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STRUCTURAL PROPERTIES OF INVERSE LINEAR SYSTEMS

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Some of the finite and infinite properties of a square inverse system are related to the corresponding properties of the original system. Equivalent invertible systems are seen to give rise to inverses which are similarly equivalent. An extension to the case of inverses of generalised transfer function matrices is also considered.

1. INTRODUCTION

Inverse systems have long held an interest ([9, 11]) for the design of linear systems since it was noted that a form of polynomial system matrix realisation of the inverse of a square invertible transfer function matrix $G(s)$ is directly derivable from a polynomial realisation of $G(s)$. The relationship between certain properties of the inverse system and the given system could then be easily derived. This relationship was described in the case of the finite frequency aspects and Kailath [3] furthered the description. Specifically Kailath established that the same form of equivalence which relates two given realisations of $G(s)$ is induced between the derived realisations of its inverse. In all of this work the focus was on the finite frequency aspects of the system's behaviour, typical of the conventional study of linear systems. In this paper the extension of these results to the case of the infinite frequency invariants and the so-called generalised theory of linear systems will be given.

2. PRELIMINARIES

Consider a linear time invariant multivariable system Σ described by

$$\begin{aligned} A(\rho)\,\beta(t) &= B(\rho)\,u(t) \\ y(t) &= C(\rho)\,\beta(t) + D(\rho)\,u(t) \end{aligned} \qquad (\rho = \mathrm{d}/\mathrm{d}t), \tag{2.1}$$

where $A(\rho) \in I\!R[\rho]^{r\times r}$ with $|A(\rho)| \neq 0$, $B(\rho) \in I\!R[\rho]^{r\times m}$, $C(\rho) \in I\!R[\rho]^{p\times r}$, $D(\rho) \in I\!R[\rho]^{p\times m}$, $\beta(t) : (0-,\infty) \to I\!R^r$ the *pseudo state* of Σ, $u(t) : (0-,\infty) \to I\!R^m$ the

control input and $y(t)$ the *output* of Σ. The Rosenbrock system matrix of Σ is

$$P(s) = \begin{bmatrix} A(s) & B(s) \\ -C(s) & D(s) \end{bmatrix} \in \mathbb{R}[s]^{(r+p)\times(r+m)} \tag{2.2}$$

and its *normalized form* ([13]) is

$$\mathcal{P}(s) = \left[\begin{array}{ccc|c} A(s) & B(s) & 0 & 0 \\ -C(s) & D(s) & I_p & 0 \\ 0 & -I_m & 0 & I_m \\ \hline 0 & 0 & -I_p & 0 \end{array}\right] =: \begin{bmatrix} \mathcal{T}(s) & \mathcal{U} \\ -\mathcal{V} & 0 \end{bmatrix}. \tag{2.3}$$

The transfer function of the system (2.1) or (2.3) is

$$G(s) = C(s)\,A^{-1}(s)\,B(s) + D(s) \equiv \mathcal{V}\mathcal{T}^{-1}\mathcal{U}. \tag{2.4}$$

For the relevant terminology concerning system matrices the reader is referred to Rosenbrock [9], Verghese [13] and Kailath [3]. We merely note

Definition 1. [4] The *input (output) dynamical indices* of Σ of (2.1) are the right (left) minimal indices of the compound matrix $[\mathcal{T}(s)\,\mathcal{U}]$ $\left(\begin{bmatrix} \mathcal{T}(s) \\ -\mathcal{V} \end{bmatrix}\right)$ derived from its normalised form (2.3).

A transformation ([2]) with important system theory implications is

Definition 2. Two Rosenbrock system matrices $P_1(s)$, $P_2(s)$ are said to be *full system equivalent* (FSE) if $\exists$ polynomial matrices $M(s)$, $N(s)$, $X(s)$ and $Y(s)$ such that

$$\begin{bmatrix} M(s) & 0 \\ X(s) & I \end{bmatrix} \begin{bmatrix} A_1(s) & B_1(s) \\ -C_1(s) & D_1(s) \end{bmatrix} = \begin{bmatrix} A_2(s) & B_2(s) \\ -C_2(s) & D_2(s) \end{bmatrix} \begin{bmatrix} N(s) & Y(s) \\ 0 & I \end{bmatrix}, \tag{2.5}$$

where the compound matrices

$$\begin{bmatrix} M(s) & 0 & A_2(s) & B_2(s) \\ X(s) & I & -C_2(s) & D_2(s) \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} A_1(s) & B_1(s) \\ -C_1(s) & D_1(s) \\ -N(s) & -Y(s) \\ 0 & -I \end{bmatrix} \tag{2.6}$$

satisfy the following conditions:

(i) they have full normal rank (2.7a)

(ii) they have no finite nor infinite zeros (2.7b)

(iii) have respective McMillan degrees $\delta_M(P_2)$ and $\delta_M(P_1)$. (2.7c)

Some important properties of this transformation are given by Hayton et al. [2] and Karampetakis and Vardulakis [4], and we merely note

Lemma 1. Under (FSE) the following are invariant:

(i) generalized order $\mathbf{f}$, the order n and the Rosenbrock degree d_R,

(ii) transfer function and so the finite and infinite transmission poles and zeros,

(iii) finite and infinite system poles and zeros,

(iv) finite and infinite invariant zeros,

(v) sets of finite and infinite input (output) decoupling zeros.

(vi) set of input (output) dynamical indices.

In the sequel we shall examine the properties of a system Σ in relation to those of the inverse system, in the case when the transfer function matrix $G(s)$ of Σ is square and invertible. The finite case has been studied by Rosenbrock and Van Der Weiden [11] and Kailath [3], and in what follows we extend these results to encompass the infinite frequency aspects of the system's behaviour.

3. SQUARE INVERSE LINEAR SYSTEMS

Consider Σ of (2.1) with $p = m$ and let the transfer function matrix $G(s) \in I\!R[s]^{p\times p}$ of Σ be invertible. Then a system matrix giving rise to $G^{-1}(s)$ is ([9, p. 172])

$$P'(s) = \left[\begin{array}{cc|c} A(s) & B(s) & 0 \\ -C(s) & D(s) & -I_p \\ \hline 0 & I_p & 0 \end{array}\right] \tag{3.1}$$

The system Σ' described by (3.1) is the *inverse system* of Σ. In the present note the finite and infinite frequency behaviour of Σ' is considered in relation to that of Σ.

Theorem 1. Σ, Σ' have the same finite and infinite input, output and input-output decoupling zeros. Additionally Σ, Σ' have the same input and output dynamical indices.

Proof. The finite and infinite input decoupling zeros of Σ' are respectively the finite and infinite zeros of the compound matrix

$$[\mathcal{T}'(s)\ \mathcal{U}'] = \left[\begin{array}{cccc|c} A(s) & B(s) & 0 & 0 & 0 \\ -C(s) & D(s) & -I_p & 0 & 0 \\ 0 & -I_p & 0 & I_p & 0 \\ 0 & 0 & -I_p & 0 & I_p \end{array}\right] \overset{\text{s.e.}}{\sim} \left[\begin{array}{cc|c} \mathcal{T}(s) & \mathcal{U} & 0 \\ \hline 0 & 0 & I_p \end{array}\right] \tag{3.2}$$

formed from the normalized form system matrix of Σ'. In (3.2), (s.e.) denotes the transformation of strict equivalence which is well known to leave invariant the finite and infinite zero structure. Note that the finite and infinite zero structure of the final matrix in (3.2) is the finite and infinite input decoupling zero structure of Σ.

Thus Σ and Σ' have the same finite and infinite input decoupling zeros. By similar arguments the result for the finite and infinite output decoupling zeros follows.

For the case of the input-output decoupling zeros, notice that removing all the finite and infinite input decoupling zeros from Σ' leaves the normalised form of Σ' as

$$\left[\begin{array}{cccc|c} A'(s) & B'(s) & 0 & 0 & 0 \\ -C'(s) & D'(s) & -E'(s) & 0 & 0 \\ 0 & -I_p & 0 & I_p & 0 \\ 0 & 0 & -I_p & 0 & I_p \\ \hline 0 & 0 & 0 & -I_p & 0 \end{array}\right]. \tag{3.3}$$

Removing these same input decoupling zeros from Σ leaves its normalised form as

$$\left[\begin{array}{ccc|c} A'(s) & B'(s) & 0 & 0 \\ -C'(s) & D'(s) & -E'(s) & 0 \\ 0 & -I_p & 0 & I_p \\ \hline 0 & 0 & -I_p & 0 \end{array}\right]. \tag{3.4}$$

Clearly the finite and infinite output decoupling zeros of (3.3) and (3.4) are identical which establishes the result concerning the input-output decoupling zeros of Σ' and Σ.

