Computation of the Generalised Inverse of a Polynomial Matrix and Applications

by

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ABSTRACT

The computation of the generalised inverse of a constant matrix (Dessel 1965) is utilized in finding the generalised inverse and its Laurent expansion of a nonregular polynomial matrix. The proposed algorithm constitutes a generalization of the algorithm proposed by Fragulis et. al. (1991) for regular polynomial matrices and gives rise to numeroous applications in multivariable system analysis.

I. INTRODUCTION

Consider the polynomial matrix

$$A(s) = A_q s^q + A_{q-1} s^{q-1} + \cdots + A_1 s + A_0 \in \Re[s]^{n\times m} \tag{1.1}$$

with $A_i \in \Re^{n\times m}, i \in q$ and n not necessary equal to m. The problem of the investigation of the generalised inverse of a polynomial matrix has been the concern of many scientists [1], [2], [4], [6], [13]-15], [17] because of the large number of its implications in multivariable system analysis i.e. computation of the transfer function matrix of a system [4], [6], inverse systems [9]-[12], solutions of systems (see section IV), controllability and observability matrices of general Polynomial Matrix Descriptions (PMDs) [3], solution of diophantique equations (see section IV) which gives rise to numeroous applications [8] e.t.c.

In case where (n=m, q=0 and $\det[A(s)] \neq 0$ i.e. $A(s) \equiv A_0 \in \Re^{n\times n}$ with $\det[A_0] \neq 0$, the above problem has been investigated by Faddeev & Faddeeva (1963), Zadeh and Desoer (1963), while in case where (n=m, q=1 and $\det[A(s)] \neq 0$) and

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(n=m, $q \in N$ and $\det[A(s)] \neq 0$) the solution proposed by the above authors has been extended by Mertzios (1984) and Fragulis et. al. (1991) respectively. It was shown by the above authors that an interesting application of the determination of the inverse of A(s) is the computation of the transfer function matrix of a linear, time invariant, multivariable system.

In the nonregular case of constant matrices $A(s) \equiv A_0 \in \Re^{n \times m}$ with n not necessary equal to m, the *generalised inverse* of A_0 has been defined by Penrose (1955) and a numerical algorithm for the computation of this matrix is later given by Decell (1965). However an open question still remains for the computation of the generalised inverse of A(s) in (1.1). The answer to this question will be the subject of this paper.

More analytically in section II we present some preliminary results concerning the definition of the generalised inverse of a constant matrix and its computation (Decell 1965). In section III we determine a two-dimension recursive algorithm which computes the generalised inverse of A(s) in (1.1), in terms of its coefficient matrices $A_i \in \Re^{n \times m}, i \in q$ while in section IV we present three applications of the generalised inverse : a) the computation of the right (left) inverse of a rational matrix, b) the investigation of the solution of an AR-representation and c) the solution in a specific feedback compensation problem. All the above implications are illustrated via examples. Finally in section V we evaluate the Laurent expansion of the generalised inverse.

II. BACKGROUND MATERIAL

Some preliminary results concerning the definition of the generalised inverse of a constant matrix and its computation, are necessary for what follows and we shall present it in the sequel.

DEFINITION 2.1 (Penrose 1955) For every matrix $A \in \Re^{n \times m}$, a unique matrix $A^{\dagger} \in \Re^{m \times n}$ exist which is called *generalised inverse* satisfying the following

(i) $AA^{\dagger}A = A$,

(ii) $A^{\dagger}AA^{\dagger} = A^{\dagger}$,

(iii) $(AA^{\dagger})^{T} = AA^{\dagger}$,

(iv) $(A^{\dagger}A)^{T} = A^{\dagger}A$

where A^{T} denotes the transpose of A. In the special case that the matrix A is square nonsingular matrix, the generalised inverse of A is simply its inverse i.e. $A^{\dagger} = A^{-1}$. In cases where there exist a matrix A^{g_1} which satisfies only the first condition is called $\{1\}$-inverse. $\{1\}$-inverses are not unique and play a fundamental role in the solution of polynomial diophantique equations as we shall see in section IV. •

In analogous way we define the generalised inverse $A(s)^{\dagger} \in \Re(s)^{m\times n}$ of a rational matrix $A(s) \in \Re(s)^{n\times m}$ as the matrix which satisfies the properties (i)-(iv) of the Definition 1.1. Consider now the following theorem :

THEOREM 2.2 (Decell 1965) Let $A \in \Re^{n\times m}$ and

$$a(s) = \det\left[sI_n - AA^T\right] = a_0 s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n \quad \text{with } a_0 = 1 \tag{2.1}$$

be the characteristic polynomial of AA^T. If $k \neq 0$ is the largest integer such that $a_k \neq 0$, then the generalised inverse of A is given by

$$A^{\dagger} = -a_k^{-1}A^T\left[\left(AA^T\right)^{k-1} + a_1\left(AA^T\right)^{k-2} + \cdots + a_{k-1}I_n\right] \tag{2.2}$$

If k=0 is the largest integer such that $a_k \neq 0$, then $A^{\dagger} = 0$. •

ALGORITHM 2.3 (Computation of the generalised inverse of a constant matrix $A \in \Re^{n\times m}$)

Step 1. Let $A \in \Re^{n\times m}$. Consider the sequences $\{a_0, a_1, \ldots, a_n\}$ and $\{B_0, B_1, \ldots, B_n\}$ constructed by the following way

$$\begin{array}{lll} A_0 = 0 & a_0 = 1 & B_0 = I_n \\ A_1 = \left(AA^T\right)B_0 & a_1 = -\dfrac{\text{trace}[A_1]}{1!} & B_1 = A_1 + a_1 I_n \\ \ldots\ldots\ldots\ldots & \ldots\ldots\ldots\ldots & \ldots\ldots\ldots\ldots \\ A_n = \left(AA^T\right)B_{n-1} & a_n = -\dfrac{\text{trace}[A_n]}{n!} & B_n = A_n + a_n I_n \end{array} \tag{2.3}$$

Step 2. If $k \neq 0$ is the largest integer such that $a_k \neq 0$, then the generalised inverse $A^{\dagger}$ of A is given by

$$A^{\dagger} = -a_k^{-1} A^T \left[\left(AA^T\right)^{k-1} + a_1 \left(AA^T\right)^{k-2} + \cdots + a_{k-1} I_n \right] = -a_k^{-1} A^T B_{k-1} \tag{2.4}$$

else (k=0 is the largest integer such that $a_k \neq 0$) $A^{\dagger} = 0$. •

III. GENERALISED INVERSE OF A POLYNOMIAL MATRIX.

Consider now the polynomial matrix

$$A(s) = A_q s^q + A_{q-1} s^{q-1} + \cdots + A_1 s + A_0 \in \Re[s]^{n \times m} \tag{3.1}$$

and therefore the transpose of A(s) is

$$A(s)^T = A_q^T s^q + A_{q-1}^T s^{q-1} + \cdots + A_1^T s + A_0^T \in \Re[s]^{m \times n} \tag{3.2}$$

