INT. J. CONTROL, 1994, VOL. 60, NO. 4, 521–541

Computation of the transfer function matrix and its Laurent expansion of generalized two-dimensional systems

N. P. KARAMPETAKIS†, B. G. MERTZIOS‡, and
A. I. G. VARDULAKIS†

An algorithm is developed for the computation of the transfer function matrix of a two-dimensional system, which is given in its generalized form. The algorithm is a recursion in terms of the original system matrix and does not require the inversion of a two-variable polynomial matrix. An algorithm for the evaluation of the Laurent expansion of the inverse of a two-variable polynomial matrix is also presented.

1. Introduction

Two-dimensional (2-D) systems have drawn considerable attention over the last few years since they provide the mathematical framework for the study of 2-D digital filters, which are finding numerous applications in image processing (medical imaging, processing of geophysical and seismic data, processing of satellite photos and video images), electrical networks with variable elements etc.

Roesser (1975), Fornasini and Marchesini (1976, 1978), Kurek (1985), and Kaczorek (1988) have presented polynomial matrix description models for (2-D) linear discrete systems with constant coefficients. The most general among them was the singular model of Kaczorek. Later, in a series of papers, Komornik *et al*. (1989), Rocha (1990), Rocha and Willems (1989, 1990) etc, have shown that the behaviour of any 2-D discrete system may be represented by a 2-D autoregressive representation (AR-representation). This representation has the advantage that extends to all the proposed models, until now.

In § 2 we present a generalized polynomial matrix description model for 2-D linear discrete systems. It is shown that the singular general model proposed by Kaczorek (1988) is a particular case of this new model. It is easily seen that this new model is very closely related to the 2-D AR (autoregressive) and 2-D ARMA (autoregressive-moving average) models proposed by Rocha and Willems (1990).

In § 3 we present a new algorithm for the direct computation of the transfer function of this model without inverting a polynomial matrix of two variables. This problem has been considered by Koo and Chen (1977) who extended Fadeeva's algorithm (Zadeh and Desoer 1963) for the inversion of the resolvent matrix $sI_n - A$ in the 2-D case. Mertzios (1984) presented a generalized algorithm for the inversion of the generalized pencil $sE - A$ where both E and A may be singular matrices and later (Mertzios 1986) applied this algorithm to

Received 22 September 1992, Revised 15 April 1993.
† Department of Mathematics, Faculty of Sciences, Aristotle University of Thessaloniki, Thessaloniki 54006, Greece.
‡ Department of Electrical Engineering, Democritus University of Thrace, 67100 Xanthi, Greece.

2-D state-space systems producing an alternative to the algorithm in Koo and Chen (1977). Moreover, explicit expressions for the transfer function matrices of polynomial matrix descriptions (PMDs) and 2-D state-space systems in terms of the system matrices have been derived by Fragulis *et al*. (1991), Mertzios and Paraskevopoulos (1981), Mertzios and Syrmos (1987), Mertzios and Lewis (1988), and Lewis and Mertzios (1992).

Finally, we present an algorithm for the computation of the Laurent expansion of the inverse of a two-variable polynomial matrix, i.e. in the case where it is unique. This problem has been considered by Mertzios and Lewis (1989) and Lewis and Mertzios (1992), who proposed an efficient algorithm for the computation of the fundamental sequence for singular 1-D and 2-D systems. Fragulis *et al*. (1991) proposed an extension of the results concerning 1-D singular systems to general polynomial matrix descriptions (PMDs). The fundamental sequence of PMDs and generalized 2-D systems has many applications in analysis and synthesis of those systems, i.e. computation of the solution of those systems, reachability and observability criteria etc. A generalized 2-D Cayley–Hamilton theorem is also proposed.

2. Generalized model of 2-D linear discrete systems

Let $\mathbb{R}[\sigma_1, \sigma_2]^{n\times n}$ ($\mathbb{R}(\sigma_1, \sigma_2)^{n\times n}$) denote the $n \times n$ real polynomial (rational) matrices in the two indeterminates σ_1, σ_2 and, generally, $\mathbb{R}[\sigma_1, \sigma_2, \ldots, \sigma_k]^{n\times n}$ ($\mathbb{R}(\sigma_1, \sigma_2, \ldots, \sigma_k)^{n\times n}$) denote the $n \times n$ real polynomial (rational) matrices in the k indeterminates $\sigma_1, \sigma_2, \ldots, \sigma_k$. A model described by the equations

$$A(\sigma_1, \sigma_2)x(t_1, t_2) = B(\sigma_1, \sigma_2)u(t_1, t_2) \tag{2.1 a}$$

$$y(t_1, t_2) = C(\sigma_1, \sigma_2)x(t_1, t_2) + D(\sigma_1, \sigma_2)u(t_1, t_2) \tag{2.1 b}$$

will be called the *generalized model* (GM) of 2-D linear discrete systems, where $A(\sigma_1, \sigma_2) \in \mathbb{R}[\sigma_1, \sigma_2]^{n\times n}$ with $\det[A(\sigma_1, \sigma_2)] \neq 0$, $B(\sigma_1, \sigma_2) \in \mathbb{R}[\sigma_1, \sigma_2]^{n\times m}$, $C(\sigma_1, \sigma_2) \in \mathbb{R}[\sigma_1, \sigma_2]^{p\times n}$, $D(\sigma_1, \sigma_2) \in \mathbb{R}[\sigma_1, \sigma_2]^{p\times m}$ t_1, t_2 are integer-valued vertical and horizontal coordinates respectively; $x(t_1, t_2)$: $\mathbb{Z}^+ \times \mathbb{Z}^+ \to \mathbb{R}^n$ is the local state vector at (t_1, t_2), $u(t_1, t_2)$: $\mathbb{Z}^+ \times \mathbb{Z}^+ \to \mathbb{R}^m$ is the input vector, $y(t_1, t_2)$: $\mathbb{Z}^+ \times \mathbb{Z}^+ \to \mathbb{R}^p$ is the output vector and σ_1 is the down shift and σ_2 the left shift operator defined by

$$\sigma_1 x(t_1, t_2) = x(t_1 + 1, t_2) \tag{2.2 a}$$

$$\sigma_2 x(t_1, t_2) = x(t_1, t_2 + 1) \tag{2.2 b}$$

The polynomial matrix of two variables $A(\sigma_1, \sigma_2)$ may be written as

$$A(\sigma_1, \sigma_2) = \sum_{i=0}^{q_1}\sum_{j=0}^{q_2} A_{ij}\sigma_1^i\sigma_2^j \tag{2.3}$$

The global boundary conditions for (2.1) are given by

$$\begin{aligned} x(i, k) &= x_{ik} \quad \text{for} \quad i = 0, 1, \ldots \quad \text{and} \quad k = 0, 1, \ldots, q_2 - 1 \\ x(m, j) &= x_{mj} \quad \text{for} \quad m = 0, 1, \ldots, q_1 - 1 \quad \text{and} \quad j = 0, 1, \ldots \end{aligned} \tag{2.4}$$

where x_{ik} and x_{mj} are known vectors. The model (2.1) may be written in the following form

$$\begin{bmatrix} A(\sigma_1, \sigma_2) & B(\sigma_1, \sigma_2) & 0 \\ -C(\sigma_1, \sigma_2) & D(\sigma_1, \sigma_2) & I_p \\ 0 & -I_m & 0 \end{bmatrix} \begin{bmatrix} x(i, j) \\ -u(i, j) \\ y(i, j) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ I_m \end{bmatrix} u(i, j) \quad (2.5\,a)$$

$$y(i, j) = [0 \quad 0 \quad I_p] \begin{bmatrix} x(i, j) \\ -u(i, j) \\ y(i, j) \end{bmatrix} \quad (2.5\,b)$$

which will be called the *normalized form* of the generalized model (2.1). In a similar way, we can define generalized models for N-D linear discrete systems, to be systems of the following form

$$A(\sigma_1, \sigma_2, \ldots, \sigma_n)x(t_1, t_2, \ldots, t_n) = B(\sigma_1, \sigma_2, \ldots, \sigma_n)u(t_1, t_2, \ldots, t_n) \quad (2.6\,a)$$

$$y(t_1, \ldots, t_n) = C(\sigma_1, \ldots, \sigma_n)x(t_1, \ldots, t_n) + D(\sigma_1, \ldots, \sigma_n)u(t_1, \ldots, t_n) \quad (2.6\,b)$$