It can be easily seen from (3.2) that the compound matrices $[\mathcal{T}'(s)\ \mathcal{U}']$ and $[\mathcal{T}(s)\ \mathcal{U}]$ taken from the normalised forms of Σ and Σ' are (s.e.) and therefore have the same right and left minimal indices. Thus Σ, Σ' have the same input (and similarly output) dynamical indices.

Lemma 2. [8] If $G(s)$ is square and invertible then the finite (resp. infinite) poles of $G(s)$ are the finite (resp. infinite) zeros of $G^{-1}(s)$, and vice versa. A slight restatement of this result gives

Corollary 1. The finite (resp. infinite) transmission zeros of Σ' are the finite (resp. infinite) transmission poles of Σ and the finite (resp. infinite) transmission poles of Σ' are the finite (resp. infinite) transmission zeros of Σ.

This can be extended to system matrix representations of Σ, Σ' as follows

Theorem 2. The number of finite (resp. infinite) system zeros of Σ' coincides with the number of finite (resp. infinite) system poles of Σ and the number of finite (resp. infinite) system poles of Σ' coincides with the number of finite (resp. infinite) system zeros of Σ.

Proof. We have that

$$\begin{aligned} &\#\{\text{system zeros in } \mathbb{C}\cup\{\infty\} \text{ of } \Sigma'\} \\ &\qquad \equiv \#\{\text{zeros of } G^{-1}(s) \text{ in } \mathbb{C}\cup\{\infty\}\} + \#\{\text{decoupling zeros of } \Sigma' \text{ in } \mathbb{C}\cup\{\infty\}\} \\ &\qquad = \#\{\text{poles of } G(s) \text{ in } \mathbb{C}\cup\{\infty\}\} + \#\{\text{decoupling zeros in } \mathbb{C}\cup\{\infty\}\} \end{aligned}$$

by Theorem 1 and Corollary 1 where $\#\{\cdot\}$ denotes the total number of the elements of the specified set, counted according to degree and multiplicity. This latter is simply $\#\{\text{system poles in } \mathbb{C} \cup \{\infty\} \text{of } \Sigma\}$.

Under the similar arguments we obtain the second result. □

An extension of this result may obtained.

Lemma 3. [12] If $G(s)$ is square and nonsingular then

$$\#\{\text{poles of } G(s) \text{ in } \mathbb{C} \cup \{\infty\}\} = \#\{\text{zeros of } G(s) \text{ in } \mathbb{C} \cup \{\infty\}\}.$$

Theorem 3. The systems Σ and Σ' have the same generalized order f, complexity c and Rosenbrock degree d_R ([10]).

Proof. For the generalised order we have by Theorem 1, Corollary 1 and Lemma 3 that

$$\begin{aligned}
f_{\Sigma'} &= \#\{\text{zeros of } \mathcal{T}'(s) \text{ in } \mathbb{C} \cup \{\infty\}\} \\
&= \#\{\text{system poles of } \Sigma' \text{ in } \mathbb{C} \cup \{\infty\}\} \\
&= \#\{\text{poles of } G^{-1}(s) \text{ in } \mathbb{C} \cup \{\infty\}\} + \#\{\text{decoupling zeros of } \Sigma' \text{ in } \mathbb{C} \cup \{\infty\}\} \\
&= \#\{\text{zeros of } G(s) \text{ in } \mathbb{C} \cup \{\infty\}\} + \#\{\text{decoupling zeros of } \Sigma \text{ in } \mathbb{C} \cup \{\infty\}\} \\
&= \#\{\text{poles of } G(s) \text{ in } \mathbb{C} \cup \{\infty\}\} + \#\{\text{decoupling zeros of } \Sigma \text{ in } \mathbb{C} \cup \{\infty\}\} \\
&= \#\{\text{system poles of } \Sigma \text{ in } \mathbb{C} \cup \{\infty\}\} \\
&= \#\{\text{zeros of } \mathcal{T}(s) \text{ in } \mathbb{C} \cup \{\infty\}\} = f_\Sigma.
\end{aligned}$$

The results for c and d_R were given by Rosenbrock and Van Der Weiden [11]. □

While the generalized order of Σ' coincides with that of Σ, the order n of Σ' does not.