Following similar lines with Theorem 2.2 we can easily obtain the following

THEOREM 3.1 Let $A(s) \in \Re[s]^{n \times m}$ as in (3.1) and

$$a(z,s) = \det\left[zI_n - A(s)A(s)^{\dagger}\right] = a_0(s)z^n + a_1(s)z^{n-1} + \cdots + a_{n-1}(s)z + a_n(s) \tag{3.3}$$

$a_0(s) = 1$, be the characteristic polynomial of $A(s)A(s)^T$. Let $a_{n-1}(s) \equiv 0, \ldots, a_{k+1}(s) \equiv 0$ while $a_k(s) \neq 0$ and $\Lambda := \{s_i \in \Re : a_k(s_i) = 0\}$. Then the generalised inverse $A(s)^{\dagger}$ of $A(s)$ for $s \in \Re - \Lambda$ is given by

$$A(s)^{\dagger} = -a_k(s)^{-1} A(s)^T \left[\left(A(s)A(s)^T\right)^{k-1} + a_1(s)\left(A(s)A(s)^T\right)^{k-2} + \cdots + a_{k-1}(s) I_n \right] \tag{3.4a}$$

If k=0 is the largest integer such that $a_k(s) \neq 0$, then $A(s)^{\dagger} = 0$. For those $s_i \in \Lambda$ find the largest integer $k_i < k$ such that $a_{k_i}(s_i) \neq 0$ and then the generalised inverse $A(s_i)^{\dagger}$ of $A(s_i)$ is given by

$$A(s_i)^{\dagger} = -a_{k_i}(s_i)^{-1} A(s_i)^T \left[\left(A(s_i)A(s_i)^T\right)^{k_i-1} + a_1(s_i)\left(A(s_i)A(s_i)^T\right)^{k_i-2} + \cdots + a_{k_i-1}(s_i) I_n \right] \tag{3.4b}$$

Proof. The proof is exactly the same with this of Decell (1965). •

The computation of $a_i(s)$ and therefore of the generalised inverse $A(s)^{\dagger}$ of $A(s)$ follows in a similar way with Algorithm 2.3 as follows

ALGORITHM 3.2 (Computation of the generalised inverse $A(s)^{\dagger} \in \Re(s)^{m \times n}$ of $A(s) \in \Re[s]^{n \times m}$)

Step 1. Let $A(s) \in \Re[s]^{n \times m}$. Consider the sequences $\{a_0(s), a_1(s), \ldots, a_n(s)\}$ and $\{B_0(s), B_1(s), \ldots, B_n(s)\}$ constructed by the following way

$$\begin{array}{lll} A_0(s)=0 & a_0(s)=1 & B_0(s)=I_n \\ A_1(s)=\left(A(s)A(s)^T\right)B_0(s) & a_1(s)=-\dfrac{\text{trace}\left[A_1(s)\right]}{1!} & B_1(s)=A_1(s)+a_1(s)I_n \\ \ldots\ldots\ldots\ldots & \ldots\ldots\ldots\ldots & \ldots\ldots\ldots\ldots \\ A_n(s)=\left(A(s)A(s)^T\right)B_{n-1}(s) & a_n(s)=-\dfrac{\text{trace}\left[A_n(s)\right]}{n!} & B_n(s)=A_n(s)+a_n(s)I_n \end{array} \quad (3.5)$$

Step 2. Let $a_{n-1}(s)\equiv 0,\ldots,a_{k+1}(s)\equiv 0$ while $a_k(s)\neq 0$ and $\Lambda:=\{s_i\in\Re : a_k(s_i)=0\}$. Then the generalised inverse $A(s)^{\dagger}$ of $A(s)$ for $s\in\Re-\Lambda$ is given by

$$\begin{aligned} A(s)^{\dagger} &= -a_k(s)^{-1}A(s)^T\left[\left(A(s)A(s)^T\right)^{k-1}+a_1(s)\left(A(s)A(s)^T\right)^{k-2}+\cdots+a_{k-1}(s)I_n\right] \\ &= -a_k(s)^{-1}A(s)^T B_{k-1}(s) \end{aligned} \quad (3.6a)$$

If k=0 is the largest integer such that $a_k(s)\neq 0$, then $A(s)^{\dagger}=0$. For those $s_i\in\Lambda$ find the largest integer $k_i<k$ such that $a_{k_i}(s_i)\neq 0$ and then the generalised inverse $A(s_i)^{\dagger}$ of $A(s_i)$ is given by

$$\begin{aligned} A(s_i)^{\dagger} &= -a_{k_i}(s_i)^{-1}A(s_i)^T\left[\left(A(s_i)A(s_i)^T\right)^{k_i-1}+a_1(s_i)\left(A(s_i)A(s_i)^T\right)^{k_i-2}+\cdots+a_{k_i-1}(s_i)I_n\right]= \\ &= -a_{k_i}(s_i)^{-1}A(s_i)^T B_{k_i-1}(s_i) \end{aligned} \quad (3.6b) \bullet$$

It is seen from (3.1), (3.2) that the greatest powers of $A_1(s)=A(s)A(s)^T$ (and thus of the trace $a_1(s)$ of $A_1(s)$) and $B_1(s)$ are equal to 2q. In the same way according to (3.1) and (3.2) the greatest power of

$$A_2(s)=\left(A(s)A(s)^T\right)B_1(s)=\left(A(s)A(s)^T\right)^2+a_1(s)\left(A(s)A(s)^T\right) \quad (3.7)$$

(and thus of $a_2(s)=-\dfrac{\text{trace}\left[A_2(s)\right]}{2!}$) and

$$B_2(s)=A_2(s)+a_2(s)I_n \overset{(3.7)}{=} \left(A(s)A(s)^T\right)^2+a_1(s)\left(A(s)A(s)^T\right)+a_2(s)I_n \quad (3.8)$$

are equal to 4q. Thus it is easily seen from (3.1) and (3.5) that $a_i(s)$ and $B_i(s)$ may be rewritten as

$$a_i(s)=\sum_{j=0}^{2iq}\hat{a}_{i,j}s^j=-\text{trace}\frac{\left[\left(A(s)A(s)^T\right)^i+a_1(s)\left(A(s)A(s)^T\right)^{i-1}+\cdots+a_{i-1}(s)\left(A(s)A(s)^T\right)\right]}{i!} \quad (3.9)$$

with $i=0,1,\ldots,k$ and

$$B_i(s)=\sum_{j=0}^{2iq}B_{i,j}s^j=\left(A(s)A(s)^T\right)^i+a_1(s)\left(A(s)A(s)^T\right)^{i-1}+\cdots+a_i(s)I_n \quad (3.10)$$

with $i=0,1,..,k-1$, where $B_{i,j},\hat{a}_{i,j}$ are constant coefficient matrices and scalar of the powers s^j. It is seen from (3.6) that for the computation of the generalised inverse of