A model described by the equations

$$\begin{aligned} Ex(t_1 + 1, t_2 + 1) = {} & A_0x(t_1, t_2) + A_1x(t_1 + 1, t_2) + A_2x(t_1, t_2 + 1) \\ & + B_0u(t_1, t_2) + B_1u(t_1 + 1, t_2) + B_2u(t_1, t_2 + 1) \end{aligned} \quad (2.7\,a)$$

$$y(t_1, t_2) = Cx(t_1, t_2) + Du(t_1, t_2) \quad (2.7\,b)$$

will be called the *singular general model* and has been presented by Kaczorek (1988). We can easily see that this model may be rewritten as

$$(E\sigma_1\sigma_2 - A_0 - A_1\sigma_1 - A_2\sigma_2)x(t_1, t_2) = (B_0 + B_1\sigma_1 + B_2\sigma_2)u(t_1, t_2) \quad (2.8\,a)$$

$$y(t_1, t_2) = Cx(t_1, t_2) + Du(t_1, t_2) \quad (2.8\,b)$$

So, for $A(\sigma_1, \sigma_2) = E\sigma_1\sigma_2 - A_0 - A_1\sigma_1 - A_2\sigma_2$, $B(\sigma_1, \sigma_2) = B_0 + B_1\sigma_1 + B_2\sigma_2$, $C(\sigma_1, \sigma_2) = C$ and $D(\sigma_1, \sigma_2) = D$, we obtain that the singular general model, which has been one of the most general polynomial matrix description models until now, is a particular model of the class of generalized models proposed here (see Kaczorek 1988). The generalized model (2.1) or, equivalently, the analogue Rosenbrock representation model for 2-D systems, may be rewritten in a 2-D ARMA form (Blomberg and Ylinen 1983) as

$$\underbrace{\begin{bmatrix} A(\sigma_1, \sigma_2) & 0 \\ -C(\sigma_1, \sigma_2) & I_p \end{bmatrix}}_{R(\sigma_1, \sigma_2)} \underbrace{\begin{bmatrix} x(t_1, t_2) \\ y(t_1, t_2) \end{bmatrix}}_{w(t_1, t_2)} = \underbrace{\begin{bmatrix} B(\sigma_1, \sigma_2) \\ D(\sigma_1, \sigma_2) \end{bmatrix}}_{M(\sigma_1, \sigma_2)} u(t_1, t_2) \quad (2.9)$$

According to Komornik *et al*. (1989), the behaviour

$$\mathcal{B} = \{w(t_1, t_2) \colon \mathbb{L}^+ \times \mathbb{L}^+ \to \mathbb{R}^{n+p} \mid \exists\, u(t_1, t_2) \text{ s.t. } (w(t_1, t_2), u(t_1, t_2)) \text{ satisfies (2.9)}\} \quad (2.10)$$

may also be described by a 2-D AR-representation of the form

$$\begin{bmatrix} A(\sigma_1, \sigma_2) & B(\sigma_1, \sigma_2) & 0 \\ -C(\sigma_1, \sigma_2) & D(\sigma_1, \sigma_2) & I_p \\ 0 & -I_m & 0 \end{bmatrix} \begin{bmatrix} x(i, j) \\ -u(i, j) \\ y(i, j) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ I_m \end{bmatrix} u(i, j) \quad (2.5\,a)$$

$$y(i, j) = [0 \quad 0 \quad I_p] \begin{bmatrix} x(i, j) \\ -u(i, j) \\ y(i, j) \end{bmatrix} \quad (2.5\,b)$$

which will be called the *normalized form* of the generalized model (2.1). In a similar way, we can define generalized models for N-D linear discrete systems, to be systems of the following form

$$A(\sigma_1, \sigma_2, \ldots, \sigma_n)x(t_1, t_2, \ldots, t_n) = B(\sigma_1, \sigma_2, \ldots, \sigma_n)u(t_1, t_2, \ldots, t_n) \quad (2.6\,a)$$

$$y(t_1, \ldots, t_n) = C(\sigma_1, \ldots, \sigma_n)x(t_1, \ldots, t_n) + D(\sigma_1, \ldots, \sigma_n)u(t_1, \ldots, t_n) \quad (2.6\,b)$$

A model described by the equations

$$\begin{aligned} Ex(t_1 + 1, t_2 + 1) = {} & A_0x(t_1, t_2) + A_1x(t_1 + 1, t_2) + A_2x(t_1, t_2 + 1) \\ & + B_0u(t_1, t_2) + B_1u(t_1 + 1, t_2) + B_2u(t_1, t_2 + 1) \end{aligned} \quad (2.7\,a)$$

$$y(t_1, t_2) = Cx(t_1, t_2) + Du(t_1, t_2) \quad (2.7\,b)$$

will be called the *singular general model* and has been presented by Kaczorek (1988). We can easily see that this model may be rewritten as

$$(E\sigma_1\sigma_2 - A_0 - A_1\sigma_1 - A_2\sigma_2)x(t_1, t_2) = (B_0 + B_1\sigma_1 + B_2\sigma_2)u(t_1, t_2) \quad (2.8\,a)$$

$$y(t_1, t_2) = Cx(t_1, t_2) + Du(t_1, t_2) \quad (2.8\,b)$$

So, for $A(\sigma_1, \sigma_2) = E\sigma_1\sigma_2 - A_0 - A_1\sigma_1 - A_2\sigma_2$, $B(\sigma_1, \sigma_2) = B_0 + B_1\sigma_1 + B_2\sigma_2$, $C(\sigma_1, \sigma_2) = C$ and $D(\sigma_1, \sigma_2) = D$, we obtain that the singular general model, which has been one of the most general polynomial matrix description models until now, is a particular model of the class of generalized models proposed here (see Kaczorek 1988). The generalized model (2.1) or, equivalently, the analogue Rosenbrock representation model for 2-D systems, may be rewritten in a 2-D ARMA form (Blomberg and Ylinen 1983) as

$$\underbrace{\begin{bmatrix} A(\sigma_1, \sigma_2) & 0 \\ -C(\sigma_1, \sigma_2) & I_p \end{bmatrix}}_{R(\sigma_1, \sigma_2)} \underbrace{\begin{bmatrix} x(t_1, t_2) \\ y(t_1, t_2) \end{bmatrix}}_{w(t_1, t_2)} = \underbrace{\begin{bmatrix} B(\sigma_1, \sigma_2) \\ D(\sigma_1, \sigma_2) \end{bmatrix}}_{M(\sigma_1, \sigma_2)} u(t_1, t_2) \quad (2.9)$$

According to Komornik *et al*. (1989), the behaviour

$$\mathcal{B} = \{w(t_1, t_2)\colon \mathbb{L}^+ \times \mathbb{L}^+ \to \mathbb{R}^{n+p} | \exists\, u(t_1, t_2) \text{ s.t. } (w(t_1, t_2), u(t_1, t_2)) \text{ satisfies } (2.9)\} \quad (2.10)$$

may also be described by a 2-D AR-representation of the form where $X(z_1, z_2) = Z[x(t_1, t_2)]$, $Y(z_1, z_2) = Z[y(t_1, t_2)]$ and $U(z_1, z_2) = Z[u(t_1, t_2)]$, which gives rise to the transfer function matrix

$$H(z_1, z_2) := C(z_1, z_2)A^{-1}(z_1, z_2)B(z_1, z_2) + D(z_1, z_2) \in \mathbb{R}(z_1, z_2)^{p\times m} \tag{3.7}$$

or, equivalently, according to (2.4)

$$H(z_1, z_2) = \mathcal{V}\mathcal{T}^{-1}(z_1, z_2)\mathcal{U} \in \mathbb{R}(z_1, z_2)^{p\times m} \tag{3.8}$$

where

$$\mathcal{T}(z_1, z_2) = \begin{bmatrix} A(z_1, z_2) & B(z_1, z_2) & 0 \\ -C(z_1, z_2) & D(z_1, z_2) & I_p \\ 0 & -I_m & 0 \end{bmatrix}; \quad \mathcal{V} = [0 \quad 0 \quad I_p]; \quad \mathcal{U} = \begin{bmatrix} 0 \\ 0 \\ I_m \end{bmatrix} \tag{3.9}$$

Therefore, in fact, the problem of computing $H(z_1, z_2)$ is reduced to the calculation of $A^{-1}(z_1, z_2)$ or, equivalently, of $\mathcal{T}^{-1}(z_1, z_2)$ in the most efficient way. In the following, a generalization of the Leverrier algorithm in polynomial matrices with more than one variable is proposed. More specifically, if we substitute the constant matrix A by $A(z_1, z_2)$, in the known Leverrier algorithm where

$$A(z_1, z_2) = \sum_{j=0}^{q_1} \sum_{k=0}^{q_2} A_{jk} z_1^j z_2^k \tag{3.10}$$

we shall obtain the following algorithm.