Example 1. Let a Rosenbrock system matrix of a system Σ be

$$P(s) = \left[\begin{array}{c|c} s^2+5s+6 & s+1 \\ \hline 2s-5 & 3s+2 \end{array}\right] \tag{3.5}$$

with order $n_\Sigma = \deg|s^2+5s+6| = 2$. The Rosenbrock system matrix of Σ' is

$$P'(s) = \left[\begin{array}{cc|c} s^2+5s+6 & s+1 & 0 \\ 2s-5 & 3s+2 & -1 \\ \hline 0 & 1 & 0 \end{array}\right] \tag{3.6}$$

with order $n'_\Sigma = \deg|P(s)| = 3 \neq 2 = n_\Sigma$. However it may easily be verified that $f'_\Sigma = 3 = f_\Sigma$ in line with Theorem 3.

Any result concerning the order and the generalized order Σ', will carry over to its least order $v(G^{-1}(s))$ and generalized least order $\delta_M(G^{-1}(s))$. Thus ([11]) $G(s)$

and $G^{-1}(s)$ have the same McMillan degree. Notice however that although $G(s)$ and $G^{-1}(s)$ do not have the same least order, it follows from the above results that Σ' has least order if and only if Σ does.

It is clear that given two systems Σ_1, Σ_2 related by (FSE) then they will have identical invariants in the manner of Lemma 1. The results above ensure that the corresponding inverse systems of Σ_1 and Σ_2 themselves possess identical pole and zero structures. This indicates that a deeper relationship exists between the inverse systems of such systems Σ_1 and Σ_2.

Theorem 4. Let Σ_1, Σ_2 be two linear systems of the form (2.1) and let Σ_1', Σ_2' be their respective inverse systems. Then

$$\Sigma_1 \overset{(\mathrm{FSE})}{\sim} \Sigma_2 \Leftrightarrow \Sigma_1' \overset{(\mathrm{FSE})}{\sim} \Sigma_2'.$$

Proof. ($\Rightarrow$) Let Σ_1, Σ_2 be (FSE), then $\exists$ polynomial matrices $M(s)$, $N(s)$, $X(s)$, $Y(s)$ such that

$$\begin{bmatrix} M(s) & 0 \\ X(s) & I \end{bmatrix} \begin{bmatrix} A_1(s) & B_1(s) \\ -C_1(s) & D_1(s) \end{bmatrix} = \begin{bmatrix} A_2(s) & B_2(s) \\ -C_2(s) & D_2(s) \end{bmatrix} \begin{bmatrix} N(s) & Y(s) \\ 0 & I \end{bmatrix}, \tag{3.7}$$

where (3.7) satisfies the conditions (2.7). Equivalently (3.7) may be written as

$$\left[\begin{array}{cc|c} M & 0 & 0 \\ X & I & 0 \\ \hline 0 & 0 & I \end{array}\right] \left[\begin{array}{cc|c} A_1 & B_1 & 0 \\ -C_1 & D_1 & -I \\ \hline 0 & I & 0 \end{array}\right] = \left[\begin{array}{cc|c} A_2 & B_2 & 0 \\ -C_2 & D_2 & -I \\ \hline 0 & I & 0 \end{array}\right] \left[\begin{array}{cc|c} N & Y & 0 \\ 0 & I & 0 \\ \hline 0 & 0 & I \end{array}\right] \tag{3.8}$$

where (3.8) satisfies (2.7). Hence (3.8) is a transformation of (FSE) between Σ_1', Σ_2'.

($\Leftarrow$) Let Σ_1', Σ_2' be (FSE), then $\exists$ polynomial matrices $M_{ij}(s)$, $N_{ij}(s)$, $X_i(s)$, $Y_i(s)$, $i = 1, 2$ such that

$$\left[\begin{array}{cc|c} M_{11} & M_{12} & 0 \\ M_{21} & M_{22} & 0 \\ \hline X_1 & X_2 & I \end{array}\right] \left[\begin{array}{cc|c} A_1 & B_1 & 0 \\ -C_1 & D_1 & -I \\ \hline 0 & I & 0 \end{array}\right] = \left[\begin{array}{cc|c} A_2 & B_2 & 0 \\ -C_2 & D_2 & -I \\ \hline 0 & I & 0 \end{array}\right] \left[\begin{array}{cc|c} N_{11} & N_{12} & Y_1 \\ N_{21} & N_{22} & Y_2 \\ \hline 0 & 0 & I \end{array}\right] \tag{3.9}$$