A(s) we need the integer k, the quantities $a_k(s)$ and $B_{k-1}(s)$ i.e. the coefficients $\hat{a}_{k,j}$ and the coefficient matrices $B_{k-1,j}$ defined by

$$a_k(s) = \sum_{j=0}^{2kq} \hat{a}_{k,j} s^j \tag{3.11}$$

and

$$B_{k-1}(s) = \sum_{j=0}^{2(k-1)q} B_{k-1,j} s^j \tag{3.12}$$

Taking into account that

$$\begin{aligned} A(s)A(s)^T B_i(s) &= \left(\sum_{j=0}^{q} A_j s^j\right)\left(\sum_{j=0}^{q} A_j^T s^j\right)\left(\sum_{j=0}^{2iq} B_{i,j} s^j\right) = \\ &= \left(\sum_{j=0}^{2q}\left(\sum_{k=0}^{j} A_{j-k} A_k\right) s^j\right)\left(\sum_{j=0}^{2iq} B_{i,j} s^j\right) = \sum_{j=0}^{2(i+1)q}\left(\sum_{k=0}^{j}\left(\sum_{\ell=0}^{j-k}\left(A_{j-k-\ell} A_\ell^T\right)\right) B_{i,k}\right) s^j \end{aligned} \tag{3.13}$$

and substituting (3.9), (3.10) and (3.13) in the recursive relations (3.5), we obtain the following recursive algorithm that determine $\hat{a}_{i+1,j}$ and $B_{i+1,j}$ for j=0,1,...,2(i+1)q.

ALGORITHM 3.3 (Computation of the generalised inverse $A(s)^{\dagger} \in \Re(s)^{m \times n}$ of $A(s) \in \Re[s]^{n \times m}$)

Initial Conditions

$$B_{0,0} = I_n \tag{3.14}$$

Boundary Conditions

$$B_{0,j} = 0 \quad \forall j > 0 \tag{3.15a}$$

$$B_{i,j} = 0 \quad j = 2iq+1, 2iq+2, \ldots, 2(n-1)q \tag{3.15b}$$

Recursive Relations for $\boxed{a_i(s)}$

$$\hat{a}_{i+1,j} = -\frac{1}{i+1}\operatorname{trace}\left(\sum_{k=0}^{j}\left(\sum_{\ell=0}^{j-k}\left(A_{j-k-\ell} A_\ell^T\right) B_{i,k}\right)\right) \quad \begin{matrix} j = 0,1,\ldots,2(i+1)q \\ i = 0,1,\ldots,n-1 \end{matrix} \tag{3.16}$$

Recursive Relations for $\boxed{B_i(s)}$

$$B_{i+1,j} = \left(\sum_{k=0}^{j}\left(\sum_{\ell=0}^{j-k}\left(A_{j-k-\ell} A_\ell^T\right) B_{i,k}\right)\right) + \hat{a}_{i+1,j} I_n \quad \begin{matrix} j = 0,1,\ldots,2(i+1)q \\ i = 0,1,\ldots,n-2 \end{matrix} \tag{3.17}$$

Terminate

FIND k : $a_{k+1}(s) \equiv a_{k+2}(s) \equiv \cdots \equiv a_n(s) \equiv 0$

or $\hat{a}_{k+1,j} = \hat{a}_{k+2,j} = \cdots = \hat{a}_{n,j} = 0 \quad \forall j \in N$

then

$$B_j := B_{k-1,j} \quad j = 0,1,\ldots,2(k-1)q \tag{3.18a}$$

$$\hat{a}_j := \hat{a}_{k,j} \quad j = 0,1,\ldots,2kq \tag{3.18b}$$

OUTPUT

$$A(s)^{\dagger} = -\left(\sum_{j=0}^{2kq} \hat{a}_j s^j\right)^{-1} \left(\sum_{j=0}^{q} A_j^T s^j\right)\left(\sum_{j=0}^{2(k-1)q} B_j s^j\right) = -\left(\sum_{j=0}^{2kq} \hat{a}_j s^j\right)^{-1} \left(\sum_{j=0}^{(2k-1)q} \sum_{\ell=0}^{j} \left(A_{j-\ell}^T B_\ell\right) s^j\right) \tag{3.19}$$ •

It is readily seen that the generalised inversersion algorithm is a two-dimensional recursive algorithm since it depends of two independent variables i,j.

REMARK 3.4 Note that **a)** we use the same algorithm for those $s_i \in \Lambda$ i.e. $\Lambda := \{s_i \in \Re : a_k(s_i) = 0\}$, by finding the largest $k_i < k$ such that $a_{k_i}(s_i) \neq 0$ and **b)** in case where k=0 is the largest integer such that $a_k(s) \neq 0$ then $A(s)^{\dagger} = 0$. •

IV. IMPLICATIONS OF THE GENERALISED INVERSE IN LINEAR SYSTEM THEORY.

An important result presented by Penrose (1955) which is used later by Decell (1965) to prove Theorem 1, will be usefull in the sequel for many important problems in Linear System Theory.

THEOREM 4.1 (Penrose 1955) The matrix equation $PXQ = C$ has a solution iff $PP^{\dagger}CQ^{\dagger}Q = C$, in which case all the solutions are given by the following formula

$$X = P^{\dagger}CQ^{\dagger} + Y - P^{\dagger}PYQQ^{\dagger} \tag{4.1}$$

where $P^{\dagger}, Q^{\dagger}$ are the generalised inverses of P and Q respectively and Y is arbitrary to within having the dimension of X. •

It can be easily seen that the same theorem may be applied to polynomial matrices without any changes and thus we shall have that

THEOREM 4.2 The polynomial matrix equation $P(s)X(s)Q(s) = C(s)$ has a solution iff $P(s)P(s)^{\dagger} C(s)Q(s)^{\dagger} Q(s) = C(s)$ $\forall s \in \Re$, in which caseall the solutions are given by the following formula

$$X(s) = P(s)^{\dagger} C(s)Q(s)^{\dagger} + Y(s) - P(s)^{\dagger} P(s)Y(s)Q(s)Q(s)^{\dagger} \qquad (4.2)$$

where $P(s)^{\dagger}, Q(s)^{\dagger}$ are the generalised inverses of $P(s)$ and $Q(s)$ respectively and $Y(s)$ is arbitrary to within having the dimension of $X(s)$.

Proof. The proof is exactly the same to the one presented in Penrose (1955). •

We have to note here that the above theorem remains the same if we substitute the generalised inverses $P(s)^{\dagger}$ and $Q(s)^{\dagger}$ with the $\{1\}$-inverses $P(s)^{g_1}$ and $Q(s)^{g_1}$ respectively. In the light of Algorithm 3.3 and Theorem 4.2 we may have numerous applications. However in this paper we shall present three only.

a. Computation of the right (left) inverse of a polynomial matrix.

