)}(s) = \begin{bmatrix} \dfrac{1}{s+1} & 0 \\ 0 & s+1 \end{bmatrix}; \quad S^{\infty}_{G(s)}(s) = \begin{bmatrix} s & 0 \\ 0 & \dfrac{1}{s} \end{bmatrix} \tag{4.21}$$

while

$$S^{C}_{G_F(s)}(s) = \begin{bmatrix} \frac{1}{s+2} & 0 \\ 0 & \frac{s+1}{s+2} \end{bmatrix}; \quad S^{\infty}_{G_F(s)}(s) = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{s} \end{bmatrix} \tag{4.22}$$

Thus, $G(s)$ and $G_F(s)$ posses the same finite and infinite zero structure but different pole structures.

Case 4: $\Omega \equiv C^+ \cup \{\infty\}$ (extended right half-plane), then $F(s)$ is Ω-polynomial, means that $F(s)$ is stable. In this case Theorem 7 yields that the zero structure of $G(s)$ in the extended right half-plane remains invariant under stable output feedback. In particular, note that every minimum phase transfer function matrix remains minimum phase under stable output feedback. □

Example 8: Consider the minimum phase transfer function matrix

$$G(s) = \begin{bmatrix} s+1 & 0 \\ -\frac{s+2}{s+1} & \frac{s+4}{s+1} \end{bmatrix} \tag{4.23}$$

under the stable output feedback

$$F(s) = \begin{bmatrix} -\frac{s^2+3s+3}{(s+1)(s+2)^2} & 0 \\ \frac{s^2+4s+2}{(s+1)(s+4)} & \frac{4s+10}{(s+2)(s+4)} \end{bmatrix} \tag{4.24}$$

Then

$$G_F(s) = \begin{bmatrix} (s+2)^2 & 0 \\ -\frac{(s+2)^3}{s+3} & \frac{s+2}{s+3} \end{bmatrix} \tag{4.25}$$

Note that

$$S^{C}_{G(s)}(s) = \begin{bmatrix} \frac{1}{s+1} & 0 \\ 0 & (s+1)(s+4) \end{bmatrix} \quad \text{and} \quad S^{\infty}_{G(s)}(s) = \begin{bmatrix} s & 0 \\ 0 & 1 \end{bmatrix} \tag{4.26}$$

while

$$S^{C}_{G_F(s)}(s) = \begin{bmatrix} \frac{s+2}{s+3} & 0 \\ 0 & (s+2)^2 \end{bmatrix} \quad \text{and} \quad S^{\infty}_{G_F(s)}(s) = \begin{bmatrix} s^2 & 0 \\ 0 & 1 \end{bmatrix} \tag{4.27}$$

and so $G_F(s)$ is minimum phase as predicted by Theorem 7. □

5. Conditions for the absence of infinite zeros of rational matrices

In this section, as a further application of Ω-equivalence, an extension of some of the known conditions for the absence of infinite zeros of polynomial matrices (see for example Pugh *et al*. 1992), to the case of rational matrices, is presented. Some additional ideas for the study of the point at infinity are needed.

Definition 5 (Vardulakis *et al.* 1982, Vardulakis 1991)*:* Let $t(s) = n(s)/d(s) \in \boldsymbol{R}(s)$, where $n(s)$, $d(s) \in \boldsymbol{R}[s]$, $d(s) \neq 0$. We define a *discrete valuation at* $s = \infty$ of $t(s)$ as the map $\delta_\infty(\cdot)$: $\boldsymbol{R}(s) \to \boldsymbol{Z} \cup \{\infty\}$

$$\delta_\infty(t(s)) := \begin{cases} \deg d(s) - \deg n(s) & t(s) \not\equiv 0 \\ +\infty & t(s) \equiv 0 \end{cases} \tag{5.1}$$

Let now $T(s) \in \boldsymbol{R}(s)^{p\times m}$, with $\operatorname{rank}_{\boldsymbol{R}} T(s) = r$. We define a *valuation at* $s = \infty$ of the matrix $T(s)$ as the map $\delta_\infty(\cdot)$: $\boldsymbol{R}(s)^{p\times m} \to \boldsymbol{Z} \cup \{\infty\}$

$$\delta_\infty(T(s)) := \begin{cases} \min\left\{ \begin{array}{l} \delta_\infty(\cdot) \text{ among the } \delta_\infty(\cdot) \text{ of all} \\ r\text{th order minors of } T(s) \end{array}\right. & \text{if } r > 0 \\ +\infty & \text{if } r = 0 \end{cases} \tag{5.2}$$

