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Generalized Inverses of Two-Variable Polynomial Matrices and Applications*

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Abstract. The main contribution of this paper is to present (a) an algorithm for the computation of the generalized inverse of a not necessarily square two-variable polynomial matrix and (b) some applications of the proposed algorithm to the solution of Diophantine equations.

1. Introduction

Consider the 2D polynomial matrix:

$$A(z_1, z_2) = \sum_{i=0}^{q_1} \sum_{j=0}^{q_2} A_{i,j} z_1^i z_2^j \in \Re[z_1, z_2]^{n\times m} \tag{1.1}$$

where $A_{i,j} \in \Re^{n\times m}$, $i = 0, 1, \ldots, q_1$, and $j = 0, 1, \ldots, q_2$. The problem of the investigation of the generalized inverse of a polynomial matrix with one or more variables has been the concern of many scientists [3], [4], [6]–[10], [17]–[22] because of its many implications [1], [8], [9], [13]–[16] in multivariable system analysis.

As concerns the inverse of regular polynomial matrices, i.e., $n = m$, in the case where $z_2 = 0$, $q_1 = 0$, and $\det[A_{0,0}] \neq 0$, this problem has been investigated in [4] and [24]. In cases where ($z_2 = 0, q_1 = 1$, and $\det[A_{0,0} + A_{1,0}z_1] \neq 0$) and ($z_2 = 0, q_1 \in N$, and $\det[A(z_1, 0)] \neq 0$), a solution has been proposed in [17a] and [6], respectively. Regarding the inverse of a regular polynomial matrix with two variables, explicit expressions have been given in [9], [12], [17b], [18]–[20].

In the nonregular case, i.e., $n \neq m$ or $n = m$ with $\det[A(z_1, z_2)] = 0$, the generalized inverse of the constant complex matrices was defined in [21], and a numerical

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algorithm for the computation of this matrix was later given in [3]. A recent approach to the investigation of the generalized inverse for one-variable polynomial matrices with real coefficient matrices has been proposed in [8], whereas a more extended theoretical approach to matrix-valued functions has been proposed in [22]. The extension of these attempts to two-variable polynomial matrices with complex coefficient matrices is proposed in this paper.

The interest of this important problem stems from its numerous applications, e.g., computation of the transfer function matrix of a system [6], [9], [17]–[20], inverse systems [13]–[16], solution of systems [1], [8], controllability and observability matrices of general polynomial matrix descriptions (PMDs) [5], solutions of Diophantine equations [8] which give rise to numerous applications [11], and the causal factorization problem [23].

This paper is divided into four sections. Section 2 contains some preliminary results concerning the definition of the generalized inverse. In Section 3 we present an algorithm for the computation of the generalized inverse of the matrix (1.1). In Section 4 this algorithm is applied to the solution of (a) the general Diophantine equation and (b) the model matching problem. The whole theory is illustrated via an example in Section 5. Section 6 presents conclusions.

2. Preliminary results

The definition of the generalized inverse of a constant matrix was originally proposed by Penrose in (1955) [21]:

Definition 1. For every matrix $A \in C^{n\times m}$, a unique matrix $A^+ \in C^{m\times n}$ exists that is called the *generalized inverse* and satisfies the following conditions:

(i) $AA^+A = A$,

(ii) $A^+AA^+ = A^+$,

(iii) $(AA^+)^* = AA^+$,

(iv) $(A^+A)^* = A^+A$

where A^* denotes the conjugate transpose of the matrix A (or simply its transpose if A is real). In the special case where the matrix A is a square nonsingular matrix, the generalized inverse of A is simply its inverse, i.e., $A^\dagger = A^{-1}$. In cases where there is a matrix A^{g_1} that satisfies only the first condition, this matrix is called a {1}-inverse. {1}-inverses are not unique and they play an important role in the solution of Diophantine equations, as we shall see in Section 4.