It is known that the right (left) inverse of a polynomial matrix $A(s) \in \Re[s]^{n\times m}$ is defined as the matrix $X(s) \in \Re(s)^{m\times n}$ which satisfies the following property

$$A(s)X(s) = I_n \qquad \left(X(s)A(s) = I_m\right) \qquad (4.3)$$

A direct application of Theorem 4.2 is the following

THEOREM 4.3 A polynomial matrix $A(s) \in \Re[s]^{n\times m}$ has a right (left) inverse $A(s)^{*}$ iff $A(s)A(s)^{\dagger} = I_n$ $\left(A(s)^{\dagger} A(s) = I_m\right)$ in which case all right (left) inverses are given by the following formula

$$A(s)^{*} = A(s)^{\dagger} + \left(I_m - A(s)^{\dagger} A(s)\right)Y(s) \qquad (4.4a)$$

$$\left(A(s)^{*} = A(s)^{\dagger} + Y(s)\left(I_n - A(s)A(s)^{\dagger}\right)\right) \qquad (4.4b)$$

where $Y(s) \in \Re(s)^{m\times n}$ $\left(Y(s) \in \Re(s)^{n\times m}\right)$ is an arbitrary rational matrix to within having the dimension of $A(s)^{*}$. •

REMARK 4.4 The ability to compute the generalised inverse of a polynomial matrix $A(s) \in \Re[s]^{n\times m}$ via the algorithm 3.2 or 3.3 give us the necessary tools for the computation of the right and left inverse of a polynomial matrix in case where this matrix exists. The above algorithms may be used for the computation of the right and left inverse of a rational matrix also instead of a polynomial. The only we have to do is to write the rational matrix $A(s)\in\Re(s)^{n\times m}$ as $\dfrac{A^{'}(s)}{d(s)}$ where $d(s)$ is the greatest common multiple of all denominator polynomials of $A(s)$. Then the problem of finding the right inverse of $A(s)$ is reduced to the problem of finding a matrix $X(s)$: $A^{'}(s)X(s)=d(s)I_n$, a problem which has a solution iff (according to Theorem 4.2) $A^{'}(s)A^{'}(s)^{\dagger}d(s)=d(s)I_n$ or equivalently iff $A^{'}(s)A^{'}(s)^{\dagger}=I_n\ \ \forall s\in\Re$. In case where the above condition is satisfied the right inverse $A(s)^{*}$ of $A(s)$ will be given by the following formula

$$A(s)^{*}=A^{'}(s)^{\dagger}d(s)+\left(I_m-A^{'}(s)^{\dagger}A^{'}(s)\right)Y(s) \tag{4.5}$$ •

An important application of the computation of the right or left inverse of a rational matrix is the investigation of the solution of the matrix diophantique equation:: Given the rational matrices $N(s)$ and $D(s)$, find $X(s)$ and $Y(s)$ of appropriate dimensions such that :

$$N(s)X(s)+D(s)Y(s)=I \tag{4.6a}$$

or equivalently

$$\begin{pmatrix}N(s) & D(s)\end{pmatrix}\begin{pmatrix}X(s)\\ Y(s)\end{pmatrix}=I \tag{4.6b}$$

The investigation of the solution space of (4.6) plays an important role in problems like parametrization of stabilizing controllers, robust stabilization, disturbance rejection, reference tracking, model matching, H_2-optimal control e.t.c. (see survey paper of Kucera 1993).

EXAMPLE 4.5 Let

$$N(s)=\begin{pmatrix}1 & s\\ 0 & 1\end{pmatrix} \quad \text{and} \quad D(s)=\begin{pmatrix}0\\ s\end{pmatrix} \tag{E.1}$$

Find $X(s)\in\Re(s)^{2\times 2}$ and $Y(s)\in\Re(s)^{1\times 2}$ such that

$$N(s)X(s)+D(s)Y(s)=I_2 \tag{E.2}$$

or equivalently

$$\underbrace{\begin{pmatrix} 1 & s & 0 \\ 0 & 1 & s \end{pmatrix}}_{A(s)} \begin{pmatrix} X(s) \\ Y(s) \end{pmatrix} = I_2 \tag{E.3}$$

Consider the polynomial matrix $A(s)$

$$A(s) = \begin{pmatrix} 1 & s & 0 \\ 0 & 1 & s \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}}_{A_0} + \underbrace{\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{A_1} s \tag{E.4}$$

We have that n=2, m=3 and q=1. We apply the algorithm 3.3 and find

$$A(s)^{\dagger} = \frac{1}{s^4 + s^2 + 1} \begin{pmatrix} s^2 + 1 & -s \\ s^3 & 1 \\ -s^2 & s + s^3 \end{pmatrix} \quad \forall s \in \Re \tag{E.5}$$

It is easily seen that $A(s)A(s)^{\dagger} = I_2$ and thus there exist a right inverse of the matrix $A(s)$ and is given by the following formula

$$A(s)^{*} \equiv \begin{pmatrix} X(s) \\ Y(s) \end{pmatrix} = A(s)^{\dagger} + \left(I_3 - A(s)^{\dagger} A(s)\right) Y(s) =$$

(E.6)

$$= \frac{1}{s^4 + s^2 + 1} \left\{ \begin{bmatrix} \begin{pmatrix} s^2 + 1 & -s \\ s^3 & 1 \\ -s^2 & s + s^3 \end{pmatrix} + \begin{pmatrix} s^4 & -s^3 & s^2 \\ -s^3 & s^2 & -s \\ s^2 & -s & 1 \end{pmatrix} Y(s) \end{bmatrix} \right\}$$

$$= \frac{1}{s^4 + s^2 + 1} \left\{ \begin{bmatrix} \begin{pmatrix} s^2 + 1 + s^2 a(s) & -s + s^2 b(s) \\ s^3 - s a(s) & 1 - s b(s) \\ -s^2 + a(s) & s + s^3 + b(s) \end{pmatrix} \text{ for arbitrary } a(s), b(s) \in \Re(s) \end{bmatrix} \right\}$$

Thus

$$X(s) = \frac{1}{s^4 + s^2 + 1} \begin{pmatrix} s^2 + 1 + s^2 a(s) & -s + s^2 b(s) \\ s^3 - s a(s) & 1 - s b(s) \end{pmatrix} \tag{E.7}$$

and

$$Y(s) = \frac{1}{s^4 + s^2 + 1} \begin{pmatrix} -s^2 + a(s) & s + s^3 + b(s) \end{pmatrix} \tag{E.8}$$

For $a(s) = s^2$ and $b(s) = -s - s^3$ we have a polynomial solution of the diophantique equation (E.3)

$$X(s) = \begin{pmatrix} 1 & -s \\ 0 & 1 \end{pmatrix} \text{ and } Y(s) = \begin{pmatrix} 0 & 0 \end{pmatrix} \tag{E.9}$$ •