Algorithm 1:
Define

$$(zI_n - A(z_1, z_2))^{-1} = \frac{1}{\Delta(z)}[R_0(z_1, z_2)z^{n-1} + \cdots + R_{n-2}(z_1, z_2)z + R_{n-1}(z_1, z_2)] \tag{3.11}$$

where

$$\Delta(z) = \det[zI_n - A(z_1, z_2)] = z^n + a_1(z_1, z_2)z^{n-1} + \cdots + a_n(z_1, z_2) \tag{3.12}$$

and $R_0(z_1, z_2)$, $R_1(z_1, z_2)$, $\ldots$, $R_{n-1}(z_1, z_2)$ are polynomial matrices with two variables. Using the Leverrier algorithm for (3.3) we obtain

$$\begin{aligned} R_0(z_1, z_2) &= I_n \\ R_1(z_1, z_2) &= A(z_1, z_2)R_0(z_1, z_2) + a_1(z_1, z_2)I_n \\ &= A(z_1, z_2) + a_1(z_1, z_2)I_n \\ R_2(z_1, z_2) &= A(z_1, z_2)R_1(z_1, z_2) + a_2(z_1, z_2)I_n \\ &= A(z_1, z_2)^2 + a_1(z_1, z_2)A(z_1, z_2) + a_2(z_1, z_2)I_n \\ &\vdots \end{aligned}$$

$$\begin{aligned}R_{n-1}(z_1, z_2) &= A(z_1, z_2)R_{n-2}(z_1, z_2) + a_{n-1}(z_1, z_2)I_n \\ &= A(z_1, z_2)^{n-1} + \cdots + a_{n-1}(z_1, z_2)I_n \\ 0 &= A(z_1, a_2)R_{n-1}(z_1, z_2) + a_n(z_1, z_2)I_n \\ &= A(z_1, z_2)^n + a_1(z_1, z_2)A(z_1, z_2)^{n-1} + \cdots + a_n(z_1, z_2)I_n\end{aligned} \tag{3.13}$$

and

$$\begin{aligned}a_1(z_1, z_2) &= -\frac{1}{1}\operatorname{tr}[A(z_1, z_2)R_0(z_1, z_2)] \\ a_2(z_1, z_2) &= -\frac{1}{2}\operatorname{tr}[A(z_1, z_2)R_1(z_1, z_2)] \\ a_3(z_1, z_2) &= -\frac{1}{3}\operatorname{tr}[A(z_1, z_2)R_2(z_1, z_2)] \\ &\vdots \\ a_{n-1}(z_1, z_2) &= -\frac{1}{n-1}\operatorname{tr}[A(z_1, z_2)R_{n-1}(z_1, z_2)] \\ a_n(z_1, z_2) &= -\frac{1}{n}\operatorname{tr}[A(z_1, z_2)R_n(z_1, z_2)] \qquad \square\end{aligned} \tag{3.14}$$

From the above algorithm, which is a generalization of the known Leverrier algorithm, we obtain that

$$\begin{aligned}&[zI_n - A(z_1, z_2)][R_0(z_1, z_2)z^{n-1} + \cdots + R_{n-2}(z_1, z_2)z + R_{n-1}(z_1, z_2)] \\ &\qquad = \Delta(z)I_n = \det[zI_n - A(z_1, z_2)]I_n\end{aligned} \tag{3.15}$$

Therefore, for $z = 0$ we obtain

$$A(z_1, z_2)R_{n-1}(z_1, z_2) = (-1)^{n-1}\det[A(z_1, z_2)] \tag{3.16}$$

Using (3.13) we also obtain that

$$A(z_1, z_2)R_{n-1}(z_1, z_2) = -a_n(z_1, z_2)I_n \tag{3.17}$$

Equating the second-right terms of (3.16) and (3.17) we have that

$$a_n(z_1, z_2) = (-1)^n\det[A(z_1, z_2)] \tag{3.18}$$

and

$$A^{-1}(z_1, z_2) = -\frac{R_{n-1}(z_1, z_2)}{a_n(z_1, z_2)} \tag{3.19}$$

It is seen from (3.10), (3.13) and (3.14) that $R_i(z_1, z_2)$ and $a_i(z_1, z_2)$ may be written as

$$R_i(z_1, z_2) = \sum_{j=0}^{iq_1}\sum_{k=0}^{iq_2} R_{i,jk} z_1^j z_2^k \quad i = 0, 1, \ldots, n-1 \tag{3.20}$$

and

$$a_i(z_1, z_2) = \sum_{j=0}^{iq_1} \sum_{k=0}^{iq_2} a_{i,jk} z_1^j z_2^k \quad i = 0, 1, \ldots, n \tag{3.21}$$

where $R_{i,jk}$, $a_{i,jk}$ are constant coefficient matrices and scalars of the powers $z_1^j z_2^k$. It can be seen from (3.19) that, for the computation of the inverse of $A(z_1, z_2)$ and therefore for the transfer function, we need only the quantities $R_{n-1}(z_1, z_2)$ and $a_n(z_1, z_2)$, i.e. the coefficient matrices $R_{n-1,jk}$ and the coefficients $a_{n,jk}$ defined by

$$R_{n-1}(z_1, z_2) = (-1)^{n-1} \operatorname{adj}[A(z_1, z_2)] = \sum_{j=0}^{(n-1)q_1} \sum_{k=0}^{(n-1)q_2} R_{n-1,jk} z_1^j z_2^k \tag{3.22}$$

and

$$a_n(z_1, z_2) = (-1)^n \det[A(z_1, z_2)] = \sum_{j=0}^{nq_1} \sum_{k=0}^{nq_2} a_{n,jk} z_1^j z_2^k \tag{3.23}$$

Taking into account that

$$A(z_1, z_2) R_i(z_1, z_2) = \left(\sum_{j=0}^{q_1} \sum_{k=0}^{q_2} A_{jk} z_1^j z_2^k \right) \left(\sum_{j=0}^{iq_1} \sum_{k=0}^{iq_2} R_{i,jk} z_1^j z_2^k \right)$$

$$= \sum_{j=0}^{(i+1)q_1} \sum_{k=0}^{(i+1)q_2} \left(\sum_{l=0}^{j} \sum_{m=0}^{k} A_{lm} R_{i,j-l,k-m} \right) z_1^j z_2^k \tag{3.24}$$

and substituting (3.20), (3.21) and (3.24) in the recursive relations (3.13) and (3.14), we obtain the following recursive algorithm that determines $a_{i+1,jk}$, $R_{i+1,jk}$ for $j = 0, 1, \ldots, (i+1)q_1$ and $k = 0, 1, \ldots, (i+1)q_2$.

Algorithm 2:

Initial conditions

$$R_{0,00} = I_n \tag{3.25}$$

Boundary conditions

$$R_{0,jk} = 0 \quad \forall j,k > 0 \tag{3.26}$$

$$R_{i,jk} = 0 \quad \begin{aligned} j &= iq_1 + 1, \ldots, (n-1)q_1 \\ k &= iq_2 + 1, \ldots, (n-1)q_2 \\ i &= 1, 2, \ldots, n-1 \end{aligned} \tag{3.27}$$

Recursive relations for a_i (z_1, z_2)

$$a_{i+1,j,k} = -\frac{1}{i+1} \operatorname{tr} \left(\sum_{l=0}^{j} \sum_{m=0}^{k} A_{lm} R_{i,j-l,k-m} \right) \tag{3.28}$$

$$\begin{aligned} j &= 0, 1, \ldots, (i+1)q_1 \\ k &= 0, 1, \ldots, (i+1)q_2 \\ i &= 0, 1, \ldots, n-1 \end{aligned}$$

Recursive relation for R_i (z_1, z_2)

$$R_{i+1,jk} = \left(\sum_{l=0}^{j}\sum_{m=0}^{k} A_{lm}R_{i,j-l,k-m}\right) + a_{i+1,jk}I_n \quad j = 0, 1, \ldots, (i+1)q_1 \tag{3.29}$$
$$k = 0, 1, \ldots, (i+1)q_2$$
$$i = 0, 1, \ldots, n-2$$

Terminate

$$R_{jk} = R_{n-1,jk} \quad j = 0, 1, \ldots, (n-1)q_1 \quad k = 0, 1, \ldots, (n-1)q_2$$
$$a_{jk} = a_{n,jk} \quad j = 0, 1, \ldots, nq_1 \quad k = 0, 1, \ldots, nq_2 \qquad \square \tag{3.30}$$

It is readily seen that the inversion algorithm is a three-dimensional algorithm since it depends on three independent variables i, j, k.

