A Fundamental Notion of Equivalence for Linear Multivariable Systems

A. C. Pugh, N. P. Karampetakis, A. I. G. Vardulakis, and G. E. Hayton

***Abstract*— A fundamental form of equivalence of polynomial matrix descriptions of linear multivariable systems is defined, based on the existence of a bijective map between the finite and infinite solution sets of the differential equations describing the two systems. The connection with the system matrix relationship of full system equivalence is established.**

I. Introduction

Strict system equivalence was proposed in [10] for the study of the finite frequency behavior of linear systems. It has the property of leaving invariant the finite structure of any polynomial matrix description (PMD) of a linear multivariable system Σ to which it is applied. Further studies of this transformation have been conducted in [2], while notably [7] showed that it is equivalent to the existence of a certain bijective map between the finite solution sets of the differential equations describing the system.

A number of important system properties are related to infinite frequency behavior, and these are not left invariant by the previous transformation. For this reason, [12] proposed the notion of strong system equivalence for systems in generalized state-space form, while [8] gave a closed-form description of this transformation termed complete system equivalence. Subsequently, [3] gave interpretations of this form of equivalence in terms of bijective maps between the finite and infinite solution sets as well as between the restricted initial conditions of the generalized state-space equations.

More recently [1, 5] have proposed extensions of these latter transformations to the general case of PMD's, which were termed respectively strong system equivalence and full system equivalence. The picture is less complete in this general case but it turns out that full system equivalence is a closed-form matrix description of the equivalence in this case. This paper takes up the views of [3], [7], [12] and gives a characterization of full system equivalence in terms of bijective maps between the finite and infinite solution sets of the differential equations underlying the PMD. Within this framework a complete explanation of the role of the conditions of full system equivalence emerges, which establishes that this transformation plays the same role in the generalised study of linear systems as strict system equivalence does in the conventional study.

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II. Preliminary Results

Consider a linear system Σ described by a PMD

$$T(\rho)\beta(t) = U(\rho)u(t) \tag{2.1a}$$

$$y(t) = V(\rho)\beta(t) + W(\rho)u(t) \tag{2.1b}$$

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where ($\rho = d/dt$), $T(\rho) \in \mathbb{R}[\rho]^{r\times r}$ with $|T(\rho)| \neq 0$, $U(\rho) \in \mathbb{R}[\rho]^{r\times m}$, $V(\rho) \in \mathbb{R}[\rho]^{p\times r}$, $W(\rho) \in \mathbb{R}[\rho]^{p\times m}$. Let

$$P(s) := \begin{bmatrix} T(s) & -U(s) \\ V(s) & W(s) \end{bmatrix} \in \mathbb{R}[s]^{(r+p)\times(r+m)} \tag{2.2}$$

be the Rosenbrock system matrix corresponding to Σ. The normalized form of $P(s)$ is [12]

$$\mathcal{P}(s) := \left[\begin{array}{ccc|c} T & -U & 0 & 0 \\ V & W & -I_p & 0 \\ 0 & I_m & 0 & -I_m \\ \hline 0 & 0 & I_p & 0 \end{array}\right] =: \begin{bmatrix} \mathcal{T}(s) & -\mathcal{U} \\ \mathcal{V} & 0 \end{bmatrix} \tag{2.3}$$

which has the advantage over (2.2) of permitting consistent definitions of finite and infinite frequency characteristics to be made.

Let p, m be fixed positive integers. Define $P_0(p, m)$ as the set of $(r+p)\times(r+m)$ Rosenbrock system matrices (i.e., $r > 0$) and $P(p, m)$ as the set of $(r+p)\times(r+m)$ polynomial matrices where $r \geq \max(-p, -m)$.