<u>b. Solution of AR-representations.</u>

Consider the following AutoRegressive (AR) representation (Willems 1991)

$$A(\rho)\beta(t) = 0 \tag{4.7}$$

where $\beta(t):(0-,+\infty)\to\Re^m$, ρ denotes the differential operator i.e. $\rho[\beta(t)]:=\frac{d\beta(t)}{dt}$,

$$A(\rho)= A_q\rho^q + A_{q-1}\rho^{q-1}+\cdots+A_1\rho + A_0 \in \Re[\rho]^{n\times m} \tag{4.8}$$

and n not necessary equal to m. (4.7) may be rewritten under Laplace transforms as

$$A(s)\hat{\beta}(s)=\begin{pmatrix} s^{q-1}I_n & s^{q-2}I_n & \cdots & I_n\end{pmatrix}\begin{pmatrix} A_q & 0 & \cdots & 0 \\ A_{q-1} & A_q & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_1 & A_2 & \cdots & A_q\end{pmatrix}\begin{pmatrix}\beta(0-) \\ \beta^{(1)}(0-) \\ \vdots \\ \beta^{(q-1)}(0-)\end{pmatrix}=:\hat{a}(s) \tag{4.9}$$

where $\hat{\beta}(s) = L[\beta(t)]$ (L denotes the Laplace transform). In the light of Theorem 4.2 we obtain the following

THEOREM 4.6 The AR-representation (4.7) has a solution iff $A(s)A(s)^{\dagger}\hat{a}(s) = \hat{a}(s)I_n$ $\forall s\in\Re$, in which case all general solutions are given by the following formula

$$\beta(t) = L^{-1}\left[\hat{\beta}(s)\right]= L^{-1}\left[A(s)^{\dagger}\hat{a}(s)+\left[I_m - A(s)^{\dagger}A(s)\right]y(s)\right] \tag{4.10}$$

where $y(s)$ is arbitrary to within having the dimension of $\hat{\beta}(s)$. •

EXAMPLE 4.7 Consider the following AR-representation

$$\underbrace{\begin{pmatrix}\rho & \rho^4 & \rho^2+\rho \\ 1 & \rho^3 & \rho+1 \\ 0 & \rho+1 & 0\end{pmatrix}}_{A(\rho)}\underbrace{\begin{pmatrix}\beta_1(t) \\ \beta_2(t) \\ \beta_3(t)\end{pmatrix}}_{\beta(t)} = 0_{3,1} \quad \rho:= d/dt \tag{E.1}$$

We are interested in the solution of the above AR-representation under the following initial conditions :

$$\beta_2^{(1)}(0-)=1 \quad \text{and} \quad \beta_i^{(j)}(0-)= 0 \ \ \forall(i,j)\neq(2,1) \tag{E.2}$$

We have that

$$A(\rho)=\begin{pmatrix}\rho & \rho^4 & \rho^2+\rho \\ 1 & \rho^3 & \rho+1 \\ 0 & \rho+1 & 0\end{pmatrix}=\underbrace{\begin{pmatrix}0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0\end{pmatrix}}_{A_0}+\underbrace{\begin{pmatrix}1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0\end{pmatrix}}_{A_1}\rho+ \tag{E.3}$$

$$+\underbrace{\begin{pmatrix}0&0&1\\0&0&0\\0&0&0\end{pmatrix}}_{A_2}\rho^2+\underbrace{\begin{pmatrix}0&0&0\\0&1&0\\0&0&0\end{pmatrix}}_{A_3}\rho^3+\underbrace{\begin{pmatrix}0&1&1\\0&0&0\\0&0&0\end{pmatrix}}_{A_4}\rho^4$$

with $\mathrm{rank}_{\Re(s)}A(s)=2<3$, n=m=3 and q=4. We apply the algorithm 3.3 and find that

For $s\in\Re-\{-1\}$

$$A(s)^{\dagger}=-\frac{1}{(s+1)^2(s^2+1)(s^2+2s+2)}\begin{pmatrix}-s(s+1)^2 & -(s+1)^2 & s^3(s+1)(s^2+1)\\ 0 & 0 & -(s+1)(s^2+1)(s^2+2s+2)\\ -s(s+1)^3 & -(s+1)^3 & s^3(s+1)^2(s^2+1)\end{pmatrix}$$

(E.4a)

For $s=-1$

$$A(-1)^{\dagger}=\begin{pmatrix}-1/4 & 1/4 & 0\\ 1/4 & -1/4 & 0\\ 0 & 0 & 0\end{pmatrix} \tag{E.4b}$$

Taking Laplace transforms into the AR-representation (E.1) we obtain the following equation

$$\underbrace{\begin{pmatrix}s & s^4 & s^2+s\\ 1 & s^3 & s+1\\ 0 & s+1 & 0\end{pmatrix}}_{A(s)}\underbrace{\begin{pmatrix}\hat{\beta}_1(s)\\ \hat{\beta}_2(s)\\ \hat{\beta}_3(s)\end{pmatrix}}_{\hat{\beta}(s)}=\underbrace{\begin{pmatrix}s^2\\ s\\ 0\end{pmatrix}}_{\hat{a}(s)} \tag{E.5}$$

where $\hat{\beta}_i(s)=L[\beta_i(t)]$ (L denotes the Laplace transform). (E.5) has a solution according to Theorem 4.6 iff

$$A(s)A(s)^{\dagger}\hat{a}(s)\equiv\hat{a}(s)\Leftrightarrow$$

For $s\in\Re-\{-1\}$

$$\underbrace{\frac{1}{s^2+1}\begin{pmatrix}s^2 & s & 0\\ s & 1 & 0\\ 0 & 0 & s^2+1\end{pmatrix}}_{A(s)A(s)^{\dagger}}\underbrace{\begin{pmatrix}s^2\\ s\\ 0\end{pmatrix}}_{\hat{a}(s)}\equiv\underbrace{\begin{pmatrix}s^2\\ s\\ 0\end{pmatrix}}_{\hat{a}(s)} \tag{E.6a}$$

For $s=-1$

$$\underbrace{\begin{pmatrix}1/2 & -1/2 & 0\\ -1/2 & 1/2 & 0\\ 0 & 0 & 0\end{pmatrix}}_{A(-1)A(-1)^{\dagger}}\underbrace{\begin{pmatrix}1\\ -1\\ 0\end{pmatrix}}_{\hat{a}(-1)}\equiv\underbrace{\begin{pmatrix}1\\ -1\\ 0\end{pmatrix}}_{\hat{a}(-1)} \tag{E.6b}$$

which are satisfied. All the solutions of the matrix equation (E.9) are given according to Theorem 4.6 by the following formula