This algorithm for the inversion of $A(z_1, z_2)$ can also be extended in the case of an n-variable polynomial matrix, and the proposed algorithm will be $(n+1)$th dimensional.

Algorithm 3:

Initial conditions

$$R_{0,0\ldots0} = I_n \tag{3.31}$$

Boundary conditions

$$R_{0,i_1,i_2\ldots i_n} = 0 \quad \forall i_1, i_2, \ldots, i_n > 0 \tag{3.32}$$

$$R_{i,i_1,i_2\ldots i_n} = 0 \quad i_1 = iq_1 + 1, \ldots, (n-1)q_1 \tag{3.33}$$
$$i_2 = iq_2 + 1, \ldots, (n-1)q_2$$
$$\cdots\cdots\cdots\cdots$$
$$i_n = iq_n + 1, \ldots, (n-1)q_n$$
$$i = 1, 2, \ldots, n-1$$

Recursive relations for $a_i(z_1, z_2, \ldots, z_n)$

$$a_{i+1,i_1,\ldots,i_n} = -\frac{1}{i+1}\operatorname{tr}\left(\sum_{l_1=0}^{i_1}\sum_{l_2=0}^{i_2}\cdots\sum_{l_n=0}^{i_n} A_{l_1l_2\ldots l_n}R_{i,i_1-l_1,\ldots,i_n-l_n}\right) \tag{3.34}$$
$$i_1 = 0, 1, \ldots, (i+1)q_1$$
$$i_2 = 0, 1, \ldots, (i+1)q_2$$
$$\cdots\cdots\cdots\cdots$$
$$i_n = 0, 1, \ldots, (i+1)q_n$$
$$i = 0, 1, \ldots, n-1$$

Recursive relation for $R_i(z_1, z_2, \ldots, z_n)$

$$R_{i+1,i_1,\ldots,i_n} = \left(\sum_{l_1=0}^{i_1}\sum_{l_2=0}^{i_2}\cdots\sum_{l_n=0}^{i_n} A_{i_1,\ldots,i_n}R_{i,i_1-l_1,\ldots,i_n-l_n}\right) + a_{i+1,i_1,\ldots,i_n}I_n \tag{3.35}$$
$$i_1 = 0, 1, \ldots, (i+1)q_1$$
$$i_2 = 0, 1, \ldots, (i+1)q_2$$
$$\cdots\cdots\cdots\cdots$$
$$i_n = 0, 1, \ldots, (i+1)q_n$$
$$i = 0, 1, \ldots, n-2$$

Terminate

$$R_{i_1,\ldots,i_n} = R_{n-1,i_1,\ldots,i_n} \quad \begin{aligned} i_1 &= 0, 1, \ldots, (n-1)q_1 \\ i_2 &= 0, 1, \ldots, (n-1)q_2 \\ &\cdots\cdots\cdots\cdots \\ i_n &= 0, 1, \ldots, (n-1)q_n \end{aligned}$$

$$a_{i_1,\ldots,i_n} = a_{n,i_1,\ldots,i_n} \quad \begin{aligned} i_1 &= 0, 1, \ldots, nq_1 \\ i_2 &= 0, 1, \ldots, nq_2 \\ &\cdots\cdots\cdots\cdots \\ i_n &= 0, \ldots, nq_n \end{aligned} \tag{3.36}$$

□

The inverse of

$$A(z_1, z_2, \ldots, z_n) = \sum_{i_1=0}^{q_1}\sum_{i_2=0}^{q_2} \ldots \sum_{i_n=0}^{q_n} A_{i_1 i_2 \ldots i_n} z_1^{i_1} z_2^{i_2} \ldots z_n^{i_n} \tag{3.37}$$

is given by

$$A^{-1}(z_1, z_2, \ldots, z_n) = -\frac{R_{n-1}(z_1, z_2, \ldots, z_n)}{a_n(z_1, z_2, \ldots, z_n)} \tag{3.38}$$

where

$$\begin{aligned} R_{n-1}(z_1, z_2, \ldots, z_n) &= (-1)^{n-1} \operatorname{adj}[A(z_1, z_2, \ldots, z_n)] \\ &= \sum_{i_1=0}^{(n-1)q_1} \ldots \sum_{i_n=0}^{(n-1)q_n} R_{n-1,i_1,\ldots,i_n} z_1^{i_1} \ldots z_n^{i_n} \end{aligned} \tag{3.39}$$

and

$$a_n(z_1, z_2, \ldots, z_n) = (-1)^n \det[A(z_1, z_2, \ldots, z_n)] = \sum_{i_1=0}^{nq_1} \ldots \sum_{i_n=0}^{nq_n} a_{n,i_1,\ldots,i_n} z_1^{i_1} \ldots z_n^{i_n} \tag{3.40}$$

Note that if the polynomial matrix has no inverse (the determinant is zero) then, from the formulae (3.34), we obtain coefficients $a_{i_1,\ldots,i_n}$, $i_k = 0, 1, \ldots, nq_k$ for $k = 0, 1, \ldots, n$ equal to zero.

The formulae (3.34) and (3.35) are readily reduced to the Leverrier's type algorithm for singular systems (Mertzios 1984) if we assume that $q_1 = 1$ and $q_2 = q_3 = \cdots = q_n = 0$, i.e. when the polynomial matrix is a singular pencil $A(s) = A_1 s + A_0$. Moreover, the formulae (3.34) and (3.35) are reduced to the Leverrier's-type algorithm for generalized dynamical systems (Fragulis *et al.* 1991) if we assume that $q_2 = q_3 = \cdots = q_n = 0$ and q_1 is an arbitrary constant; this is the case where the polynomial matrix has only one variable, $A(s) = A_0 + A_1 s + \cdots + A_{q_1} s^{q_1}$.

Suppose now that

$$C(z_1, z_2) = \sum_{i=0}^{c_1}\sum_{j=0}^{c_2} C_{ij} z_1^i z_2^j \tag{3.41 a}$$

$$B(z_1, z_2) = \sum_{i=0}^{b_1}\sum_{j=0}^{b_2} B_{ij} z_1^i z_2^j \tag{3.41 b}$$

and

$$D(z_1, z_2) = \sum_{i=0}^{d_1} \sum_{j=0}^{d_2} D_{ij} z_1^i z_2^j \tag{3.41 c}$$