This may be written in the equivalent form

$$\begin{bmatrix} M_{11} & M_{12} & 0 & A_2 & B_2 & 0 \\ M_{21} & M_{22} & 0 & -C_2 & D_2 & -I \\ X_1 & X_2 & I & 0 & I & 0 \end{bmatrix} \begin{bmatrix} A_1 & B_1 & 0 \\ -C_1 & D_1 & -I \\ 0 & 0 & I \\ -N_{11} & -N_{12} & -Y_1 \\ -N_{21} & -N_{22} & -Y_2 \\ 0 & 0 & -I \end{bmatrix} = 0. \tag{3.10}$$

where (3.10) satisfies the conditions (2.7 a, b, c) of (FSE). From the McMillan degree conditions (2.7c) it follows that the matrices X_1, X_2 and Y_1, Y_2 are at most constant and that the following matrices are inverses of one another

$$\left[\begin{array}{ccc|ccc} I & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 & 0 \\ \hline 0 & Y_1 & 0 & I & 0 & 0 \\ -X_1 & -X_2 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & 0 & I \end{array}\right] \quad \text{and} \quad \left[\begin{array}{ccc|ccc} I & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 & 0 \\ \hline 0 & -Y_1 & 0 & I & 0 & 0 \\ X_1 & -Y_2 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & 0 & I \end{array}\right]. \tag{3.11}$$

Consider then the equations taken from (3.10)

$$\begin{array}{ll} (1,3): & -M_{12} - A_2Y_1 + B_2Y_2 = 0 \\ (2,3): & -M_{22} + C_2Y_1 - D_2Y_2 + I = 0 \\ (3,1): & X_1A_1 - X_2C_1 - N_{21} = 0 \\ (3,2): & X_1B_1 + X_2D_1 + I - N_{22} = 0 \\ (3,3): & -X_2 - Y_2 = 0. \end{array} \tag{3.12}$$

In view of (3.12), internally recoordinating ([2]) the transformation (3.10) by means of the matrices (3.11) yields

$$\begin{bmatrix} M_{11} - B_2X_1 & 0 & 0 & A_2 & B_2 & 0 \\ M_{21} - D_2X_1 & I & 0 & -C_2 & -D_2 & -I \\ 0 & 0 & I & 0 & I & 0 \end{bmatrix} \begin{bmatrix} A_1 & B_1 & 0 \\ -C_1 & -D_1 & -I \\ 0 & I & 0 \\ Y_1C_1 - N_{11} & -Y_1D_1 - N_{12} & 0 \\ 0 & -I & 0 \\ 0 & 0 & -I \end{bmatrix} = 0. \tag{3.13}$$

Since the matrices (3.11) are constant and nonsingular, the compound matrices in (3.13) still satisfy all the requirements of Definition 3. Hence by restriction

$$\begin{bmatrix} M_{11} - B_2X_1 & 0 \\ M_{21} - D_2X_1 & I \end{bmatrix} \begin{bmatrix} A_1 & B_1 \\ -C_1 & D_1 \end{bmatrix} = \begin{bmatrix} A_2 & B_2 \\ -C_2 & D_2 \end{bmatrix} \begin{bmatrix} -Y_1C_1 + N_{11} & Y_1D_1 + N_{12} \\ 0 & I \end{bmatrix} \tag{3.14}$$

is a transformation satisfying conditions (2.7 a, b, c) and so Σ_1 and Σ_2 are (FSE). □