For $s \in \Re - \{-1\}$

$$\hat{\beta}(s) = A(s)^{\dagger}\hat{a}(s) + \left(I_3 - A(s)^{\dagger}A(s)\right)y(s) =$$

$$= -\frac{1}{(s+1)^2(s^2+1)(s^2+2s+2)}\begin{pmatrix} -s(s+1)^2 & -(s+1)^2 & s^3(s+1)(s^2+1) \\ 0 & 0 & -(s+1)(s^2+1)(s^2+2s+2) \\ -s(s+1)^3 & -(s+1)^3 & s^3(s+1)^2(s^2+1) \end{pmatrix}\begin{pmatrix} s^2 \\ s \\ 0 \end{pmatrix} +$$

$$+\frac{1}{s^2+2s+2}\begin{pmatrix} (s+1)^2 & 0 & -(s+1) \\ 0 & 0 & 0 \\ -(s+1) & 0 & 1 \end{pmatrix}\begin{pmatrix} y_1(s) \\ y_2(s) \\ y_3(s) \end{pmatrix} \Leftrightarrow \hat{\beta}(s) = \frac{1}{s^2+2s+2}\begin{pmatrix} s-(s+1)a(s) \\ 0 \\ s(s+1)+a(s) \end{pmatrix} \tag{E.7a}$$

For $s = -1$

$$\hat{\beta}(-1) = A(-1)^{\dagger}\hat{a}(-1) + \left(I_3 - A(-1)^{\dagger}A(-1)\right)Y =$$

$$= \begin{pmatrix} -1/4 & 1/4 & 0 \\ 1/4 & -1/4 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + \begin{pmatrix} 1/2 & 1/2 & 0 \\ 1/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \end{pmatrix} \Leftrightarrow \hat{\beta}(-1) = \begin{pmatrix} -\frac{1}{2}(1-Y_1-Y_2) \\ \frac{1}{2}(1+Y_1+Y_2) \\ Y_3 \end{pmatrix} \tag{E.7b}$$

where $a(s) := -y_1(s)(s+1) + y_3(s)$ is an arbitrary rational function. Assuming that $Y_1 = -Y_2 - 1$ and $Y_3 = a(-1)(\equiv y_3(-1))$ the solution $\hat{\beta}(s)$ is continuous at $s = -1$ and its inverse Laplace transform is the following

$$\beta(t) = \begin{pmatrix} e^{-t}[\cos(t) - \sin(t)] + \int_{0-}^{t} e^{-(t-\tau)}\cos(t-\tau)x(\tau)d\tau \\ 0 \\ \delta(t) - e^{-t}[\cos(t) + \sin(t)] + \int_{0-}^{t} e^{-(t-\tau)}\sin(t-\tau)x(\tau)d\tau \end{pmatrix} \tag{E.8}$$

where $x(t)$ denotes the inverse Laplace transform of $a(s)$ i.e. $x(t) = L^{-1}[a(s)]$ and thus it is an arbitrary function. •

c. Feedback compensation.

Consider an open loop system with transfer function matrix $G(s) \in \Re(s)^{n\times m}$.

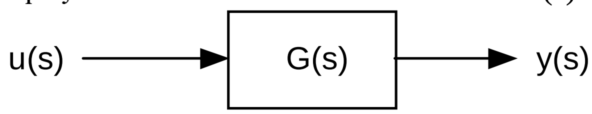

Diagram 1. Open loop system.

We would like to find out when there exist an output feedback of the form

$$u(s) = -F(s)y(s) + v(s) \text{ with } F(s) \in \Re(s)^{m\times n} \tag{4.11}$$

such that the closed loop system

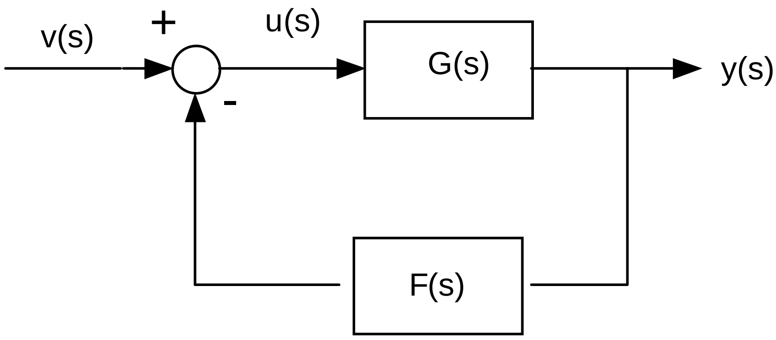

Diagram 2. Closed loop system.

has transfer function H(s). We would like therefore to find a rational matrix $F(s) \in \Re(s)^{m\times n}$ which satisfies the following equation

$$H(s) = \left(I_n + G(s)F(s)\right)^{-1} G(s) \Leftrightarrow G(s)F(s)H(s) = G(s) - H(s) \tag{4.12}$$

Let $G(s) = \dfrac{G'(s)}{g(s)}$ where $G'(s) \in \Re[s]^{n\times m}$ and $g(s)$ is the least common multiple of all the common denominators of the matrix G(s). In the same way, let $H(s) = \dfrac{H'(s)}{h(s)}$ where $H'(s) \in \Re[s]^{m\times n}$ and $h(s)$ is the least common multiple of all the denominators of the matrix H(s). Then equation (4.12) may be rewritten as

$$\frac{G'(s)}{g(s)} F(s) \frac{H'(s)}{h(s)} = \frac{G'(s)}{g(s)} - \frac{H'(s)}{h(s)} \Leftrightarrow$$

$$G'(s)F(s)H'(s) = G'(s)h(s) - H'(s)g(s) \tag{4.13}$$

In the light of Theorem 4.2 we can easily obtain a necessary and sufficient condition for the existence of solution of the equation (4.13) by the following

THEOREM 4.8 The equation (4.13) has a solution iff

$$G'(s)G'(s)^{\dagger}\left[G'(s)h(s)-H'(s)g(s)\right]H'(s)^{\dagger}H'(s)=G'(s)h(s)-H'(s)g(s) \qquad (4.14)$$

in which case all the compensators are given by the following formula

$$F(s)=G'(s)^{\dagger}\left[G'(s)h(s)-H'(s)g(s)\right]H'(s)^{\dagger}+Y(s)-G'(s)^{\dagger}G'(s)Y(s)H'(s)H'(s)^{\dagger} \qquad (4.15)$$

where Y(s) is arbitrary to within having the dimension of F(s).

Proof. Let $P(s)=G'(s)$, $X(s)=F(s)$, $Q(s)=H'(s)$ and $C(s)=G'(s)h(s)-H'(s)g(s)$ in Theorem 4.2. Then the proof of Theorem 4.8 follows. •

EXAMPLE 4.9 Let an open loop system with transfer function matrix

$$G(s)=\begin{pmatrix} \frac{1}{s-1} & 0 & 0 \\ 0 & \frac{1}{s-2} & 0 \end{pmatrix} \in \Re(s)^{2\times 3} \qquad (E.1)$$

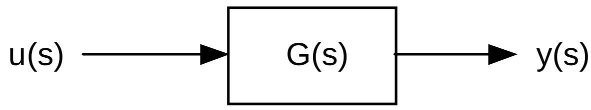

Diagram 3. Open loop system.