Definition 1: [4] $T_1(s)$, $T_2(s) \in P(p, m)$ are fully equivalent (FE) in case $\exists$ polynomial matrices $M(s)$, $N(s)$ of appropriate dimensions such that

$$[M \quad T_2]\begin{bmatrix} T_1 \\ -N \end{bmatrix} = 0 \tag{2.4}$$

and where the compound matrices

$$[M \quad T_2]; \begin{bmatrix} T_1 \\ -N \end{bmatrix} \tag{2.5}$$

are such that

1) they have full normal rank (2.6a)
2) they have no finite nor infinite zeros (2.6b)
3) the following McMillan degree conditions hold

$$\delta_M([M \quad T_2] = \delta_M(T_2); \; \delta_M\left(\begin{bmatrix} T_1 \\ -N \end{bmatrix}\right) = \delta_M(T_1). \tag{2.6c}$$

□

One notable form of FE is obtained when $T_1(s)$, $T_2(s)$ are regular pencils and $M(s)$, $N(s)$ are constant matrices. This is termed [8] complete equivalence (CE). If FE is restricted to PMD's then the following are obtained

Definition 2: $(P_1(s), P_2(s)) \in P_0(p, m) \times P_0(p, m)$ are full system equivalent (FSE) if $\exists$ polynomial matrices $M(s)$, $N(s)$, $X(s)$, $Y(s)$ such that

$$\begin{bmatrix} M & 0 \\ X & I \end{bmatrix}\begin{bmatrix} T_1 & -U_1 \\ V_1 & W_1 \end{bmatrix} = \begin{bmatrix} T_2 & -U_2 \\ V_2 & W_2 \end{bmatrix}\begin{bmatrix} N & Y \\ 0 & I \end{bmatrix} \tag{2.7}$$

is a transformation of FE. □

Definition 3: The normalized forms $\mathcal{P}_1(s)$, $\mathcal{P}_2(s)$ of $(P_1(s), P_2(s)) \in P_0(p, m) \times P_0(p, m)$ are normal full system equivalent (NFSE) if $\exists$ polynomial matrices $M(s)$, $N(s)$, $X(s)$, $Y(s)$, such that $\mathcal{P}_1(s)$ and $\mathcal{P}_2(s)$ are related by

$$\begin{bmatrix} M & 0 \\ X & I \end{bmatrix}\begin{bmatrix} \mathcal{T}_1(s) & -\mathcal{U}_1 \\ \mathcal{V}_1 & 0 \end{bmatrix} = \begin{bmatrix} \mathcal{T}_2(s) & -\mathcal{U}_2 \\ \mathcal{V}_2 & 0 \end{bmatrix}\begin{bmatrix} N & Y \\ 0 & I \end{bmatrix} \tag{2.8}$$

where (2.8) is a transformation of FE. □

FSE and NFSE define equivalence relations on $P_0(p, m)$ which give rise to identical equivalence classes [5], although the distinction is preserved here. Further properties of this equivalence which underscore its importance in linear system theory are [5, 6],

Lemma 1: Under FSE, NFSE the following are invariant:

1) the generalized order $f := \delta_M(\mathcal{T}(s))$ and the Rosenbrock degree d_R,
2) the transfer function matrix $G(s)$ and so the finite and infinite transmission poles and zeros of $G(s)$,
3) the finite and infinite system poles and zeros,
4) the finite and infinite input (output) decoupling zeros. □

If CE is restricted to the case of generalised state-space systems in the same way that FSE is obtained from FE, it produces a transformation which is termed complete system equivalence (CSE), [8]. This has been shown to possess a mapping interpretation which indicates that CSE is a natural form of equivalence for generalised state-space systems, [3]. We therefore seek an extension of these ideas to produce a mapping interpretation of FSE which would similarly indicate that FSE is the natural transformation for the generalised study of polynomial system matrix descriptions.

III. Mappings of Solution Sets

Define X_u as the set of all solutions of (2.1a) corresponding to the input $u(t)$ for all possible initial conditions on $\beta(t)$ and its $q-1$ derivatives where q is the highest power of s occurring in $P(s)$. Let X_u be related to the set X_u^1, the range of the relation

$$\begin{bmatrix} \beta_1(t) \\ u(t) \end{bmatrix} = \begin{bmatrix} \overline{N}(\rho) & \overline{Y}(\rho) \\ 0 & I_m \end{bmatrix}\begin{bmatrix} \beta(t) \\ u(t) \end{bmatrix} \tag{3.1}$$

where $\overline{N}(\rho) \in \mathbb{R}[\rho]^{r_1\times r}$ and $\overline{Y}(\rho) \in \mathbb{R}[\rho]^{r_1\times m}$. (3.1) may be written as