□

Lemma 3 (Vardulakis *et al.* 1982, Vardulakis 1991)**:** *Let* $T(s) \in \boldsymbol{R}(s)^{p\times m}$, *with* $\operatorname{rank}_{\boldsymbol{R}} T(s) = r$ *and consider the Smith–McMillan form at* $s = \infty$ *of* $T(s)$

$$S^{\infty}_{T(s)} := \operatorname{blockdiag}\left(s^{q_1}, s^{q_2}, \ldots, s^{q_k}, \frac{1}{s^{\hat{q}_{k+1}}}, \ldots, \frac{1}{s^{\hat{q}_r}}, 0_{p-r,m-r}\right) \tag{5.3}$$

where $q_1 \geqslant q_2 \geqslant \cdots \geqslant q_k$ and $\hat{q}_r \geqslant \hat{q}_{r-1} \geqslant \cdots \geqslant \hat{q}_{k+1}$ *are respectively the orders of the poles and the zeros of* $T(s)$ *at* $s = \infty$. *Then*

$$\delta_\infty(T(s)) = \sum_{i=k+1}^{r} \hat{q}_i - \sum_{i=1}^{k} q_i \tag{5.4}$$

Lemma 3 and the notion of $\{\infty\}$-least order (Definition 4) may be combined to give the following lemma.

Lemma 4: $T(s) \in \boldsymbol{R}(s)^{p\times m}$, *with* $\operatorname{rank}_{\boldsymbol{R}} T(s) = r$. $T(s)$ *has no infinite zeros if and only if*

$$\delta_\infty(T(s)) = -\nu_\infty(T(s)) \tag{5.5}$$

Proof: Notice that from the definition of $\{\infty\}$-least order

$$\nu_\infty(T(s)) = \sum_{i=1}^{k} q_i \tag{5.6}$$

and so from (5.4) and (5.6) we obtain that

$$\delta_\infty(T(s)) = -\nu_\infty(T(s)) \Leftrightarrow \sum_{i=k+1}^{r} \hat{q}_i = 0 \Leftrightarrow \hat{q}_i = 0 \quad i = k+1, \ldots, r \tag{5.7}$$

Since $\hat{q}_i$ are the orders of the infinite zeros, the results follow. □

Let $\delta_{\infty i}(\cdot)$ denote the ith row valuation at $s = \infty$ of the indicated matrix, then any rational matrix $T(s) \in \boldsymbol{R}^{p\times m}$ can be written (Vardulakis 1991) as

$$T(s) = \begin{bmatrix} \left(\frac{1}{s}\right)^{\delta_{\infty 1}(T)} & 0 & \cdots & 0 \\ 0 & \left(\frac{1}{s}\right)^{\delta_{\infty 2}(T)} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \left(\frac{1}{s}\right)^{\delta_{\infty p}(T)} \end{bmatrix} C + \hat{T}(s) \qquad (5.8)$$

where $C \in \boldsymbol{R}^{p\times m}$ and $\hat{T}(s) \in \boldsymbol{R}^{p\times m}(s)$ with $\delta_{\infty i}(\hat{T}(s)) \geqslant \delta_{\infty i}(T(s))$, $i \in \boldsymbol{p}$.

Definition 6 (Verghese and Kailath 1981, Vardulakis (1991): The constant matrix C in (5.8) is defined as the *least row valuation* (at $s = \infty$) *coefficient matrix* of the rational matrix $T(s)$ and is denoted also by $[T(s)]_{\mathrm{r}}^{\mathrm{l}}$, i.e. $[T(s)]_{\mathrm{r}}^{\mathrm{l}} := C \in \boldsymbol{R}^{p\times m}$. $T(s) \in \boldsymbol{R}^{p\times m}$ is called *row reduced* at $s = \infty$ if

$$\operatorname{rank}_{\boldsymbol{R}} [T(s)]_{\mathrm{r}}^{\mathrm{l}} = \min(p, m) \qquad (5.9)$$

□

An important property of row reducedness is given by the following lemma.