We would like to find out when there exist an output feedback of the form

$$u(s)=-F(s)y(s)+v(s) \text{ with } F(s)\in\Re(s)^{3\times 2} \qquad (E.2)$$

such that the closed loop system

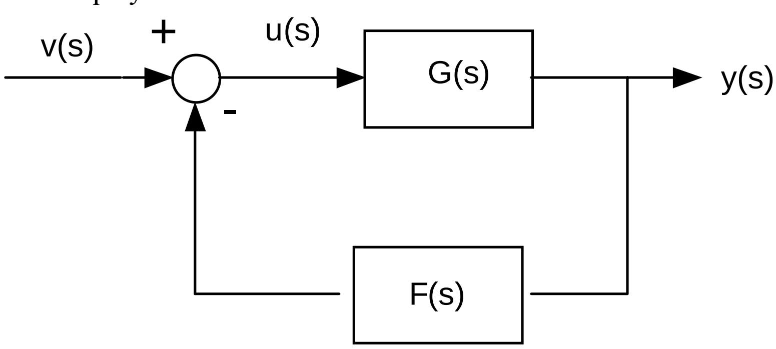

Diagram 4. Closed loop system.

has transfer function matrix

$$H(s)=\begin{pmatrix} 0 & \frac{1}{s+1} & 0 \\ \frac{1}{(s+1)^2} & 0 & 0 \end{pmatrix} \in \Re(s)^{2\times 3} \qquad (E.3)$$

It is easily seen that

$$G(s)=\frac{G^{'}(s)}{g(s)} \quad \text{and} \quad H(s)=\frac{H^{'}(s)}{h(s)} \tag{E.4}$$

where

$$G^{'}(s)=\begin{pmatrix} s-2 & 0 & 0 \\ 0 & s-1 & 0 \end{pmatrix}=\underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}}_{G_1^{'}}s+\underbrace{\begin{pmatrix} -2 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}}_{G_0^{'}} \tag{E.5a}$$

$$H^{'}(s)=\begin{pmatrix} 0 & s+1 & 0 \\ 1 & 0 & 0 \end{pmatrix}=\underbrace{\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{H_1^{'}}s+\underbrace{\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}}_{H_0^{'}} \tag{E.5b}$$

$$d(s)=(s-1)(s-2) \quad \text{and} \quad h(s)=(s+1)^2 \tag{E.5c}$$

It is easily seen (according to Algorithm 3.3) that the generalised inverses of $G^{'}(s)$ and $H^{'}(s)$ are the following

$$G^{'}(s)^{\dagger}=\begin{pmatrix} \frac{1}{s-2} & 0 \\ 0 & \frac{1}{s-1} \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad H^{'}(s)^{\dagger}=\begin{pmatrix} 0 & 1 \\ \frac{1}{s+1} & 0 \\ 0 & 0 \end{pmatrix} \tag{E.6}$$

(We are not interest for the values of $s\in\{1,-1,2\}$ because of the definition of G(s) and H(s)) A necessary and sufficient condition for the existence of F(s) according to Theorem 4.8 is that

$$G^{'}(s)G^{'}(s)^{\dagger}\left[G^{'}(s)h(s)-H^{'}(s)g(s)\right]H^{'}(s)^{\dagger}H^{'}(s)=G^{'}(s)h(s)-H^{'}(s)g(s) \tag{E.7}$$

which can be easily seen to be satisfied, in which case all the feedback compensators F(s) which gives rise to H(s) are given by the following formula

$$F(s)=G^{'}(s)^{\dagger}\left[G^{'}(s)h(s)-H^{'}(s)g(s)\right]H^{'}(s)^{\dagger}+Y(s)-G^{'}(s)^{\dagger}G^{'}(s)Y(s)H^{'}(s)H^{'}(s)^{\dagger}=$$

$$\begin{pmatrix} \frac{1}{s-2} & 0 \\ 0 & \frac{1}{s-1} \\ 0 & 0 \end{pmatrix}\left\{\begin{pmatrix} s-2 & 0 & 0 \\ 0 & s-1 & 0 \end{pmatrix}(s+1)^2-\begin{pmatrix} 0 & s+1 & 0 \\ 1 & 0 & 0 \end{pmatrix}(s-1)(s-2)\right\}\begin{pmatrix} 0 & 1 \\ \frac{1}{s+1} & 0 \\ 0 & 0 \end{pmatrix}+$$

$$+Y(s)-\begin{pmatrix} \frac{1}{s-2} & 0 \\ 0 & \frac{1}{s-1} \\ 0 & 0 \end{pmatrix}\begin{pmatrix} s-2 & 0 & 0 \\ 0 & s-1 & 0 \end{pmatrix}Y(s)\begin{pmatrix} 0 & s+1 & 0 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ \frac{1}{s+1} & 0 \\ 0 & 0 \end{pmatrix}=$$

$$=\begin{pmatrix} \frac{1}{s-2} & 0 \\ 0 & \frac{1}{s-1} \\ 0 & 0 \end{pmatrix}\begin{pmatrix} (s-2)(s+1)^2 & -(s+1)(s-1)(s-2) & 0 \\ -(s-1)(s-2) & (s-1)(s+1)^2 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ \frac{1}{s+1} & 0 \\ 0 & 0 \end{pmatrix}+$$

$$+Y(s)-\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}Y(s)I_2 \Leftrightarrow F(s)=\begin{pmatrix} -(s-1) & (s+1)^2 \\ (s+1) & -(s-2) \\ y_{31}(s) & y_{32}(s) \end{pmatrix} \quad \text{(E.8)}$$

where $y_{31}(s), y_{32}(s)$ are arbitrary functions of s. •

V. EVALUATION OF THE LAURENT EXPANSION.

It is easily seen from section III that the generalised inverse $A(s)^{\dagger}$ of $A(s)$ is unique and is given by the following formula

$$A(s)^{\dagger} = -a_k(s)^{-1}A(s)^T B_{k-1}(s) \quad (5.1)$$

In the sequel we shall compute the Laurent expansion of the generalised inverse $A(s)^{\dagger}$. From (5.1) we obtain that

$$a_k(s)A(s)^{\dagger} = -A(s)^T B_{k-1}(s) \quad (5.2)$$

Let

$$A(s)^{\dagger} = H_u s^u + H_{u-1}s^{u-1} + \cdots + H_1 s + H_0 + H_{-1}\frac{1}{s} + \cdots \quad (5.3)$$