In an analogous way we define the generalized inverse of a polynomial matrix $A(z_1, z_2) \in C[z_1, z_2]^{n\times m}$ to be a generalized inverse $A(z_1, z_2)^+ \in C(z_1, z_2)^{m\times n}$ such that $A(z_1, z_2)^+ \in C(z_1, z_2)^{m\times n}$ is a generalized inverse of $A(z_1, z_2) \in C[z_1, z_2]^{n\times m}$ for each $(z_1, z_2) \in C^2$. The uniqueness of this matrix is obvious because the proof of the uniqueness in [21] is independent of the form of the matrix $A(z_1, z_2) \in C[z_1, z_2]^{m\times n}$. Consider now the following:

Theorem 2. [3] *Let* $A \in C^{n\times m}$ *and*

$$a(s) := \det[sI_n - A\times A^*] = (-1)^n[a_0 s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n] \quad \text{with } a_0 = 1 \tag{2.1}$$

be the characteristic polynomial of $(A \times A^*)$. *If* $k \neq 0$ *is the largest integer such that* $a_k \neq 0$, *then the generalized inverse of* A *is given by*

$$A^+ = -a_k^{-1}A^*[(A \times A^*)^{k-1} + a_1(A \times A^*)^{k-2} + \cdots + a_{k-1}I_n] \,. \tag{2.2}$$

If $k = 0$ *is the largest integer such that* $a_k \neq 0$, *then* $A^+ = 0$.

A numerical algorithm for the implementation of this theorem given by [3] is the following:

Algorithm 3 (Computation of the generalized inverse of a constant matrix $A \in C^{n\times m}$)

Step 1. Let $A \in C^{n\times m}$. Consider the sequences $\{a_0, a_1, \ldots, a_n\}$, $\{B_0, B_1, \ldots, B_n\}$ constructed in the following way:

$$\begin{array}{lll}
A_0 = 0 & 1 = a_0 & B_0 = I_n \\
A_1 = (A \times A^*)B_0 & -\frac{\text{trace}(A_1)}{1} = a_1 & B_1 = A_1 + a_1 I_n \\
A_2 = (A \times A^*)B_1 & -\frac{\text{trace}(A_2)}{2} = a_2 & B_2 = A_2 + a_2 I_n \\
\vdots & \vdots & \vdots \\
A_{n-1} = (A \times A^*)B_{n-2} & -\frac{\text{trace}(A_{n-1})}{n-1} = a_{n-1} & B_{n-1} = A_{n-1} + a_{n-1} I_n \\
A_n = (A \times A^*)B_{n-1} & -\frac{\text{trace}(A_n)}{n} = a_n & B_n = A_n + a_n I_n
\end{array}$$

Step 2. If $k \neq 0$ is the largest integer such that $a_k \neq 0$, then the generalized inverse of A is given by

$$A^+ = -a_k^{-1}A^*[(A\times A^*)^{k-1} + a_1(A\times A^*)^{k-2} + \cdots + a_{k-1}I_n] = -a_k^{-1}A^* B_{k-1} \tag{2.4}$$

else ($k = 0$ is the largest integer such that $a_k \neq 0$) $A^+ = 0$.

3. Generalized inverse of a two-variable polynomial matrix

Consider now the two-variable polynomial matrix

$$A(z_1, z_2) = \sum_{i=0}^{q_1}\sum_{j=0}^{q_2} A_{i,j} z_1^i z_2^j \in C[z_1, z_2]^{n\times m} \tag{3.1}$$

and its conjugate transpose

$$A(z_1, z_2)^* = \sum_{i=0}^{q_1}\sum_{j=0}^{q_2} A_{i,j}^* \bar{z}_1^i \bar{z}_2^j \in C[z_1, z_2]^{n\times m} . \tag{3.2}$$

Following similar lines to Theorem 2 we can easily show the following:

Theorem 4. *Let* $A(z_1, z_2) \in C[z_1, z_2]^{n\times m}$ *and*

$$\begin{aligned} a(s, z_1, z_2) :=& \det[sI_n - A(z_1, z_2) \times A(z_1, z_2)^*] \\ =&(-1)^n[a_0(z_1, z_2)s^n + a_1(z_1, z_2)s^{n-1} + \cdots + a_n(z_1, z_2)] \\ & \text{with } a_0 = 1 \end{aligned} \tag{3.3}$$

be the characteristic polynomial of $(A(z_1, z_2) \times A(z_1, z_2)^*)$. *If* $k \neq 0$ *is the largest integer such that* $a_k(z_1, z_2) \neq 0$ *for some* $(z_1, z_2) \in L(\neq \{\emptyset\}) \subseteq C^2$, *then the generalized inverse of* $A(z_1, z_2)$ *for those* $(z_1, z_2) \in L(\neq \{\emptyset\}) \subseteq C^2$ *is given by*

$$\begin{aligned} A(z_1, z_2)^+ =& -a_k(z_1, z_2)^{-1}A(z_1, z_2)^* \\ & \times \left[\left(A(z_1, z_2) \times A(z_1, z_2)^*\right)^{k-1} + \cdots + a_{k-1}(z_1, z_2)I_n\right] . \end{aligned} \tag{3.4}$$

If $k = 0$ *is the largest integer such that* $a_k(z_1, z_2) \neq 0$, *then* $A(z_1, z_2)^+ = 0$. *For those* $(z_1, z_2) \in C^2 - L$ *we use the same algorithm.*

Proof. It can be readily seen that the proof is the same as that of Theorem 2 because it is independent of the variables of the matrix $A(z_1, z_2)$. □

An algorithm, very useful for symbolic computational packages, for the implementation of Theorem 4 is now given. It is similar to Algorithm 3 (the proofs are exactly the same).

Algorithm 5 (Computation of the generalized inverse of $A(z_1, z_2) \in C[z_1, z_2]^{n\times m}$)

Step 1. Consider the sequences $\{a_0(z_1, z_2), a_1(z_1, z_2), \ldots, a_n(z_1, z_2)\}$, $\{B_0(z_1, z_2), B_1(z_1, z_2), \ldots, B_n(z_1, z_2)\}$ constructed in the following way:

$$\begin{array}{lr} A_0(z_1, z_2) = 0 & 1 = a_0(z_1, z_2) \\ A_1(z_1, z_2) = (A(z_1, z_2) \times A(z_1, z_2)^*)B_0(z_1, z_2) & -\frac{\operatorname{trace}(A_1(z_1,z_2))}{1} = a_1(z_1, z_2) \\ A_2(z_1, z_2) = (A(z_1, z_2) \times A(z_1, z_2)^*)\, B_1(z_1, z_2) & -\frac{\operatorname{trace}(A_2(z_1,z_2))}{2} = a_2(z_1, z_2) \\ \vdots & \vdots \\ A_n(z_1, z_2) = (A(z_1, z_2) \times A(z_1, z_2)^*)\, B_{n-1}(z_1, z_2) & -\frac{\operatorname{trace}(A_n(z_1,z_2))}{n} = a_n(z_1, z_2) \end{array}$$

$$\begin{aligned} B_0(z_1, z_2) &= I_n \\ B_1(z_1, z_2) &= A_1(z_1, z_2) + a_1(z_1, z_2)I_n \\ B_2(z_1, z_2) &= A_2(z_1, z_2) + a_2(z_1, z_2)I_n \\ &\vdots \\ B_n(z_1, z_2) &= A_n(z_1, z_2) + a_n(z_1, z_2)I_n . \end{aligned} \tag{3.5}$$

Step 2. If $k \neq 0$ is the largest integer such that $a_k(z_1, z_2) \neq 0$ for $(z_1, z_2) \in L(\neq \{\emptyset\}) \subseteq C^2$, then the generalized inverse of $A(z_1, z_2)$ for those $(z_1, z_2) \in L(\neq \{\emptyset\}) \subseteq C^2$ is given by

$$A(z_1, z_2)^+ = -a_k(z_1, z_2)^{-1} A(z_1, z_2)^* B_{k-1}(z_1, z_2) \tag{3.6}$$

else ($k = 0$ is the largest integer such that $a_k(z_1, z_2) \neq 0$) $A(z_1, z_2)^+ = 0$. For those $(z_1, z_2) \in C^2 - L$ we use the same algorithm.