Now substituting (3.38), (3.41) and (3.22) in (3.7) we obtain that

$$H(z_1, z_2) = \frac{\sum_{i=0}^{m_1} \sum_{k=0}^{m_2} \left[-\left(\sum_{l=0}^{j} \sum_{m=0}^{k} F_{lm} B_{j-1,k-m} \right) + D_{jk} \right] z_1^j z_2^k}{a_n(z_1, z_2)} \tag{3.42}$$

where $m_1 = \max\{(n-1)q_1 + c_1 + b_1, d_1\}$, $m_2 = \max\{(n-1)q_2 + c_2 + b_2, d_2\}$ and

$$F_{ij} = \sum_{l=0}^{i} \sum_{m=0}^{j} C_{lm} R_{n-1,i-l,j-m} \tag{3.43}$$

Analogous to (3.42) we have N-D systems where, if we suppose that

$$C(z_1, z_2, \ldots, z_n) = \sum_{i_1=0}^{c_1} \sum_{i_2=0}^{c_2} \cdots \sum_{i_n=0}^{c_n} C_{i_1 i_2 \ldots i_n} z_1^{i_1} z_2^{i_2} \ldots z_n^{i_n} \tag{3.44 a}$$

$$B(z_1, z_2, \ldots, z_n) = \sum_{i_1=0}^{b_1} \sum_{i_2=0}^{b_2} \cdots \sum_{i_n=0}^{b_n} B_{i_1 i_2 \ldots i_n} z_1^{i_1} z_2^{i_2} \ldots z_n^{i_n} \tag{3.44 b}$$

and

$$D(z_1, z_2, \ldots, z_n) = \sum_{i_1=0}^{d_1} \sum_{i_2=0}^{d_2} \cdots \sum_{i_n=0}^{d_n} D_{i_1 i_2 \ldots i_n} z_1^{i_1} z_2^{i_2} \ldots z_n^{i_n} \tag{3.44 c}$$

then the transfer function of the N-D system (2.6) will be

$$H(z_1, \ldots, z_n) = \frac{\sum_{i_1=0}^{m_1} \cdots \sum_{i_n=0}^{m_n} \left(\left(-\sum_{l_1=0}^{i_1} \cdots \sum_{l_n=0}^{i_n} F_{l_1 \ldots l_n} B_{i_1-l_1, \ldots, i_n-l_n} \right) + D_{i_1 \ldots i_n} \right)}{a_n(z_1, z_2, \ldots, z_n)} \tag{3.45}$$

where $m_i = \max\{(n-1)q_i + c_i + b_i, d_i\}$, $i = 1, \ldots, n$ and

$$F_{i_1, \ldots, i_n} = \left(\sum_{l_1=0}^{i_1} \sum_{l_2=0}^{i_2} \cdots \sum_{l_n=0}^{i_n} C_{l_1, \ldots, l_n} R_{n-1, i_1-l_1, \ldots, i_n-l_n} \right) \tag{3.46}$$

So we have found the transfer function of the generalized model of N-D linear discrete systems which we have defined in § 2.

The procedure for the computation of the inverse of an n-variable polynomial matrix provides the means for the computation of the inversion of square rational matrices.

Let $T(z_1, z_2, \ldots, z_n) \in \mathbb{R}(z_1, z_2, \ldots, z_n)^{n \times n}$ with $\det[T(z_1, z_2, \ldots, z_n)] \neq 0$. Then $T(z_1, z_2, \ldots, z_n)$ may be written in the following form

$$T(z_1, z_2, \ldots, z_n) = \frac{A(z_1, z_2, \ldots, z_n)}{d(z_1, z_2, \ldots, z_n)} \tag{3.47}$$

where $d(z_1, z_2, \ldots, z_n)$ is the least common denominator of all rational functions of $A(z_1, z_2, \ldots, z_n)$. Therefore

$$T^{-1}(z_1, z_2, \ldots, z_n) = -d(z_1, z_2, \ldots, z_n)\frac{R_{n-1}(z_1, z_2, \ldots, z_n)}{a_n(z_1, z_2, \ldots, z_n)} \tag{3.48}$$

where

$$A^{-1}(z_1, z_2, \ldots, z_n) = -\frac{R_{n-1}(z_1, z_2, \ldots, z_n)}{a_n(z_1, z_2, \ldots, z_n)} \tag{3.49}$$

Example 1: Let a generalized 2-D linear discrete system have the form (2.1), or under z-transforms with zero initial conditions have the form (3.6) where

$$C(z_1, z_2) = \begin{bmatrix} z_1 & 0 & 1 \\ 0 & z_1 - z_1 & 0 \end{bmatrix}; \quad A(z_1, z_2) = \begin{bmatrix} z_1 & z_1^2 & 0 \\ z_1 - z_2 & 0 & z_2 \\ 0 & z_2^2 & z_1 z_2 \end{bmatrix};$$

$$B(z_1, z_2) = \begin{bmatrix} z_1 \\ z_2 \\ 0 \end{bmatrix}; \quad D(z_1, z_2) = 0 \tag{E1.1}$$

For this system we have $q_1 = 2$, $q_2 = 2$, $c_1 = c_2 = 1$, $b_1 = b_2 = 1$ and $n = 3$. Therefore, we are seeking $R_2(z_1, z_2)$ and $a_3(z_1, z_2)$ in order to evaluate $A^{-1}(z_1, z_2)$. Applying the presented algorithm, we obtain

$$R_{2,0,3} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \quad R_{2,1,2} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & -1 & 0 \end{bmatrix}, \quad R_{2,1,1} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$R_{2,2,1} = \begin{bmatrix} 0 & 0 & 1 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad R_{2,3,1} = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad R_{2,3,0} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \tag{E1.2}$$

$$R_{2,i,j} = 0\{\forall (i, j) \text{ s.t. } (i, j) \notin [(0, 3), (1, 2), (1, 1), (2, 1), (3, 1), (3, 0)],$$
$$i = 0, 1, 2, 3, 4 \quad \text{and} \quad j = 0, 1, 2, 3, 4\}$$

and

$$a_{3,13} = 1, \; a_{3,41} = 1, \; a_{3,32} = -1 \tag{E1.3}$$

$$a_{3,ij} = 0\{\forall (i, j) \text{ such that } (i, j) \notin [(1, 3), (4, 1), (3, 2)] \text{ and}$$
$$i = 0, 1, \ldots, 6 \; j = 0, 1, \ldots, 6\}$$

Hence

$$R_2(z_1, z_2) = \sum_{j=0}^{6} \sum_{k=0}^{6} R_{2,jk} z_1^j z_2^k = \begin{bmatrix} -z_2^3 & -z_1^3 z_2 & z_1^2 z_2 \\ z_1 z_2^2 - z_1^2 z_2 & z_1^2 z_2 & -z_1 z_2 \\ z_1 z_2^2 - z_2^3 & -z_1 z_2^2 & -z_1^3 + z_1^2 z_2 \end{bmatrix} \tag{E1.4}$$

and

$$a_3(z_1, z_2) = \sum_{j=0}^{6} \sum_{k=0}^{6} a_{3,jk} z_1^j z_2^k = z_1 z_2^3 - z_1^3 z_2^2 + z_1^4 z_2 \tag{E1.5}$$

Finally it is found that

$$A^{-1}(z_1, z_2) = \frac{\begin{bmatrix} z_2^3 & z_1^3 z_2 & -z_1^2 z_2 \\ -z_1 z_2^2 + z_1^2 z_2 & -z_1^2 z_2 & z_1 z_2 \\ -z_1 z_2^2 + z_2^3 & z_1 z_2^2 & z_1^3 - z_1^2 z_2 \end{bmatrix}}{z_1 z_2^3 - z_1^3 z_2^2 + z_1^4 z_2} \tag{E1.6}$$

Combining (3.43) and (E1.2) we obtain that

$$F_{0,3} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad F_{1,2} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$
$$F_{1,3} = \begin{bmatrix} -1 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \quad F_{2,1} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix}$$
$$F_{2,2} = \begin{bmatrix} 0 & 0 & 0 \\ 2 & -1 & 0 \end{bmatrix}, \quad F_{3,0} = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}, \tag{E1.7}$$
$$F_{3,1} = \begin{bmatrix} 0 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix}, \quad F_{4,1} = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$
$$F_{i,j} = 0\{\forall (i, j) \notin [(0, 3), (1, 2), (1, 3), (2, 1), (2, 2), (3, 0), (3, 1), (4, 1)]$$
$$i = 0, 1, 2,, 3, 4, 5 \quad \text{and} \quad j = 0, 1, 2, 3, 4, 5\}$$

Applying (3.42) we finally obtain the transfer function of the system (E1.1)

$$H(z_1, z_2) = \frac{\begin{bmatrix} z_1^4 z_2^2 + 2z_1 z_2^3 + z_1^2 z_2^3 - z_1^2 z_2^2 \\ z_1^4 z_2^2 - 3z_1^3 z_2^2 + 2z_1^2 z_2^3 \end{bmatrix}}{z_1 z_2^3 - z_1^3 z_2^2 + z_1^4 z_2} \tag{E1.8}$$

4. Evaluation of the Laurent expansion of $\mathbf{A}^{-1}(z_1, z_2)$

In this section we derive an explicit formula for the computation of the Laurent expansion of the inverse of a two variable polynomial matrix, i.e. $A^{-1}(z_1, z_2)$, in terms of the matrices A_{ij}, $i = 0, 1, \ldots, q_1$ and $j = 0, 1, \ldots, q_2$.