$$\begin{bmatrix} \beta_1(t) \\ u(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} \overline{N}(\rho) & \overline{Y}(\rho) & 0 \\ 0 & I_m & 0 \\ 0 & 0 & I_p \end{bmatrix}\begin{bmatrix} \beta(t) \\ u(t) \\ y(t) \end{bmatrix} \Rightarrow \xi_1(t) = N(\rho)\xi(t) \tag{3.2}$$

where $\xi_1(t) = [\beta_1(t)^T, u(t)^T, y(t)^T]^T$, $N(\rho) \in \mathbb{R}[\rho]^{\overline{r}_1\times\overline{r}}$, $\overline{r}_1 := r_1 + p + m$, $\overline{r} := r + p + m$ and $\xi(t) = [\beta(t)^T, u(t)^T, y(t)^T]^T$ is the pseudostate of the normalized form of (2.1)

$$\mathcal{T}(\rho)\xi(t) = \mathcal{U}u(t) \tag{3.3a}$$

$$y(t) = \mathcal{V}\xi(t). \tag{3.3b}$$

Define $\mathcal{X}_u$ as the set of all solutions $\xi(t)$, with (3.3a) corresponding to a given input $u(t)$ for all possible initial conditions on $\xi(t)$ and its $q-1$ derivatives. Suppose $\mathcal{X}_u$ is mapped onto the set $\mathcal{X}_u^1$ defined as the range of the relation

$$\begin{bmatrix} \xi_1(t) \\ u(t) \end{bmatrix} = \begin{bmatrix} N(\rho) & Y_1(\rho) \\ 0 & I_m \end{bmatrix}\begin{bmatrix} \xi(t) \\ u(t) \end{bmatrix} \tag{3.4}$$

where $N(\rho) \in \mathbb{R}[\rho]^{\overline{r}_1\times\overline{r}}$ and $Y_1(\rho) \in \mathbb{R}[\rho]^{\overline{r}_1\times m}$. Now writing

$$N(\rho) = [N_1(\rho) \quad N_2(\rho) \quad N_3(\rho)] \tag{3.5}$$

it follows from (3.4) that

$$\xi_1(t) = [N_1(\rho) \quad N_2(\rho) + Y_1(\rho) \quad N_3(\rho)]\xi(t) = N'(\rho)\xi(t). \tag{3.6}$$

For the above reasons we shall be interested only in relations of the form (3.2). Many authors [1], [7], [11] have considered the action of such a linear mapping on the solution space of a given PMD. The effects have only been partially quantified, generally as a result of focusing attention simply on the finite frequency behavior. Even when the impulsive behavior has been taken into account [7], however, full consideration has not been given, specifically in regard to the issue of the map being well constituted. A complete analysis will be given here, and the following example indicates one difficulty hitherto overlooked.

Example 1: Consider the homogeneous system of differential equations

$$\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & \rho\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}\xi_1(t)\\ \xi_2(t)\\ \xi_3(t)\end{bmatrix} = 0 \tag{E.1.1}$$

or equivalently in the frequency domain

$$\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & s\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}\overline{\xi}_1(s)\\ \overline{\xi}_2(s)\\ \overline{\xi}_3(s)\end{bmatrix} = \begin{bmatrix}0\\ \xi_3(0-)\\ 0\end{bmatrix}.$$

Hence

$$\begin{bmatrix}\xi_1(t)\\ \xi_2(t)\\ \xi_3(t)\end{bmatrix} = \begin{bmatrix}0\\ \xi_3(0-)\delta(t)\\ 0\end{bmatrix} \tag{E.1.2}$$

and so the solutions of (E.1.1) are parametrized by the single parameter $\xi_3(0-)$. Consider the relation

$$y(t) = \begin{bmatrix}y_1(t)\\ y_2(t)\end{bmatrix} = \begin{bmatrix}L_1\rho + L_0 & 0 & 1\\ 0 & 1 & 0\end{bmatrix}\begin{bmatrix}\xi_1(t)\\ \xi_2(t)\\ \xi_3(t)\end{bmatrix} \tag{E.1.3}$$

which in the frequency domain takes the form

$$\begin{bmatrix}\overline{y}_1(s)\\ \overline{y}_2(s)\end{bmatrix} = \begin{bmatrix}L_1 s + L_0 & 0 & 1\\ 0 & 1 & 0\end{bmatrix}\begin{bmatrix}\overline{\xi}_1(s)\\ \overline{\xi}_2(s)\\ \overline{\xi}_3(s)\end{bmatrix} + \begin{bmatrix}-L_1\xi_1(0-)\\ 0\end{bmatrix}.$$