The same algorithm holds when $A(z_1, z_2)$ is rational, polynomial, or indeed constant. It also holds for n-dimensional matrices of the form $A(z_1, z_2, \ldots, z_n) \in C(z_1, z_2, \ldots, z_n)^{n\times m}$. This special property of the algorithm, i.e., the fact that only one computer procedure is required, has allowed us to write an algorithm in the symbolic language of Maple [7] for the investigation of the generalized inverse of any n-dimensional matrix. Several remarks concerning the algorithm and its computational implication in Maple (see also [7]) can be made:

(1) It is a simple recursion in which each of the three sequences is added to at each step. Hence it is very *computationally attractive.*

(2) The storage requirement can be significantly reduced in two ways. First, the sequence $\{A_i(z_1, z_2)\}$, $i = 0, 1, \ldots, n$, is not required in the final result and therefore we can update the sequence at *each* successive step. Second, we only need to store the last *nonzero* $a_i(z_1, z_2)$ term. These savings will become more evident as the row order, n, of $A(z_1, z_2)$ is increased.

(3) There is *no matrix inversion* required, and therefore the algorithm can be considered stable in this respect.

(4) The dimensions of the matrix sequences involved remain *fixed* throughout the algorithm and do not increase at any step.

(5) The matrix $A(z_1, z_2)A(z_1, z_2)^*$ iN (3.5) is constant $\forall i \geq 1$ and subsequently only needs to be calculated once (i.e., outside the loop).

(6) For $n > m$ we can form the conjugate transpose of $A(z_1, z_2)$, denoted by $A(z_1, z_2)^*$, and compute the generalized inverse of this matrix instead. Then $A(z_1, z_2)^+ = [A^*(z_1, z_2)^+]^*$. This will in general be computed more quickly, as the algorithm will now be one of m rather than n steps.

Example 6. Let

$$A(z_1, z_2) := \begin{pmatrix} 1 & z_1 & 0 \\ 0 & 1 & z_2 \end{pmatrix}$$

be the polynomial matrix whose generalized inverse we seek. Then applying Algorithm 5, we have:

Step 1. Consider the sequences $\{a_0(z_1, z_2), a_1(z_1, z_2), a_2(z_1, z_2)\}$, $\{B_0(z_1, z_2), B_1(z_1, z_2), B_2(z_1, z_2)\}$ constructed in the following way:

$$A_0 = 0_{2,2}$$

$$A_1 = (A \times A^*) \times B_0 = \begin{pmatrix} 1 & z_1 & 0 \\ 0 & 1 & z_2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \overline{z}_1 & 1 \\ 0 & \overline{z}_2 \end{pmatrix} I_2 = \begin{pmatrix} 1+|z_1|^2 & z_1 \\ \overline{z}_1 & 1+|z_2|^2 \end{pmatrix}$$

$$A_2 = (A \times A^*) \times B_1 = \begin{pmatrix} 1+|z_1|^2 & z_1 \\ \overline{z}_1 & 1+|z_2|^2 \end{pmatrix} \begin{pmatrix} -1-|z_2|^2 & z_1 \\ \overline{z}_1 & -1-|z_1|^2 \end{pmatrix}$$
$$= \begin{pmatrix} -1-|z_2|^2-|z_1|^2|z_2|^2 & 0 \\ 0 & -1-|z_2|^2-|z_1|^2|z_2|^2 \end{pmatrix}$$

$$a_0 = 1$$

$$a_1 = -\text{trace}\frac{A_1}{1} = -\text{trace}\begin{pmatrix} 1+|z_1|^2 & z_1 \\ \overline{z}_1 & 1+|z_2|^2 \end{pmatrix} = -2-|z_1|^2-|z_2|^2$$

$$a_2 = -\text{trace}\frac{A_2}{2} = -\frac{1}{2}\text{trace}\begin{pmatrix} -1-|z_2|^2-|z_1|^2|z_2|^2 & 0 \\ 0 & -1-|z_2|^2-|z_1|^2|z_2|^2 \end{pmatrix}$$
$$= 1+|z_2|^2+|z_1|^2|z_2|^2$$