From (3.19) we obtain that

$$A^{-1}(z_1, z_2) = -\frac{R_{n-1}(z_1, z_2)}{a_n(z_1, z_2)} \tag{4.1}$$

or equivalently

$$a_n(z_1, z_2) A^{-1}(z_1, z_2) = -R_{n-1}(z_1, z_2) \tag{4.2}$$

For a two-variable polynomial $p(z_1, z_2)$ define $\deg_{z_1}[p(z_1, z_2)]$ as the degree of $p(z_1, z_2)$ in z_1 and similarly define $\deg_{z_2}[p(z_1, z_2)]$. Denote also by $\deg[p(z_1, z_2)]$ the degree in z of $p(z, z)$. Then, define the integers

$$\begin{aligned} r_i &= \deg_{z_i}[a_n(z_1, z_2)] \quad i = 1, 2 \\ f_i &= \deg_{z_i}[R_{n-1}(z_1, z_2)] \quad i = 1, 2 \\ r &= \deg_z[a_n(z_1, z_2)] \end{aligned} \tag{4.3}$$

Expand $A^{-1}(z_1, z_2)$, element by element, in a Laurent series expansion about infinity, convergent for large enough $|z_1|$ and $|z_2|$

$$A^{-1}(z_1, z_2) = \sum_{i=v_1}^{-\infty} \sum_{j=v_2}^{-\infty} H_{ij} z_1^i z_2^j \tag{4.4}$$

where $v_i = \deg_{z_i}[A^{-1}(z_1, z_2)]$, $i = 1, 2$. It is easily seen from (4.2), (4.3) and (4.4) that

$$v_i = f_i - r_i \quad i = 1, 2 \tag{4.5}$$

Substituting $R_{n-1}(z_1, z_2)$, $a_n(z_1, z_2)$ and $A^{-1}(z_1, z_2)$ by (3.39), (3.40) and (4.4) respectively in (4.2), and equating the coefficient matrices of the corresponding powers of $z_1^i z_2^j$, on both sides of the resulting equation, yields

$$R_{n-1,i,j} = \sum_{l=0}^{r_1} \sum_{m=0}^{r_2} a_{n,l,m} H_{i-l,j-m} \quad (0, 0) \leqslant (i, j) \leqslant (f_1, f_2) \tag{4.6}$$

and

$$0 = \sum_{l=0}^{r_1} \sum_{m=0}^{r_2} a_{n,l,m} H_{i-l,j-m} \{i < -r_1 \wedge j \leqslant v_2\} \vee \{i \leqslant v_1 \wedge j < -r_2\} \tag{4.7}$$

which allows the computation of H_{ij} in the stated region in terms of its values for smaller (i, j). Thus, (4.7) constitutes the Cayley–Hamilton theorem for the 2-D generalized models with special cases of the generalized 1-D systems and PMDs and the 2-D state-space systems (Roesser 1975, Ciftsibasi and Yuksel 1982, Lewis 1986, Mertzios and Christodoulou 1986, Lewis and Mertzios 1992).

In the case where $a_{n,r_1,r_2} = 0$, the Laurent expansion of $A^{-1}(z_1, z_2)$ may not be unique as we can see in the following.

Example 2: Let

$$A(z_1, z_2) = z_1 - z_2 \tag{E2.1}$$

It is easily seen that

$$r_1 = \deg_{z_1}[z_1 - z_2] = 1 = \deg_{z_2}[z_1 - z_2] = r_2 \tag{E2.2}$$

while

$$r = \deg[z_1 - z_2] = \deg_z[z - z] = 0 \neq r_1 + r_2 \tag{E2.3}$$

We have also that

$$A^{-1}(z_1, z_2) = \frac{1}{z_1 - z_2} = \begin{cases} \displaystyle\sum_{i=1}^{\infty} z_1^{-i} z_2^{i-1} & \left|\dfrac{z_1}{z_2}\right| > 1 \\ \displaystyle -\sum_{i=1}^{\infty} z_1^{i-1} z_2^{-i} & \left|\dfrac{z_1}{z_2}\right| < 1 \end{cases} \tag{E2.4}$$

and so the Laurent expansion of $A^{-1}(z_1, z_2)$ is not unique. □

The following theorem indicates a case where the Laurent expansion of $A^{-1}(z_1, z_2)$ is unique and provides an explicit formula for the computation of this expansion.

Theorem 3: *Suppose that*

$$r = r_1 + r_2 \tag{4.8}$$

or, equivalently, that $a_{n,r_1,r_2} \neq 0$. *Then the Laurent expansion of* $A^{-1}(z_1, z_2)$ *is unique.*

Proof: The proof of this theorem follows from the unique construction of an explicit formula for the computation of the Laurent expansion of $A^{-1}(z_1, z_2)$.

Equation (4.6) may be rewritten in the following two forms

$$\begin{bmatrix} a_{n,r_1,r_2}I_n & 0 & & & \\ a_{n,r_1-1,r_2}I_n & a_{n,r_1,r_2}I_n & \ddots & \mathbf{0} & \\ \vdots & & & & \\ a_{n,0,r_2}I_n & a_{n,1,r_2}I_n & & \ddots & \\ 0 & a_{n,0,r_2}I_n & & & \\ \mathbf{0} & & \ddots & & \\ & & a_{n,0,r_2}I_n & \cdots & a_{n,r_1,r_2}I_n \end{bmatrix} \begin{bmatrix} H_{v_1,v_2} \\ H_{v_1-1,v_2} \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ H_{-r_1,v_2} \end{bmatrix} = - \begin{bmatrix} R_{n-1,f_1,f_2} \\ R_{n-1,f_1-1,f_2} \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ R_{n-1,0,f_2} \end{bmatrix}$$

$$\Leftrightarrow P_{r_2} \cdot H_{v_2} = -R_{f_2} \tag{4.9}$$

and

$$\begin{bmatrix} P_{r_2} & 0 & & & \\ P_{r_2-1} & P_{r_2} & \ddots & \mathbf{0} & \\ \vdots & & & & \\ P_0 & P_1 & & \ddots & \\ & P_0 & & & \\ \mathbf{0} & & \ddots & & \\ & & P_0 & \cdots & P_{r_2} \end{bmatrix} \begin{bmatrix} H_{v_2} \\ H_{v_2-1} \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ H_{-r_2} \end{bmatrix} = - \begin{bmatrix} R_{f_2} \\ R_{f_2-1} \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ R_0 \end{bmatrix} \Leftrightarrow$$

$$P \cdot H = -R \tag{4.10}$$

where

$$P_i = \begin{bmatrix} a_{n,r_1,i}I_n & 0 & & & \\ a_{n,r_1-1,i}I_n & a_{n,r_1,i}I_n & \ddots & \mathbf{0} & \\ \vdots & & & & \\ a_{n,0,i}I_n & a_{n,1,i}I_n & & \ddots & \\ 0 & a_{n,0,i}I_n & & & \\ \mathbf{0} & & \ddots & & \\ & & a_{n,0,i}I_n & \cdots & a_{n,r_1,i}I_n \end{bmatrix} \in \mathbb{R}^{n(f_1+1)\times n(f_1+1)}$$

$$i = 0, 1, \ldots, r_2$$

$$H_i = \begin{bmatrix} H_{v_1,i} \\ H_{v_1-1,i} \\ \vdots \\ \vdots \\ \vdots \\ H_{-r_1,i} \end{bmatrix} \in \mathbb{R}^{n(f_1+1)\times n} \quad (i = v_2, v_2 - 1, \ldots, -r_2) \qquad \text{and}$$

$$R_i = \begin{bmatrix} R_{f_1,i} \\ R_{f_1-1,i} \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ R_{0,i} \end{bmatrix} \in \mathbb{R}^{n(f_1+1)\times n} \quad (i = 0, 1, \ldots, f_2) \tag{4.11}$$