The image under this "map" of any solution of (E.1.1) is then

$$y(t) = \begin{bmatrix}y_1(t)\\ y_2(t)\end{bmatrix} = \begin{bmatrix}-L_1\xi_1(0-)\delta(t)\\ \xi_3(0-)\delta(t)\end{bmatrix}. \tag{E.1.4}$$

It is clear therefore that the relation (E.1.3) is not a map in the strict sense because any one solution $\xi(t)$ of (E.1.1), which is parametrized solely by $\xi_3(0-)$, gives rise to many images $y(t)$ depending on the initial value $\xi_1(0-)$. □

Consider

$$\xi_1(t) = N(\rho)\xi(t). \tag{3.7}$$

The first question regarding (3.7) is therefore whether it formally represents a mapping of $\mathcal{X}_u$ to $\mathcal{X}_u^1$, or whether it is merely a relation. Without loss of generality it will be assumed that $u(t) = 0$. Thus consider the solution space $\mathcal{X}_0$ of the unforced system

$$\mathcal{T}(\rho)\xi(t) = 0 \tag{3.8}$$

under the relation (3.7). Equivalently consider the equations

$$\begin{bmatrix}0\\ \xi_1(t)\end{bmatrix} = \begin{bmatrix}\mathcal{T}(\rho)\\ N(\rho)\end{bmatrix}\xi(t). \tag{3.9}$$

Without loss of generality it may be assumed that

$$\mathcal{T}(\rho) := \mathcal{T}_k\rho^k + \cdots + \mathcal{T}_1\rho + \mathcal{T}_0$$

$$N(\rho) := N_k\rho^k + \cdots + N_1\rho + N_0 \tag{3.10}$$

where at least one of $\mathcal{T}_k, N_k \neq 0$. Taking Laplace transforms of (3.9) with assumed values $\xi^{(i)}(0-)$, $i = 1, \cdots, k-1$ gives

$$\begin{bmatrix}0\\ \hat{\xi}_1(s)\end{bmatrix} = \begin{bmatrix}\mathcal{T}(s)\\ N(s)\end{bmatrix}\hat{\xi}(s) - \begin{bmatrix}\hat{a}_\mathcal{T}(s)\\ \hat{a}_N(s)\end{bmatrix} \tag{3.11}$$

where the polynomial vectors $\hat{a}_\mathcal{T}(s)$, $\hat{a}_N(s)$ are given by

$$\hat{a}_\mathcal{Q}(s) = [s^{k-1}I, \cdots, sI, I]\begin{bmatrix}\mathcal{Q}_k & 0 & \cdots & 0\\ \vdots & \vdots & \cdots & \vdots\\ \mathcal{Q}_2 & \mathcal{Q}_3 & \ddots & 0\\ \mathcal{Q}_1 & \mathcal{Q}_2 & \cdots & \mathcal{Q}_k\end{bmatrix}\begin{bmatrix}\xi(0-)\\ \vdots\\ \xi^{(k-2)}(0-)\\ \xi^{(k-1)}(0-)\end{bmatrix}$$

$$:= \mathcal{S}_{k-1}\mathcal{X}_\mathcal{Q}\overline{\xi}(0-)\ \mathcal{Q} = \mathcal{T},\ N. \tag{3.12}$$

Theorem 1: (3.7) is a mapping in the formal sense iff

$$\delta_M\begin{bmatrix}\mathcal{T}(s)\\ N(s)\end{bmatrix} = \delta_M(\mathcal{T}(s)). \tag{3.13}$$