$$B_0 = I_2$$

$$B_1 = A_1 + a_1 I_2 = \begin{pmatrix} 1+|z_1|^2 & z_1 \\ \overline{z}_1 & 1+|z_2|^2 \end{pmatrix} + \left(-2-|z_1|^2-|z_2|^2\right) \times I_2$$
$$= \begin{pmatrix} -1-|z_2|^2 & z_1 \\ \overline{z}_1 & -1-|z_1|^2 \end{pmatrix}$$

$$B_2 = A_2 + a_2 I_2 = 0$$

Step 2. We observe that $a_2(z_1, z_2) \neq 0\ \forall (z_1, z_2) \in C^2$, and therefore

$$A(z_1, z_2)^+ = -a_2(z_1, z_2)^{-1} A(z_1, z_2)^* B_1(z_1, z_2)$$
$$= -\frac{1}{1+|z_2|^2+|z_1|^2|z_2|^2} \begin{pmatrix} 1 & 0 \\ \overline{z}_1 & 1 \\ 0 & \overline{z}_2 \end{pmatrix} \begin{pmatrix} -1-|z_2|^2 & z_1 \\ \overline{z}_1 & -1-|z_1|^2 \end{pmatrix}$$
$$= \frac{1}{1+|z_2|^2+|z_1|^2|z_2|^2} \begin{pmatrix} 1+|z_2|^2 & -z_1 \\ \overline{z}_1|z_2|^2 & 1 \\ -\overline{z}_1\overline{z}_2 & \overline{z}_2(1+|z_1|^2) \end{pmatrix}.$$

Now by setting $z_3 = \overline{z}_1$ and $z_4 = \overline{z}_2$, we can rewrite $A(z_1, z_2)$ and $A(z_1, z_2)^*$ as

$$A(z_1, z_2) = \sum_{j_1=0}^{q_1} \sum_{j_2=0}^{q_2} \sum_{j_3=0}^{0} \sum_{j_4=0}^{0} A_{j_1,j_2,j_3,j_4} z_1^{j_1} z_2^{j_2} z_3^{j_3} z_4^{j_4} \ \text{with} \tag{3.7a}$$
$$A_{j_1,j_2,0,0} \equiv A_{j_1,j_2} \ \text{and} \ A_{j_1,j_2,j_3,j_4} \equiv 0\ \forall (j_3, j_4) \neq (0,0)$$

$$A(z_1, z_2)^* = \sum_{j_1=0}^{0}\sum_{j_2=0}^{0}\sum_{j_3=0}^{q_1}\sum_{j_4=0}^{q_2} A^*_{j_1,j_2,j_3,j_4} z_1^{j_1} z_2^{j_2} z_3^{j_3} z_4^{j_4} \quad \text{with}$$
$$A^*_{0,0,j_3,j_4} \equiv A^*_{j_3,j_4} \quad \text{and} \quad A^*_{j_1,j_2,j_3,j_4} \equiv 0 \ \forall (j_1, j_2) \neq (0,0) \tag{3.7b}$$

and $A(z_1, z_2)A(z_1, z_2)^*$ as

$$\begin{aligned} &A(z_1, z_2)A(z_1, z_2)^* \\ &= \sum_{j_1=0}^{q_1}\sum_{j_2=0}^{q_2}\sum_{j_3=0}^{q_1}\sum_{j_4=0}^{q_2}\left(\sum_{n_1=0}^{j_1}\sum_{n_2=0}^{j_2}\sum_{n_3=0}^{j_3}\sum_{n_4=0}^{j_4} A_{j_1-n_1,j_2-n_2,j_3-n_3,j_4-n_4} A^*_{n_1,n_2,n_3,n_4}\right) \\ &\quad \times z_1^{j_1} z_2^{j_2} z_3^{j_3} z_4^{j_4} \\ &=: \sum_{j_1=0}^{q_1}\sum_{j_2=0}^{q_2}\sum_{j_3=0}^{q_1}\sum_{j_4=0}^{q_2} C_{j_1,j_2,j_3,j_4} z_1^{j_1} z_2^{j_2} z_3^{j_3} z_4^{j_4}\,. \end{aligned} \tag{3.8}$$