Due to the special Toeplitz form of P_{r_2}, we find that the unique (i.e. $\det[P_{r_2}] \neq 0$) inverse of $P_{r_2}^{-1}$ is

$$D = P_{r_2}^{-1} = \begin{bmatrix} d_0 I_n & & & \\ d_1 I_n & d_0 I_n & & 0 \\ \vdots & \vdots & \ddots & \\ d_{f_1} I_n & d_{f_1-1} I_n & \cdots & d_0 I_n \end{bmatrix} \tag{4.12}$$

where

$$d_0 = \frac{1}{a_{n,r_1,r_2}} \tag{4.13}$$

and

$$d_j = (-1)^j \left(\frac{1}{a_{n,r_1,r_2}}\right)^j \det \begin{bmatrix} a_{n,r_1-1,r_2} & a_{n,r_1-2,r_2} & \cdots & a_{n,r_1-j,r_2} \\ a_{n,r_1,r_2} & a_{n,r_1-1,r_2} & \cdots & a_{n,r_1-j+1,r_2} \\ & a_{n,r_1,r_2} & \cdots & a_{n,r_1-j+2,r_2} \\ 0 & & \ddots & \vdots \\ & & a_{n,r_1,r_2} & a_{n,r_1-1,r_2} \end{bmatrix} \tag{4.14}$$

or equivalently

$$d_j = -\frac{1}{a_{n,r_1,r_2}} \sum_{i=0}^{j-1} [a_{n,r_1-j+i,r_2} \times d_i] \quad \text{for } j = 1, 2, \ldots, f_1 \tag{4.15}$$

so that we may write for the elements of H_{v_2} the expressions

$$H_{i,v_2} = -\sum_{j=0}^{v_1-i} d_j R_{n-1,r_1+i+j,f_2} \qquad i = -r_1, -r_1+1, \ldots, v_1 \tag{4.16}$$

Due to the special Toeplitz form of P in (4.10), we find also that the unique P^{-1} is

$$P^{-1} = \begin{bmatrix} D_0 & & & \\ D_1 & D_0 & & 0 \\ \vdots & \vdots & \ddots & \\ D_{f_2} & D_{f_2-1} & \cdots & D_0 \end{bmatrix} \tag{4.17}$$

where

$$D_0 = P_{r_2}^{-1} \tag{4.18}$$

and

$$D_i = -\left(\sum_{j=0}^{i-1} D_j P_{r_2-i+j}\right) P_{r_2}^{-1} \quad i = 1, 2, \ldots, f_2 \tag{4.19}$$

Thus, from (4.10), (4.17), (4.18) and (4.19) we obtain that

$$H_i = -\sum_{j=0}^{\nu_2-i} D_j R_{r_2+i+j} \quad i=-r_2, -r_2+1, \ldots, \nu_2 \tag{4.20}$$

For the calculation of H_{ij} for less values of (i, j) we rewrite (4.7) as

$$0 = \sum_{l=0}^{r_1} \sum_{m=0}^{r_2} a_{n,l,m} H_{i+r_1-l,j+r_2-m}$$
$$\{i < -r_1 \wedge j \leqslant \nu_2\} \vee \{i \leqslant \nu_1 \wedge j < -r_2\} \tag{4.21}$$

and thus

$$H_{ij} = -\frac{1}{a_{n,r_1,r_2}} \sum_{l=0}^{r_1} \sum_{m=0}^{r_2} a_{n,l,m} H_{i+r_1-l,j+r_2-m}$$
$$(l, m) \neq (r_1, r_2)$$
$$\{i < -r_1 \wedge j \leqslant \nu_2\} \vee \{i \leqslant \nu_1 \wedge j < -r_2\} \tag{4.22}$$

From (4.16), (4.21) and (4.22) we obtain a unique form of the Laurent expansion of $A^{-1}(z_1, z_2)$ and thus the theorem has been proved. □

Summarizing the above results we obtain the following algorithm.

Algorithm 4

Step 1. Compute $f_1, f_2, r_1, r_2, r, \nu_1, \nu_2$.

Step 2. If $r_1 + r_2 \neq r$ then Stop.

Step 3. $d_0 = \dfrac{1}{a_{n,r_1,r_2}}$

For $i = 1$ to f_1

$$d_i = -\frac{1}{a_{n,r_1,r_2}} \sum_{j=0}^{i-1} [a_{n,r_1-i+j,r_2} \times d_j]$$

Next i

Step 4. For $i = -r_1$ to ν_1

$$H_{i,\nu_2} = -\sum_{j=0}^{\nu_1-i} d_j R_{n-1,r_1+i+j,f_2}$$

Next i.

Step 5. $D_0 = \begin{bmatrix} d_0 I_n & & & \\ d_1 I_n & d_0 I_n & & 0 \\ \vdots & \vdots & \ddots & \\ d_{f_1} I_n & d_{f_1-1} I_n & \cdots & d_0 I_n \end{bmatrix}$

For $i = 1$ to f_2

$$D_i = -\left(\sum_{j=0}^{i-1} D_j P_{r_2-i+j}\right) D_0$$

Next i

Step 6. For $i = -r_2$ to v_2

$$H_i = -\sum_{j=0}^{v_2-i} D_j R_{r_2+i+j}$$

Next i

Step 7. $\{i < -r_1 \wedge j \leqslant v_2\} \vee \{i \leqslant v_1 \wedge j < -r_2\}$

$$H_{ij} = -\frac{1}{a_{n,r_1,r_2}} \sum_{l=0}^{r_1} \sum_{m=0}^{r_2} a_{n,l,m} H_{i+r_1-l,j+r_2-m}$$

$(l, m) \neq (r_1, r_2)$

Step 8. $A^{-1}(z_1, z_2) = \sum_{i=v_1}^{-\infty} \sum_{j=v_2}^{-\infty} H_{ij} z_1^i z_2^j$

End

Following similar lines we can develop an algorithm for the computation of the Laurent expansion of $A^{-1}(z_1, z_2, \ldots, z_n)$. Some applications of the computation of the fundamental sequence H_{ij} are the exact solution of the system (2.1) following the same method as Mertzios and Lewis (1989), Lewis and Mertzios (1990, 1992), as well as the computation of the Laurent expansion of rational matrices under the technique in (3.47)–(3.49).

Example 3: Let

$$A(z_1, z_2) = \begin{bmatrix} z_1 + 1 & z_1^2 \\ 0 & z_2 + 1 \end{bmatrix} \tag{E3.1}$$

We shall determine the matrices H_{ij} using the proposed algorithm. It is found using Algorithm 2 that

$$\begin{aligned} R_1(z_1, z_2) &= \sum_{i=0}^{2} \sum_{j=0}^{1} R_{1,ij} z_1^i z_2^j \\ &= \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} + \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix} z_2 + \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} z_1 + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} z_1^2 \\ &= \begin{bmatrix} -z_2 - 1 & z_1^2 \\ 0 & -z_1 - 1 \end{bmatrix} \end{aligned} \tag{E3.2}$$

and

$$a_2(z_1, z_2) = \sum_{i=0}^{4} \sum_{j=0}^{2} a_{2ij} z_1^i z_2^j = 1 + z_2 + z_1 + z_1 z_2 \tag{E3.3}$$

Step 1. $r_1 = \deg_{z_1}[a_2(z_1, z_2)] = 1$

$r_2 = \deg_{z_2}[a_2(z_1, z_2)] = 1$

$$r = \deg_z[a_2(z_1, z_2)] = \deg_z[z^2 + 2z + 1] = 2$$
$$f_1 = \deg_{z_1}[R_1(z_1, z_2)] = 2$$
$$f_2 = \deg_{z_2}[R_1(z_1, z_2)] = 1$$
$$v_1 = f_1 - r_1 = 1, \quad v_2 = f_2 - r_2 = 0$$

Step 2. $2 = r = r_1 + r_2 = 1 + 1$, so Continue.