Proof: The relation (3.7) is a mapping in the formal sense iff it uniquely specifies an image, $\xi_1(t)$, for each solution $\xi(t)$ of (3.8). Accordingly, in respect of (3.11), the relation (3.7) is a mapping iff for each $\hat{\xi}(s)$ determined by

$$\mathcal{T}(s)\hat{\xi}(s) = \hat{a}_\mathcal{T}(s) \tag{3.14}$$

the relation

$$\hat{\xi}_1(s) = N(s)\hat{\xi}(s) - \hat{a}_N(s) \tag{3.15}$$

determines $\hat{\xi}_1(s)$ uniquely. Clearly that if a given $\hat{\xi}(s)$ has two images $\hat{\xi}_1(s)$ and $\hat{\xi}_1'(s)$ under (3.15), then this is due to $\hat{a}_N(s)$ or more particularly to $\overline{\xi}(0-)$ of (3.12). Let $\overline{\xi}(0-) \neq \overline{\xi}'(0-)$ be two initial condition vectors then these determine the same solution $\xi(t)$ provided that $\overline{\xi}(0-) - \overline{\xi}'(0-) \in \ker\mathcal{X}_\mathcal{T}$. If this condition is satisfied, then $\hat{\xi}_1(s)$ is determined via (3.15) as the unique image of $\hat{\xi}(s)$ provided $\overline{\xi}(0-) - \overline{\xi}'(0-) \in \ker\mathcal{X}_N$. Hence (3.7) is a map in the formal sense iff

$$\ker\mathcal{X}_\mathcal{T} \subseteq \ker\mathcal{X}_N \tag{3.16}$$

or equivalently

$$\ker\begin{bmatrix}\mathcal{X}_\mathcal{T}\\ \mathcal{X}_N\end{bmatrix} = \ker\mathcal{X}_\mathcal{T}. \tag{3.17}$$

From the dimension theorem of linear maps, (3.17) holds iff

$$\mathrm{Rank}_\mathbf{R}\begin{bmatrix}\mathcal{X}_\mathcal{T}\\ \mathcal{X}_N\end{bmatrix} = \mathrm{Rank}_\mathbf{R}\,\mathcal{X}_\mathcal{T}. \tag{3.18}$$

Hence from the characterisation of the McMillan degree

$$\delta_M\left(\begin{bmatrix}\mathcal{T}(s)\\ N(s)\end{bmatrix}\right) = \delta_M(\mathcal{T}(s)). \tag{3.19}$$

□

Remark: In Example 1 it can be seen that (E.1.3) is a map iff $L_1 = 0$. Further

$$\delta_M\begin{bmatrix}\mathcal{T}(s)\\ N(s)\end{bmatrix} \geq \delta_M(\mathcal{T}(s)) \tag{3.20}$$

with equality holding iff $L_1 = 0$, which verifies the theorem. □

The existence condition for the map (3.7) to be properly constituted is thus revealed as the McMillan degree condition appearing in the definition of FE. Thus far from being a condition somewhat arbitrarily attached to the transformation, it is in fact a vital condition if the "map" (3.7) is considered in relation to the complete solution space $\mathcal{X}_u$.

Now by definition of $\mathcal{X}_u^1$ above, (3.7) is an onto mapping of $\mathcal{X}_u$ to $\mathcal{X}_u^1$ whenever it is properly constituted. Consider when it might be injective and hence invertible.

Theorem 2: For any fixed $u(t)$, (3.7) is an injective mapping of $\mathcal{X}_u$ to $\mathcal{X}_u^1$ iff the existence condition (3.13) holds and $N(s)$ and $\mathcal{T}(s)$ have no common finite or infinite zeros.

Proof: Again there is no loss of generality in considering the unforced system (3.8). Note that (3.13) is a necessary and sufficient condition for the relation (3.7) to be a mapping.

Assume two solutions $\hat{\xi}(t)$, $\hat{\xi}'(t)$ from $\mathcal{X}_u$ map to $\hat{\xi}_1(t) \in \mathcal{X}_u^1$. Then from (3.14), (3.15)

$$\mathcal{T}(s)[\hat{\xi}(s) - \hat{\xi}'(s)] = \hat{a}_\mathcal{T}(s) - \hat{a}_\mathcal{T}'(s) \tag{3.21}$$

$$0 = N(s)[\hat{\xi}(s) - \hat{\xi}'(s)] - [\hat{a}_N(s) - \hat{a}_N'(s)]. \tag{3.22}$$

A necessary and sufficient condition for $\mathcal{T}(s)$, $N(s)$ to have no common finite zeros is that $\exists$ polynomial matrices $\overline{R}_1(s)$, $\overline{R}_2(s)$ of the appropriate dimensions such that

$$\overline{R}_1(s)\mathcal{T}(s) + \overline{R}_2(s)N(s) = I_{\overline{r}}. \tag{3.23}$$