It is easily seen from (3.5) that $a_i(z_1, z_2)$ and $B_i(z_1, z_2)$ may be rewritten as

$$a_i(z_1, z_2) = \sum_{j_1=0}^{iq_1}\sum_{j_2=0}^{iq_2}\sum_{j_3=0}^{iq_1}\sum_{j_4=0}^{iq_2} \hat{a}_{i,j_1,j_2,j_3,j_4} z_1^{j_1} z_2^{j_2} z_3^{j_3} z_4^{j_4}\,, \quad i = 0, 1, \ldots, n \tag{3.9}$$

and

$$B_i(z_1, z_2) = \sum_{j_1=0}^{iq_1}\sum_{j_2=0}^{iq_2}\sum_{j_3=0}^{iq_1}\sum_{j_4=0}^{iq_2} \hat{B}_{i,j_1,j_2,j_3,j_4} z_1^{j_1} z_2^{j_2} z_3^{j_3} z_4^{j_4}\,, \quad i = 0, 1, \ldots, n-1 \tag{3.10}$$

where $\hat{B}_{i,j_1,j_2,j_3,j_4}$, $\hat{a}_{i,j_1,j_2,j_3,j_4}$ are constant coefficient matrices and scalars of the powers $z_1^{j_1} z_2^{j_2} z_3^{j_3} z_4^{j_4}$. It is seen from Algorithm 5 that for the computation of the generalized inverse of $A(z_1, z_2)$ we need the integer k, and the quantities $a_k(z_1, z_2)$ and $B_{k-1}(z_1, z_2)$, i.e., the coefficients $\hat{a}_{k,j_1,j_2,j_3,j_4}$ and the coefficient matrices $\hat{B}_{k-1,j_1,j_2,j_3,j_4}$ defined by

$$a_k(z_1, z_2) = \sum_{j_1=0}^{kq_1}\sum_{j_2=0}^{kq_2}\sum_{j_3=0}^{kq_1}\sum_{j_4=0}^{kq_2} \hat{a}_{k,j_1,j_2,j_3,j_4} z_1^{j_1} z_2^{j_2} z_3^{j_3} z_4^{j_4} \tag{3.11}$$

and

$$B_{k-1}(z_1, z_2) = \sum_{j_1=0}^{(k-1)q_1}\sum_{j_2=0}^{(k-1)q_2}\sum_{j_3=0}^{(k-1)q_1}\sum_{j_4=0}^{(k-1)q_2} \hat{B}_{k-1,j_1,j_2,j_3,j_4} z_1^{j_1} z_2^{j_2} z_3^{j_3} z_4^{j_4}\,. \tag{3.12}$$

Now taking into account that

$$\begin{aligned}
&A(z_1, z_2)A(z_1, z_2)^* B_i(z_1, z_2)\\
&= \left(\sum_{j_1=0}^{q_1}\sum_{j_2=0}^{q_2}\sum_{j_3=0}^{q_1}\sum_{j_4=0}^{q_2} C_{j_1,j_2,j_3,j_4} z_1^{j_1} z_2^{j_2} z_1^{j_3} z_2^{j_4}\right)\\
&\quad\times \left(\sum_{j_1=0}^{iq_1}\sum_{j_2=0}^{iq_2}\sum_{j_3=0}^{iq_1}\sum_{j_4=0}^{iq_2} \hat{B}_{i,j_1,j_2,j_3,j_4} z_1^{j_1} z_2^{j_2} z_1^{j_3} z_2^{j_4}\right)\\
&= \sum_{j_1=0}^{(i+1)q_1}\sum_{j_2=0}^{(i+1)q_2}\sum_{j_3=0}^{(i+1)q_1}\sum_{j_4=0}^{(i+1)q_2}\\
&\quad\times\left(\sum_{n_1=0}^{j_1}\sum_{n_2=0}^{j_2}\sum_{n_3=0}^{j_3}\sum_{n_4=0}^{j_4} C_{j_1-n_1,j_2-n_2,j_3-n_3,j_4-n_4}\hat{B}_{i,n_1,n_2,n_3,n_4}\right) z_1^{j_1} z_2^{j_2} z_3^{j_3} z_4^{j_4}
\end{aligned} \tag{3.13}$$