Lemma 5 (Verghese and Kailath 1981, Vardulakis 1991): *Let* $T(s) \in \boldsymbol{R}^{p\times m}$ *with* $\operatorname{rank}_{\boldsymbol{R}(s)} T(s) = p$. *Then* $T(s)$ *is row reduced at* $s = \infty$ *if and only if the pole–zero structure at* $s = \infty$ *of* $T(s)$ *is given by the pole–zero structure of its rows taken separately.*

Let now $T(s) \in \boldsymbol{R}(s)^{p\times m}$, with $p < m$ and let $V(s)$ be any $p \times p$ rational matrix such that

$$\bar{T}(s) = V(s)T(s) \qquad (5.10)$$

where $\bar{T}(s)$ is row reduced at $s = \infty$ and $\delta_{\infty i}(\bar{T}(s)) \leqslant 0$ $\forall i \in \boldsymbol{p}$. Notice that the condition $\delta_{\infty i}(\bar{T}(s)) \leqslant 0$ $\forall i \in \boldsymbol{p}$ guarantees that the row reduced matrix $\bar{T}(s)$ has no infinite zeros. (5.10) may be written in the form

$$[V(s) \quad \bar{T}(s)] \begin{bmatrix} T(s) \\ -I \end{bmatrix} = 0 \qquad (5.11)$$

Theorem 8: *With the above notation and assumptions,* (5.11) *is an* $\{\infty\}$*-equivalence transformation if and only if*

$$\delta_{\infty i}(V(s)) \geqslant \delta_{\infty i}(\bar{T}(s)) \qquad (5.12)$$

Proof:

($\Leftarrow$) Assume that (5.11) is an $\{\infty\}$-equivalence transformation. The $\{\infty\}$-least order conditions of this transformation give

$$v_{\infty}(V(s) \quad \bar{T}(s)) = v_{\infty}(\bar{T}(s)) \qquad (5.13)$$

Since $\bar{T}(s)$ is row reduced at $s = \infty$ and $\delta_{\infty i}(\bar{T}(s)) \leqslant 0$ $\forall i \in \boldsymbol{p}$ it follows from Lemma 5 that it has no infinite zeros and so satisfies the condition of Lemma 4, i.e.

$$-v_{\infty}(\bar{T}(s)) = \delta_{\infty}(\bar{T}(s)) \qquad (5.14)$$

From (5.13) and (5.14) it then follows

$$-\nu_\infty(V(s)\ \ \bar{T}(s)) = \delta_\infty(\bar{T}(s)) \tag{5.15}$$

which implies that

$$\delta_{\infty i}(V(s)) \geqslant \delta_{\infty i}(\bar{T}(s)) \quad \text{since} \quad \delta_{\infty i}(\bar{T}(s)) \leqslant 0 \tag{5.16}$$

Hence the condition (5.12) is satisfied.

($\Rightarrow$) Suppose that (5.11) and (5.12) hold. It is obvious that the compound matrix

$$\begin{bmatrix} T(s) \\ -I \end{bmatrix} \tag{5.17}$$

has no infinite zeros and satisfies the $\{\infty\}$-least order conditions. Consider then the compound matrix

$$[V(s) \quad \bar{T}(s)] \tag{5.18}$$

The condition (5.12) together with the assumption $\delta_{\infty i}(\bar{T}(s)) \leqslant 0\ \forall i \in \boldsymbol{p}$ implies that

$$\delta_{\infty i}(V(s)\ \ \bar{T}(s)) = \delta_{\infty i}(\bar{T}(s)) \leqslant 0 \tag{5.19}$$

and since $\bar{T}(s)$ is row reduced at $s = \infty$, it follows therefore that $(V(s)\ \ \bar{T}(s))$ is also row reduced at $s = \infty$ and has no infinite zeros. Thus

$$\begin{aligned} \nu_\infty([V(s) \quad \bar{T}(s)]) &\overset{\text{(Definition 4)}}{=} \sum q_i([V(s) \quad \bar{T}(s)]) \\ &\overset{\text{(Lemma 5)}}{=} -\sum \delta_{\infty i}([V(s) \quad \bar{T}(s)]) \\ &\overset{(5.19)}{=} -\sum \delta_{\infty i}(\bar{T}(s)) \overset{\text{(Lemma 5)}}{=} \sum q_i(\bar{T}(s)) \\ &\overset{\text{(Definition 4)}}{=} \nu_\infty(\bar{T}(s)) \end{aligned} \tag{5.20}$$

where $q_i(\cdot)$ denotes the orders of the poles at $s = \infty$ of the indicated matrix, and so (5.11) is an $\{\infty\}$-equivalence transformation. □