Postmultiplying (3.23) by $\hat{\xi}(s) - \hat{\xi}'(s)$ and substituting from (3.21) and (3.22) gives

$$\hat{\xi}(s) - \hat{\xi}'(s) = \overline{R}_1(s)\{\hat{a}_T(s) - \hat{a}'_T(s)\} + \overline{R}_2(s)\{\hat{a}_N(s) - \hat{a}'_N(s)\} \tag{3.24}$$

However, the r.h.s. of (3.24) is polynomial, thus

$$\text{s.p.}(\overline{\xi}(s) - \overline{\xi}'(s)) = 0 \tag{3.25}$$

where s.p. denotes "strictly proper part." (3.25) is a necessary condition which $\hat{\xi}(t)$, $\hat{\xi}'(t)$ must satisfy if they are to map onto the same element $\hat{\xi}_1(s)$, under (3.15).

Now $\mathcal{T}(s)$ and $N(s)$ have no common infinite zeros. A necessary and sufficient condition for this is that $\exists$ proper rational matrices $R_1(s)$, $R_2(s)$ such that

$$R_1(s)\mathcal{T}(s) + R_2(s)N(s) = I_{\overline{r}}. \tag{3.26}$$

Since $R_1(s)$ and $R_2(s)$ are proper, they possess Laurent expansions in s^{-1} which consist only of positive powers i.e., for $i = 1, 2$

$$R_i(s) = R_{0i} + R_{1i}\frac{1}{s} + R_{2i}\frac{1}{s^2} + \cdots. \tag{3.27}$$

Substituting from (3.10) and (3.27) into (3.26) then gives, on multiplying out and comparing positive powers of s

$$\begin{aligned}
s^k:&\quad R_{01}\mathcal{T}_k + R_{02}N_k = 0\\
s^{k-1}:&\quad R_{11}\mathcal{T}_k + R_{01}\mathcal{T}_{k-1} + R_{12}N_k + R_{02}N_{k-1} = 0\\
\vdots&\quad \cdots\\
s:&\quad R_{k-1,1}\mathcal{T}_k + R_{k-2,1}\mathcal{T}_{k-1} + \cdots + R_{01}\mathcal{T}_1\\
&\quad + R_{k-1,2}N_k + R_{k-2,2}N_{k-1} + \cdots + R_{02}N_1 = 0
\end{aligned}$$

which may be written more concisely as

$$\mathcal{X}_{R_1}\mathcal{X}_T + \mathcal{X}_{R_2}\mathcal{X}_N = 0 \tag{3.28}$$

where

$$\mathcal{X}_{R_i} = \begin{bmatrix} R_{0i} & 0 & \cdots & 0\\ R_{1i} & R_{0i} & \cdots & 0\\ \vdots & & \ddots & \vdots\\ R_{k-1,i} & R_{k-2,i} & \cdots & R_{0i}\end{bmatrix}.$$

If pol denotes "polynomial part" then

$$\begin{aligned}
&\text{pol}\,(R_i(s)[s^{k-1}I, s^{k-2}I, \cdots, I])\\
&= [R_{0i}s^{k-1} + R_{1i}s^{k-2} + \cdots + R_{k-1,i},\\
&\quad R_{0i}s^{k-2} + R_{1i}s^{k-3} + \cdots + R_{k-2,i}, \cdots, R_{0i}]\\
&= [s^{k-1}I, s^{k-2}I, \cdots, I]\mathcal{X}_{R_i} \qquad i = 1, 2.
\end{aligned} \tag{3.29}$$

Premultiplying (3.28) by $[s^{k-1}I, s^{k-2}I, \cdots, I]$ and postmultiplying by $(\overline{\xi}(0-) - \overline{\xi}'(0-))$ yields, by virtue of (3.29), that

$$\text{pol}\,(R_1(s)\{\hat{a}_T(s) - \hat{a}'_T(s)\} + R_2(s)\{\hat{a}_N(s) - \hat{a}'_N(s)\}) = 0. \tag{3.30}$$

Now from (3.26), postmultiplication by $\hat{\xi}(s) - \hat{\xi}'(s)$ yields

$$\hat{\xi}(s) - \hat{\xi}'(s) = R_1(s)\mathcal{T}(s)\{\hat{\xi}(s) - \hat{\xi}'(s)\} + R_2(s)N(s)\{\hat{\xi}(s) - \hat{\xi}'(s)\}.$$