and substituting (3.9), (3.10), and (3.13) in the recursive relations (3.5), we obtain the following recursive algorithm, which determines $\hat{a}_{i+1,j_1,j_2,j_3,j_4}$ and $\hat{B}_{i+1,j_1,j_2,j_3,j_4}$ for $j_z = 0, 1, \ldots, (i+1)q_z$ with $z = 1, 2$ and $j_z = 0, 1, \ldots, (i+1)q_{z-2}$ with $z = 3, 4$.

Algorithm 7 (Generalized inverse $A(z_1, z_2)^+$ of $A(z_1, z_2)$)

Initial conditions

$$\hat{B}_{0,0,0,0,0} = I_n$$
$$A_{j_1,j_2,0,0} = A_{j_1,j_2} \text{ and } A^*_{0,0,j_3,j_4} = A^*_{j_3,j_4}. \tag{3.14}$$

Boundary conditions

$$\hat{B}_{0,j_1,j_2,j_3,j_4} = 0 \ \forall j_z > 0, \quad z = 1, 2, 3, 4$$
$$B_{i,j_1,j_2,j_3,j_4} = 0, \quad j_z = iq_z + 1, iq_z + 2, \ldots, (n-1)q_z, \quad z = 1, 2$$
$$B_{i,j_1,j_2,j_3,j_4} = 0, \ j_z = iq_{z-2} + 1, iq_{z-2} + 2, \ldots, (n-1)q_{z-2}, \quad z = 3, 4$$
$$A_{j_1,j_2,j_3,j_4} = 0 \ \forall (j_3, j_4) \neq (0,0) \text{ and } A^*_{j_1,j_2,j_3,j_4} = 0 \ \forall (j_1, j_2) \neq (0,0)$$
$$C_{j_1,j_2,j_3,j_4} = \sum_{j_1=0}^{n_1}\sum_{j_2=0}^{n_2}\sum_{j_3=0}^{n_3}\sum_{j_4=0}^{n_4} A_{j_1-n_1,j_2-n_2,j_3-n_3,j_4-n_4} A^*_{n_1,n_2,n_3,n_4} \tag{3.15}$$

Recursive relations for $\boxed{a_i(z_1, z_2)}$

$$\begin{aligned}
&\hat{a}_{i+1,j_1,j_2,j_3,j_4}\\
&= -\frac{1}{i+1}\operatorname{trace}\left\{\sum_{n_1=0}^{j_1}\sum_{n_2=0}^{j_2}\sum_{n_3=0}^{j_3}\sum_{n_4=0}^{j_4} C_{j_1-n_1,j_2-n_2,j_3-n_3,j_4-n_4}\hat{B}_{i,n_1,n_2,n_3,n_4}\right\}
\end{aligned}$$

$$j_z = 0, 1, \ldots, (i+1)q_z, \quad z = 1, 2 \text{ and } j_z = 0, 1, \ldots, (i+1)q_{z-2}, \quad z = 3, 4$$
$$i = 0, 1, \ldots, n-1 \tag{3.16}$$

Recursive relations for $\boxed{B_i(z_1, z_2)}$

$$\hat{B}_{i+1,j_1,j_2,j_3,j_4} = \sum_{n_1=0}^{j_1}\sum_{n_2=0}^{j_2}\sum_{n_3=0}^{j_3}\sum_{n_4=0}^{j_4} C_{j_1-n_1,j_2-n_2,j_3-n_3,j_4-n_4}\hat{B}_{i,n_1,n_2,n_3,n_4} + \hat{a}_{i+1,j_1,j_2,j_3,j_4}I_n$$

$$j_z = 0, 1, \ldots, (i+1)q_z, \quad z = 1, 2 \text{ and } j_z = 0, 1, \ldots, (i+1)q_{z-2}, \quad z = 3, 4 \tag{3.17}$$