Step 3. $d_0 = \dfrac{1}{a_{211}} = 1$

$$d_1 = -\frac{d_0 a_{201}}{a_{211}} = -\frac{1\cdot 1}{1} = -1$$
$$d_2 = -\frac{d_1 a_{201}}{a_{211}} = -\frac{(-1)\cdot 1}{1} = 1$$

Step 4. $H_{-10} = -d_0 R_{101} - d_1 R_{111} - d_2 R_{121} =$

$$= -1\cdot\begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix} - (-1)\cdot 0_2 - 1\cdot 0_2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$
$$H_{00} = -d_0 R_{111} - d_1 R_{121} = -1 0_2 - (-1)\cdot 0_2 = 0_2$$
$$H_{10} = -d_0 R_{121} = -1\cdot 0_2 = 0_2$$

Step 5

$$D_0 = \begin{bmatrix} d_0\cdot I_2 & 0_2 & 0_2 \\ d_1\cdot I_2 & d_0\cdot I_2 & 0_2 \\ d_2\cdot I_2 & d_1\cdot I_2 & d_0\cdot I_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \end{bmatrix} = P_1^{-1}$$

$$D_1 = -[D_0 P_0] D_0 = -D_0 \begin{bmatrix} a_{210}\cdot I_2 & 0_2 & 0_2 \\ a_{200}\cdot I_2 & a_{210}\cdot I_2 & 0_2 \\ 0_2\cdot I_2 & a_{200}\cdot I_2 & a_{210}\cdot I_2 \end{bmatrix} D_0$$

$$= -\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{bmatrix}$$

$$\times \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \end{bmatrix}$$

$$= -I_6 D_0 = -D_0$$

Step 6

$$H_{-1} = -D_0R_0 - D_1R_1 = -D_0R_0 + D_0R_1 = D_0[-R_0 + R_1]$$

$$= D_0\left[-\begin{bmatrix} R_{120} \\ R_{110} \\ R_{100} \end{bmatrix} + \begin{bmatrix} R_{121} \\ R_{111} \\ R_{101} \end{bmatrix}\right]$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \end{bmatrix}\left[-\begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & -1 \\ -1 & 0 \\ 0 & -1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ -1 & 0 \\ 0 & 0 \end{bmatrix}\right]$$

$$= \begin{bmatrix} 0 & -1 \\ 0 & 0 \\ \hdashline 0 & 1 \\ 0 & 1 \\ \hdashline 0 & -1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} H_{1\ -1} \\ H_{0\ -1} \\ H_{-1\ -1} \end{bmatrix}$$

$$H_0 = -D_0R_1 \Rightarrow \begin{bmatrix} H_{10} \\ H_{00} \\ H_{-10} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \hdashline 0 & 0 \\ 0 & 0 \\ \hdashline 1 & 0 \\ 0 & 0 \end{bmatrix}$$

Step 7. $\{i < -1 \wedge j \leqslant 0\} \vee \{i \leqslant 1 \wedge j < -1\}$

$$H_{ij} = -\frac{1}{a_{211}}\sum_{l=0}^{1}\sum_{m=0}^{1} a_{2lm}H_{i+1-l,j+1-m}$$

$$(l,\ m) \neq (1,\ 1)$$

$$= -\frac{1}{1}[a_{200}H_{i+1j+1} + a_{201}H_{i+1j} + a_{210}H_{ij+1}]$$

$$= -[1 \cdot H_{i+1j+1} + 1 \cdot H_{i+1j} + 1 \cdot H_{ij+1}]$$

$$= -H_{i+1j+1} - H_{i+1j} - H_{ij+1}$$

$$H_{-20} = -H_{-11} - H_{-10} - H_{-21} = -0_2 - \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} - 0_2 = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$H_{-2\ -1} = -H_{-10} - H_{-1\ -1} - H_{-20}$$

$$= -\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$
\begin{aligned}
H_{1\,-2} &= -H_{2\,-1} - H_{2\,-2} - H_{1\,-1} \\
&= -0_2 - 0_2 - \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \\
&\ \vdots
\end{aligned}
$$

Step 8

$$
A^{-1}(z_1, z_2) = \sum_{i=v_1}^{-\infty} \sum_{j=v_2}^{-\infty} H_{ij} z_1^i z_2^j
$$

$$
= \begin{bmatrix} \sum_{i=1}^{\infty} (-1)^{i+1} z_1^{-i} & \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} (-1)^{i+j+1} z_1^{2-i} z_2^{-j} \\ 0 & \sum_{i=1}^{\infty} (-1)^{i+1} z_2^{-i} \end{bmatrix}
$$

End

5. Conclusions

Generalized models of 2-D linear discrete systems have been defined as extensions of all known PMD models of 2-D linear discrete systems. An algorithm for the inverse of a n-variable polynomial matrix, which helps us to find the transfer function of 2-D systems and of N-D systems generally, has also been presented. This algorithm constitutes an extension of the known Leverrier-type algorithm that has been developed for singular 2-D systems. Also, an efficient algorithm has been presented for the evaluation of the Laurent expansion of a two-variable polynomial matrix under specific conditions of uniqueness. The exact computation of the Laurent expansion of the considered generalized 2-D system is very important for various analysis and synthesis problems. An extension of the known Cayley–Hamilton theorem to generalized 2-D systems has also been provided.

Acknowledgement

The research of N. P. Karampetakis was supported by the Greek National Foundation.

References

Blomberg, H., and Ylinen, R., 1983, *Algebraic Theory of Multivariable Linear Systems* (London, U.K.: Academic Press).

Ciftsibasi, T., and Yuksel, O., 1982, On the Cayley–Hamilton theorem for two-dimensional systems. *IEEE Transactions on Automatic Control*, **27**, 193–194.

Fornasini, E., and Marchesini, G., 1976, State-space realization theory of two-dimensional filters. *IEEE Transactions on Automatic Control*, **21**, 484–491; 1978, Doubly indexed dynamical systems; state space models and structural properties. *Mathematical Systems Theory*, **12**(1).

Fragulis, G., Mertzios, B., and Vardulakis, A. I. G., 1991, Computation of the inverse of a polynomial matrix and evaluation of its Laurent expansion. *International Journal of Control*, **53**, 431–433.

Kaczorek, T., 1988, The singular general model of 2-D systems and its solutions. *IEEE Transactions on Automatic Control*, **33**, 1060–62.

KOMORNIK, J., ROCHA, P., and WILLEMS, J. C., 1989, Closed subspaces, polynomial operators in the shift and ARMA representations. *Proceedings of the IFAC Workshop on System Structure and Control*, Prague, pp. 13–15.

KOO, C. S., and CHEN, C. T., 1977, Faddeeva's algorithm for spatial dynamic equations. *Proceedings of IEEE*, **65**, 975–976.

KUREK, J., 1985, The generalized state space model for a two-dimensional linear digital systems. *IEEE Transactions on Automatic Control*, **30**, 600–602.

LEWIS, F. L., 1986, Further remarks on the Cayley–Hamilton theorem and Leverrier method for the matrix pencil $(sE - A)$. *IEEE Transactions on Automatic Control*, **31**, 869–970.

LEWIS, F. L., and MERTZIOS, B. G., 1990, On the analysis of discrete linear time-invariant singular systems. *IEEE Transactions on Automatic Control*, **35**, 506–511; 1992, On the analysis of two-dimensional discrete singular systems. *Circuit Systems and Signal Process*, **11**, 399–419.

MERTZIOS, B. G., 1984, Leverrier's algorithm for singular systems. *IEEE Transactions on Automatic Control*, **29**, 652–653; 1986, An algorithm for the computation of the transfer function matrix of two dimensional systems. *Journal of Franklin Institute*, **32**, 74–80.

MERTZIOS, B. G., and CHRISTODOULOU, M. A., 1986, On the generalized Cayley–Hamilton theorem. *IEEE Transactions on Automatic Control*, **31**, 156–157.

MERTZIOS, B. G., and LEWIS, F. L., 1988, An algorithm for the computation of the transfer function matrix of generalized two-dimensional systems. *Circuit Systems and Signal Process*, **7**, 459–466; 1989, Fundamental matrix of discrete singular systems. *Circuit Systems and Signal Process*, **8**, 341–355.

MERTZIOS, B. G., and PARASKEVOPOULOS, P. N., 1981, Transfer function matrix of 2-D systems. *IEEE Transactions on Automatic Control*, **26**, 722–724.

MERTZIOS, B. G., and SYRMOS, B. L., 1987, Transfer function matrix of singular systems. *IEEE Transactions on Automatic Control*, **31**, 829–831.

ROCHA, M. P. M., 1990, Structure and Representation of 2-D Systems. Ph.D. thesis, University Groningen, The Netherlands.

ROCHA, P., and WILLEMS, J. C., 1989, On the representation of 2-D systems. *Proceedings of the IFAC Workshop on System Structure and Control*, Prague, pp. 123–125; 1990, Canonical computational forms for an AR 2-D Systems. *Multidimensional Systems and Signal Processing*, **1**, 251–278.

ROESSER, R. P., 1975, A discrete space model for linear image processing. *IEEE Transactions on Automatic Control*, **20**, 1–10.

ZADEH, L. A., and DESOER, C. A., 1963, *Linear System Theory* (New York: McGraw-Hill).