4. THE GENERALISED TRANSFER FUNCTION MATRIX AND ITS INVERSE

Taking Laplace transforms in (2.1) gives

$$\begin{aligned} A(s)\beta(s) &= B(s)u(s) + a_0(s) \\ y(s) &= C(s)\beta(s) + D(s)u(s) + \beta_0(s) \end{aligned}, \tag{4.1}$$

where $a_0(s)$, $\beta_0(s)$ are polynomial vectors, whose coefficients are determined by the initial values of $\{\beta(\cdot), u(\cdot)\}$ and their derivatives. In connection with (4.1) it is possible to introduce ([6, 3]) a *generalized transfer function matrix* $H(s)$

$$\begin{bmatrix} y(s) \\ \beta(s) \\ u(s) \end{bmatrix} = H(s) \begin{bmatrix} \beta_0(s) \\ a_0(s) \\ u(s) \end{bmatrix}; \quad H(s) = \begin{bmatrix} I_p & C(s)A(s)^{-1} & G(s) \\ 0 & A(s)^{-1} & A(s)^{-1}B(s) \\ 0 & 0 & I_m \end{bmatrix}. \tag{4.2}$$

Interestingly $H(s)$ is invertible and its inverse is the polynomial matrix

$$P_M(s) = \begin{bmatrix} I_p & -C(s) & -D(s) \\ 0 & A(s) & -B(s) \\ 0 & 0 & I_m \end{bmatrix}. \tag{4.3}$$

Morf [6] proposed an equivalence for such matrices based on strict system equivalence ([1, 5]) established its identity as regards strict system equivalence. We propose a generalisation of Morf's transformation based on (FSE).

Definition 3. Two systems Σ_1, Σ_2 of the form (2.1) are said to be *Generalized Morf System Equivalent* (GMSE) if $\exists$ polynomial matrices $X(s)$, $Y(s)$, $K(s)$, $L(s)$ such that

$$\begin{bmatrix} K(s) & 0 \\ X(s) & I \end{bmatrix} P_{M_1} = P_{M_2} \begin{bmatrix} I & 0 \\ Y(s) & L(s) \end{bmatrix}. \tag{4.4}$$

where P_{M_i}, $i = 1, 2$ is defined in (4.3) and where the compound matrices

$$\left[\begin{array}{cc|c} K(s) & 0 & \\ & & P_{M_2} \\ X(s) & I & \end{array}\right] \text{ and } \left[\begin{array}{cc} \multicolumn{2}{c}{P_{M_1}} \\ \hline -I & 0 \\ -Y(s) & -L(s) \end{array}\right] \tag{4.5}$$

satisfy the conditions of (FSE).

The connection between the equivalence classes of (FSE) and (GMSE) is as follows

Theorem 5. Let Σ_1 and Σ_2 be of the form (2.1) then

$$\Sigma_1 \overset{\text{(FSE)}}{\sim} \Sigma_2 \Leftrightarrow \Sigma_1 \overset{\text{(GMSE)}}{\sim} \Sigma_2.$$

Proof. ($\Rightarrow$) Let Σ_1, Σ_2 be (FSE), then $\exists$ polynomial matrices $M(s)$, $N(s)$, $X(s)$, $Y(s)$ such that

$$\begin{bmatrix} M(s) & 0 \\ X(s) & I \end{bmatrix} \begin{bmatrix} A_1(s) & B_1(s) \\ -C_1(s) & D_1(s) \end{bmatrix} = \begin{bmatrix} A_2(s) & B_2(s) \\ -C_2(s) & D_2(s) \end{bmatrix} \begin{bmatrix} N(s) & Y(s) \\ 0 & I \end{bmatrix}, \tag{4.6}$$

where (4.6) satisfies the conditions (2.7). In the manner of the proof of Theorem 4, the relationship (4.6) may be written in the alternative form

$$\left[\begin{array}{cc|c} I & X & 0 \\ 0 & M & 0 \\ \hline 0 & 0 & I \end{array}\right] \begin{bmatrix} I & -C_1 & -D_1 \\ 0 & A_1 & -B_1 \\ 0 & 0 & I \end{bmatrix} = \begin{bmatrix} I & -C_2 & -D_2 \\ 0 & A_2 & -B_2 \\ 0 & 0 & I \end{bmatrix} \left[\begin{array}{c|cc} I & 0 & 0 \\ \hline 0 & N & -Y \\ 0 & 0 & I \end{array}\right] \tag{4.7}$$

which can be seen to be a (FSE) transformation. Thus Σ_1 and Σ_2 are (GMSE).