Substituting from (3.21) gives

$$\hat{\xi}(s) - \hat{\xi}'(s) = R_1(s)\{\hat{a}_T(s) - \hat{a}'_T(s)\} + R_2(s)\{\hat{a}_N(s) - \hat{a}'_N(s)\}.$$

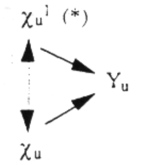

Fig. 1. Y_u is the set of outputs corresponding to $u(t)$.

Now (3.30) indicates that the polynomial part of the right hand side of this is zero and so

$$\text{pol}\,(\hat{\xi}(s) - \hat{\xi}'(s)) = 0. \tag{3.31}$$

Thus (3.31) is the necessary condition which $\hat{\xi}(s)$, $\hat{\xi}'(s)$ must satisfy if they are to map onto the same element $\hat{\xi}_1(s)$ under (3.15) when $\mathcal{T}(s)$, $N(s)$ have no common infinite zeros. Taking (3.19), (3.25), and (3.31) together yields the necessary condition under which the relation (3.7) is an injective mapping. The sufficiency of the conditions is more simply established and the details may be found in [9]. □

Corollary 1: The absence of finite zeros in $[\mathcal{T}(s)^T N(s)^T]^T$ is necessary and sufficient for the strictly proper part of the inverse map of (3.7) to be uniquely determined. In the same way the polynomial part of the inverse map of (3.7) is uniquely determined iff $[\mathcal{T}(s)^T N(s)^T]^T$ has no infinite zeros and the McMillan degree condition (3.13) holds. □

Corollary 2: In the case $N(\rho) = N$, a constant matrix, then no initial conditions are embedded in the relation (3.7) and so it always represents a map in the formal sense. □

IV. Fundamental Equivalence of Polynomial Matrix Descriptions

If $\mathcal{T}(\rho)$ and $N(\rho)$ satisfy the conditions of Theorem 2 then (3.7) is particularly interesting. This is because the set of solutions $\mathcal{X}_0$ of the homogeneous equations (3.6) forms a vector space with dimension $f := \delta_M(\mathcal{T}(s))$. Since (3.7) is a vector space isomorphism between $\mathcal{X}_0$ and $\mathcal{X}_0^1$, it will preserve the generalized order f. There is a subspace of $\mathcal{X}_0$ which can be defined as the strongly controllable subspace of the system (3.3). Such subspaces are preserved by this isomorphism. Suppose therefore that Σ_1 is a normalized system which has $\mathcal{X}_u^1$ as its set of solutions corresponding to a given $u(t)$ for all possible initial conditions, and has an output relation (*) such that the diagram 4.1 commutes. Then the system Σ_1 has not only the same generalized order and input-output relations as Σ but also the same controllability and observability properties. It is then reasonable to call these two systems equivalent and this is the basis of the definition.

Definition 4: Let $\mathcal{P}_1(s)$, $\mathcal{P}_2(s)$ be the normalized forms of $(P_1(s), P_2(s)) \in P_0(p, m) \times P_0(p, m)$. $\mathcal{P}_1(s)$ and $\mathcal{P}_2(s)$ are said to be fundamentally equivalent iff the following hold

1) $\exists$ a bijective map between the pseudostates $\xi_i(t)$, $i = 1, 2$ of $\mathcal{P}_1(s)$ and $\mathcal{P}_2(s)$

$$\xi_2(t) = N(p)\xi_1(t) \tag{4.1}$$

and

2) they have the same output $y(t)$. □

It will be seen that Definition 4 is the complete extension of the definition of fundamental equivalence of Pernebo [7]. The main difference is that, this new definition additionally refers to the impulsive solution set of the system of differential equations (3.3a).

It is now possible to establish the connection between the equivalence expressed in Definition 4 and that of Definitions 2, 3.