Corollary 8: *Conditions* (5.11) *and* (5.12) *give a sufficient condition for the absence of the infinite zeros of a rational matrix.*

Proof: It is shown in Theorem 8 that under the conditions (5.11) and (5.12), the rational matrix $T(s)$ is $\{\infty\}$-equivalent to a rational matrix $\bar{T}(s)$ row reduced at $s = \infty$. Since $\delta_{\infty i}(\bar{T}(s)) \leqslant 0\ \forall i \in \boldsymbol{p}$, $\bar{T}(s)$ has no infinite zeros, it follows from Corollary 6, that $T(s)$ has no infinite zeros. □

It is noted that in the case where $\bar{T}(s)$ and $T(s)$ are polynomial matrices (and so $\delta_{\infty i}(\bar{T}(s)) \leqslant 0\ \forall i \in \boldsymbol{p}$ is automatically satisfied), and $V(s)$ is a unimodular matrix then the conditions (5.11) and (5.12) reduce to those proposed by Zhang (1989 a, b). In this case, Corollary 8 and Theorem 8 give a sufficient condition for the absence of the infinite zeros of a polynomial matrix, which is consistent with that proposed by Pugh *et al*. (1992).

From Lemma 4, Theorem 8 and Corollary 8, we can now give a number of conditions for a rational matrix to possess no infinite zeros.

Theorem 9: *Let $T(s) \in \mathbf{R}(s)^{p\times m}$, with $p < m$ and* $\operatorname{rank}_{\mathbf{R}(s)} T(s) = p$. *Let also the Laurent expansion of $T(s)$ be*

$$T(s) = T_q s^q + \cdots + T_1 s + T_0 + T_{-1}\frac{1}{s} + \cdots \tag{5.21}$$

Then $T(s)$ has no infinite zeros if one of the following conditions is satisfied

(*a*) *$T(s)$ has a right proper inverse (Kailath 1980).*

(*b*) *The minimum among the $\delta_\infty(\cdot)$ of all* kth *order minors of $T(s)$ coincides with $-v_\infty(T(s))$ (Lemma 4) i.e. $\delta_\infty(T(s)) = -v_\infty(T(s))$.*

(*c*) *There exist a $p \times p$ rational matrix $V(s)$ such that*

$$\bar{T}(s) = V(s)T(s) \tag{5.22}$$

where $\bar{T}(s)$ is row reduced at $s = \infty$, $\delta_{\infty i}(\bar{T}(s)) \leqslant 0 \ \forall i \in \mathbf{p}$, and the following conditions are satisfied

$$\delta_{\infty i}(V(s)) \geqslant \delta_{\infty i}(\bar{T}(s)) \tag{5.23}$$

(*d*) (Pugh *et al*. 1989)

$$\operatorname{rank}_{\mathbf{R}}\begin{bmatrix} T_q & 0 & \cdots & 0 \\ T_{q-1} & T_q & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ T_0 & T_1 & \cdots & T_q \end{bmatrix} = \operatorname{rank}_{\mathbf{R}}\begin{bmatrix} T_q & 0 & \cdots & 0 \\ T_{q-1} & T_q & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ T_1 & T_2 & \cdots & T_q \end{bmatrix} + p \tag{5.24}$$

The theorem notes the sufficiency of the four conditions it proposes but it should also be noted that conditions (*a*), (*b*) and (*d*) are in fact necessary and sufficient conditions for the absence of infinite zeros of a rational matrix.

6. Conclusions

A transformation, called Ω-equivalence, between rational matrices of different dimensions has been defined. It is shown that Ω-equivalence preserves the zero structure of the given matrices, within the region $\Omega \subseteq \mathbf{C} \cup \{\infty\}$, and defines an equivalence relation on the set $P(p, m)$ in the case where $\Omega \subset \mathbf{C} \cup \{\infty\}$. It is observed that Ω-equivalence is a generalization of many known transformations, for example (e.u.e.), (e.c.e.), local equivalence and (f.e.). The notion of Ω-equivalence provides a neat explanation of the invariance of the zero structure in $\Omega \subseteq \mathbf{C} \cup \{\infty\}$ of a given rational transfer function matrix under Ω-polynomial output feedback, and a number of interesting special cases arise. The notion of Ω-equivalence has further been utilized to provide conditions for the absence of infinite zeros in a rational matrix, one of which generalizes the work of Zhang (1989 a, b) from polynomial matrices to the rational matrix case.

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