($\Leftarrow$) Let Σ_1', Σ_2' be (FSE), then $\exists$ polynomial matrices $M_{ij}(s)$, $N_{ij}(s)$, $X_i(s)$, $Y_i(s)$, $i = 1, 2$ such that

$$\begin{bmatrix} M_{11} & M_{12} & 0 & I & -C_2 & -D_2 \\ M_{21} & M_{22} & 0 & 0 & A_2 & -B_2 \\ X_1 & X_2 & I & 0 & 0 & I \end{bmatrix} \begin{bmatrix} I & -C_1 & -D_1 \\ 0 & A_1 & -B_1 \\ 0 & 0 & I \\ -I & 0 & 0 \\ -Y_1 & -N_{11} & -N_{12} \\ -Y_2 & -N_{21} & -N_{22} \end{bmatrix} = 0, \tag{4.8}$$

where (4.8) satisfies the conditions (2.7 a, b, c) of (FSE). Again in the manner of Theorem 4 this may be reduced to

$$\begin{bmatrix} M_{22} + B_2X_2 & 0 \\ M_{12} + D_2X_2 & I \end{bmatrix} \begin{bmatrix} A_1 & B_1 \\ -C_1 & D_1 \end{bmatrix} = \begin{bmatrix} A_2 & B_2 \\ -C_2 & D_2 \end{bmatrix} \begin{bmatrix} Y_1C_1 + N_{11} & -Y_1D_1 - N_{12} \\ 0 & I \end{bmatrix} \tag{4.9}$$

which can be shown to satisfy the conditions (2.7 a, b, c) and so Σ_1, Σ_2 are (FSE). □

Thus (FSE) coincides with (GMSE) and the invariants of (GMSE) are as in Lemma 1.

5. CONCLUSIONS

It has been shown that the results of Rosenbrock and Van Der Weiden [11] concerning the relationship between certain invariants of a square invertible system and its inverse system can be extended to include the infinite frequency aspects. Kailath [3] has established in the finite case that the same form of equivalence (i. e. strict system equivalence) which relates two polynomial realisations of a given $G(s)$ is induced between the derived polynomial realisations of the inverse of $G(s)$. It is established here that this relationship extends to the notion of (FSE) ([2]) which has been established ([7]) as the basic underlying notion of equivalence for the generalised study of well formed linear systems. Thus a more complete explanation of the

relationship between the invariants of a square invertible system and those of its corresponding inverse emerges. These observations further extend to the generalised transfer function matrix inverse of Morf ([6]) as revealed in Section 4.

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REFERENCES

[1] P. A. Fuhrmann: On strict system equivalence and similarity. Internat. J. Control *25* (1977), 5–10.

[2] G. E. Hayton, A. B. Walker and A. C. Pugh: Infinite frequency structure preserving transformations for general polynomial system matrices. Internat. J. Control *52* (1990), 1–14.

[3] T. Kailath: Linear Systems. Prentice–Hall, Englewood Cliffs, N. J. 1980.

[4] N. P. Karampetakis and A. I. Vardulakis: Generalized state space system matrix equivalents of a Rosenbrock system matrix. IMA J. Math. Control Inform. *9* (1992), 147–160

[5] B. Levy, S. Kung, T. Kailath and M. Morf: A unification of system equivalence definitions. In: Proceedings of the 1977 IEEE CDC, New Orleans 1977, pp. 795–800.

[6] M. Morf: Extended system and transfer function matrices and system equivalence. In: Proceedings of the 1975 IEEE CDC, Houston 1975, pp. 199–206.

[7] A. C. Pugh, N. P. Karampetakis, A. I. G. Vardulakis and G. E. Hayton: A fundamental notion of equivalence for linear multivariable systems. IEEE Trans. Automat. Control *39* (1994), 1141–1145.

[8] A. C. Pugh and P. A. Ratclife: On the zeros and poles of a rational matrix. Internat. J. Control *30* (1979), 213–226.

[9] H. H. Rosenbrock: State-Space and Multivariable Theory. Wiley, New York 1970.

[10] H. H. Rosenbrock: Order, degree and complexity. Internat. J. Control *19* (1974), 323–331.

[11] H. H. Rosenbrock and A. J. J. Van der Weiden: Inverse systems. Internat. J. Control *25* (1977), 389–392.

[12] A. I. G. Vardulakis: Linear Multivariable Control, Algebraic Analysis and Synthesis Methods. Nelson–Wiley, London 1991.

[13] G. C. Verghese: Infinite Frequency Behaviour in Generalized Dynamical Systems. Ph.D. Dissertation, Stanford University, Stanford, California, USA 1979.

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