Theorem 3: Let $\mathcal{P}_1(s)$, $\mathcal{P}_2(s)$ be the normalized forms of two systems Σ_1 and Σ_2, respectively. If $\mathcal{P}_1(s)$ and $\mathcal{P}_2(s)$ are NFSE, then they are fundamentally equivalent. □

Proof: Let $\mathcal{P}_1(s)$, $\mathcal{P}_2(s)$ be NFSE, then according to [5] $\exists$ polynomial matrices $N(\rho)$ and $M(\rho)$ such that

$$\begin{bmatrix} M(\rho) & 0 \\ 0 & I \end{bmatrix}\begin{bmatrix} \mathcal{T}_1(\rho) & -\mathcal{U}_1 \\ \mathcal{V}_1 & 0 \end{bmatrix} = \begin{bmatrix} \mathcal{T}_2(\rho) & -\mathcal{U}_2 \\ \mathcal{V}_2 & 0 \end{bmatrix}\begin{bmatrix} N(\rho) & 0 \\ 0 & I \end{bmatrix} \quad (4.2)$$

is an FE transformation. Now NFSE preserves the transfer function matrix and so the two systems have the same output $y(t)$. Thus it is only necessary to establish condition i) of fundamental equivalence. Postmultiplying (4.2) by $[\xi_1^T(t)u(t)^T]^T$ gives,

$$\begin{bmatrix} M(\rho) & 0 \\ 0 & I \end{bmatrix}\begin{bmatrix} 0 \\ y(t) \end{bmatrix} = \begin{bmatrix} \mathcal{T}_2(\rho) & -\mathcal{U}_2 \\ \mathcal{V}_2 & 0 \end{bmatrix}\begin{bmatrix} N(\rho)\xi_1(t) \\ u(t) \end{bmatrix} \Rightarrow$$

$$\begin{bmatrix} 0 \\ y(t) \end{bmatrix} = \begin{bmatrix} \mathcal{T}_2(\rho) & -\mathcal{U}_2 \\ \mathcal{V}_2 & 0 \end{bmatrix}\begin{bmatrix} N(\rho)\xi_1(t) \\ u(t) \end{bmatrix} \quad (4.3)$$

and so the relation

$$\xi_2(t) = N(\rho)\xi_1(t) \quad (4.4)$$

gives a solution of the system Σ_2 whenever $\xi_1(t)$ is a solution of Σ_1. From the FE conditions governing (4.2)

$$\delta_M \begin{bmatrix} \mathcal{T}_1(\rho) \\ N(\rho) \end{bmatrix} = \delta_M(\mathcal{T}_1(\rho)) \quad (4.5)$$

and so Theorem 1 implies that (4.4) is a map in the formal sense. The fact that

$$\begin{bmatrix} \mathcal{T}_1(\rho) \\ N(\rho) \end{bmatrix} \quad (4.6)$$

has no finite nor infinite zeros implies, by Theorem 2, that the map (4.4) is injective and thus a monomorphism between the solution (vector) spaces $\mathcal{X}_u^1$, $\mathcal{X}_u^2$ respectively of the systems Σ_1 and Σ_2. In the same way the symmetry property of NFSE may be exploited to establish a monomorphism between $\mathcal{X}_u^2$, $\mathcal{X}_u^1$ respectively. It then follows that the solution spaces $\mathcal{X}_u^1$, $\mathcal{X}_u^2$ are isomorphic and so $\exists$ a bijective map of the form (4.1), as required. □

The converse of the above theorem is also true, and the details can be found in [9].

V. Conclusion

A neat characterization of the transformation of full system equivalence has been given in terms of the existence of a bijective map between the finite and infinite solution sets of PMD's under a fixed input $u(t)$. The characterization has enabled the true nature and role of the conditions of full system equivalence to emerge. Thus for example the McMillan degree conditions (2.6c), which have previously appeared somewhat arbitrarily attached to the transformation, are seen to be vital in a quite fundamental way. The importance of the appropriately defined restricted initial conditions of a general PMD in this new characterization has also been noted.

Fundamental equivalence is therefore the complete mapping interpretation of the matrix transformation of NFSE and hence FSE. Thus whereas FSE is a closed form matrix expression of this notion of equivalence which applies to the system matrix representation (or what is the same thing, their normalized forms), fundamental equivalence gives a more natural characterization in terms of the existence of a bijective mapping between the solution sets of the defining differential equations. It is concluded that FSE with its various characterizations, is the basic transformational tool for the simultaneous study of the finite and infinite frequency behavior of general linear multivariable systems.

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