Substituting $a_k(s), A(s)^{\dagger}, A(s)^T$ and $B_{k-1}(s)$ by (3.11), (5.3), (3.2) and (3.12) respectively in (5.2) we obtain

$$\left(\sum_{i=0}^{2kq}\hat{a}_{k,i}s^i\right)\left(\sum_{i=-\infty}^{u}H_i s^i\right) = -\left(\sum_{i=0}^{q}A_i^T\right)\left(\sum_{i=0}^{2(k-1)q}B_{k-1,i}s^i\right) \equiv$$
$$\equiv \sum_{i=0}^{(2k-1)q}\left(-\sum_{j=0}^{i}A_{i-j}^T B_{k-1,j}\right)s^i =: \sum_{i=0}^{(2k-1)q}B_i' s^i \quad (5.4)$$

Let also an integer f such that : $\hat{a}_{k,f} \neq 0$ and $\hat{a}_{k,i} = 0 \ \forall i > f$. Equating the coefficient matrices of each power of s in (5.4), we obtain the following relations :

$$\begin{aligned} &\hat{a}_{k,f}H_u = B_{f+u}' \\ &\hat{a}_{k,f-1}H_u + \hat{a}_{k,f}H_{u-1} = B_{f+u-1}' \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &\hat{a}_{k,0}H_0 + \hat{a}_{k,1}H_{-1} + \cdots + \hat{a}_{k,f}H_{-f} = B_0' \end{aligned} \quad (5.5)$$

and

$$\sum_{i=0}^{f} \hat{a}_{k,i} H_{-i-j} = 0 \quad \text{for } j = -1,-2,\ldots \tag{5.6}$$

Equations (5.5) may be rewritten in matrix form as

$$\underbrace{\begin{pmatrix} \hat{a}_{k,f}I_m & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \hat{a}_{k,f-1}I_m & \hat{a}_{k,f}I_m & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \hat{a}_{k,0}I_m & \hat{a}_{k,1}I_m & \cdots & \hat{a}_{k,f}I_m & 0 & \cdots & 0 \\ 0 & \hat{a}_{k,0}I_m & \cdots & \hat{a}_{k,f-1}I_m & \hat{a}_{k,f}I_m & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \hat{a}_{k,0}I_m & \hat{a}_{k,1}I_m & \cdots & \hat{a}_{k,f}I_m \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} H_u \\ H_{u-1} \\ \vdots \\ H_{u-f} \\ \vdots \\ H_{-f+1} \\ H_{-f} \end{pmatrix}}_{H} = \underbrace{\begin{pmatrix} B'_{f+u} \\ B'_{f+u-1} \\ \vdots \\ B'_u \\ \vdots \\ B'_1 \\ B'_0 \end{pmatrix}}_{B} \Leftrightarrow \tag{5.7}$$

$$A \times H = B$$

From our assumption that $\hat{a}_{k,f} \neq 0$ we conclude that the Toeplitz matrix A is always nonsingular. Hence we can find a unique solution $H = A^{-1}B$ that determines the first $u+f+1$ matrices $H_i,\ i = -f, -f+1, \ldots, u$. The inverse A^{-1} may be written in the form (Mertzios & Lewis 1989, Fragulis et. al. 1991) :

$$D = A^{-1} = \begin{pmatrix} d_0 I_m & 0 & \cdots & 0 \\ d_1 I_m & d_0 I_m & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ d_{f+u} I_m & d_{f+u-1} I_m & \cdots & d_0 I_m \end{pmatrix} \tag{5.8}$$

where

$$d_0 = \frac{1}{\hat{a}_{k,f}} \tag{5.9}$$

and

$$d_j = (-1)^j \frac{1}{\hat{a}_{k,f}} \det \begin{pmatrix} \hat{a}_{k,f-1} & \hat{a}_{k,f-2} & \cdots & \hat{a}_{k,f-j+1} & \hat{a}_{k,f-j} \\ \hat{a}_{k,f} & \hat{a}_{k,f-1} & \cdots & \hat{a}_{k,f-j} & \hat{a}_{k,f-j-1} \\ 0 & \hat{a}_{k,f} & \cdots & \hat{a}_{k,f-j-1} & \hat{a}_{k,f-j-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \hat{a}_{k,f} & \hat{a}_{k,f-1} \end{pmatrix} \tag{5.10a}$$

or equivalently

$$d_j = -\frac{1}{\hat{a}_{k,f}} \sum_{i=0}^{j-1} \left(\hat{a}_{k,f-j+i} d_i \right) \quad \text{for } j = 1,2,\ldots,f+u \tag{5.10b}$$

From (5.7) and (5.8) we obtain that

$$H = A^{-1}B = DB \tag{5.11a}$$

or equivalently

$$H_i = -\sum_{j=0}^{u-i}\left(d_j B'_{f+i+j}\right) \text{ for } i = -f, -f+1, \ldots, u \tag{5.11b}$$

which gives the first $f+u+1$ matrices H_i, $i = -f, -f+1, \ldots, u$. From equation (5.6) we obtain

$$H_{-i-f} = -\frac{1}{\hat{a}_{k,f}}\left(\sum_{j=0}^{f-1}\left(\hat{a}_{k,j} H_{-i-j}\right)\right) \text{ for } i = -1, -2, \ldots \tag{5.12}$$

This equation gives rise to the values of the rest matrices H_i, $i = -f-1, -f-2\ldots, -\infty$. The whole theory described above for the computation of the generalised inverse of A(s) is summarized in the following

ALGORITHM 5.1 (Computation of the Laurent expansion of generalised inverse $A(s)^{\dagger}$ of A(s))

***Step* 1.** Compute $f := \deg[a_k(s)]$.

***Step* 2.** $d_0 = \dfrac{1}{\hat{a}_{k,f}}$

For i=1 to f+u

$$d_i = -\frac{1}{\hat{a}_{k,f}}\sum_{j=0}^{i-1}\left(\hat{a}_{k,f-i+j} d_j\right)$$

Next i

***Step* 3.** For i=-f to u

$$H_i = -\sum_{j=0}^{u-i}\left(d_j B'_{f+i+j}\right)$$

Next i

***Step* 4.** For i=-1 to -∞ step -1

$$H_{-i-f} = -\frac{1}{\hat{a}_{k,f}}\left(\sum_{j=0}^{f-1}\left(\hat{a}_{k,j} H_{-i-j}\right)\right)$$

Next i

OUTPUT $A(s)^{\dagger} = \sum_{i=-\infty}^{u} H_i s^i$ •

VI. CONCLUSIONS

A two-dimension recursive algorithm is determined for the computation of the generalised inverse of a polynomial matrix $A(s) = A_q s^q + \cdots + A_1 s + A_0 \in \Re[s]^{n\times m}$ in terms of the coefficient matrices $A_i \in \Re^{n\times m}$ and its Laurent expansion has also been evaluated. The whole theory has been illustrated via three implications in Linear System Theory. The above results may also be extended to the multivariable polynomial matrices using the same framework with this paper (see Karampetakis 1995) with analogous implications in analysis of multidimensional systems.

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