$$i = 0, 1, \ldots, n-2$$

Terminate

$$\text{FIND } k : a_{k+1}(z_1, z_2) = a_{k+2}(z_1, z_2) = \cdots = a_n(z_1, z_2) = 0$$

$$\text{or } \hat{a}_{k+1,j_1,j_2,j_3,j_4} = \hat{a}_{k+2,j_1,j_2,j_3,j_4} = \cdots = \hat{a}_{n,j_1,j_2,j_3,j_4} = 0 \ \forall j_z \in N \tag{3.18a}$$

then

$$B_{j_1,j_2,j_3,j_4} := \hat{B}_{k-1,j_1,j_2,j_3,j_4}, \quad j_z = 0, 1, \ldots, (k-1)q_z, \quad z = 1, 2$$
$$\text{and } j_z = 0, 1, \ldots, (k-1)q_{z-2}, \quad z = 3, 4$$

$$a_{j_1,j_2,j_3,j_4} := \hat{a}_{k,j_1,j_2,j_3,j_4}, \quad j_z = 0, 1, \ldots, kq_z, \quad z = 1, 2$$
$$\text{and } j_z = 0, 1, \ldots, kq_{z-2}, \quad z = 3, 4 \tag{3.18b}$$

Output

$$\begin{aligned} A(z_1, z_2)^+ = &-\left(\sum_{j_1=0}^{kq_1}\sum_{j_2=0}^{kq_2}\sum_{j_3=0}^{kq_1}\sum_{j_4=0}^{kq_2} a_{j_1,j_2,j_3,j_4} z_1^{j_1} z_2^{j_2} z_3^{j_3} z_4^{j_4}\right)^{-1} \\ &\times \left(\sum_{j_1=0}^{0}\sum_{j_2=0}^{0}\sum_{j_3=0}^{q_1}\sum_{j_4=0}^{q_2} A^*_{j_1,j_2,j_3,j_4} z_1^{j_1} z_2^{j_2} z_3^{j_3} z_4^{j_4}\right) \\ &\times \left(\sum_{j_1=0}^{(k-1)q_1}\sum_{j_2=0}^{(k-1)q_2}\sum_{j_3=0}^{(k-1)q_1}\sum_{j_4=0}^{(k-1)q_2} B_{j_1,j_2,j_3,j_4} z_1^{j_1} z_2^{j_2} z_3^{j_3} z_4^{j_4}\right) \end{aligned} \tag{3.19}$$

If $j_3 > j_1$ then
  substitute $z_1^{j_1} z_3^{j_3}$ in (3.19) for $|z_1|^{j_1} \bar{z}_1^{j_3-j_1}$
else
  substitute $z_1^{j_1} z_3^{j_3}$ in (3.19) for $|z_1|^{j_3} z_1^{j_1-j_3}$
endif

If $j_4 > j_2$ then
  substitute $z_2^{j_2} z_4^{j_4}$ in (3.19) for $|z_2|^{j_2} \bar{z}_2^{j_4-j_2}$
else
  substitute $z_2^{j_2} z_4^{j_4}$ in (3.19) for $|z_2|^{j_4} z_2^{j_2-j_4}$
endif

It is readily seen that the generalized inversion algorithm is a five-dimensional algorithm because it depends on five independent variables i, j_1, j_2, j_3, j_4 and can be easily implemented by the FORTRAN 77 programming language, which supports complex number operations. Note also that (a) we use the same algorithm for those $(z_1, z_2) \in C^2 - L$, i.e., $a_k(z_1, z_2) = 0$, by finding another i such that $a_i(z_1, z_2) \neq 0$ for those $(z_1, z_2) \in C^2 - L$, and (b) in the case where $k = 0$ is the largest integer such that $a_k(z_1, z_2) \neq 0$, then $A(z_1, z_2)^+ = 0$. The main disadvantage of the preceding five-dimensional algorithm is that it is numerically unstable because it is implicitly taking powers of matrices (a major handicap in all Cayley-Hamilton-theorem-based techniques). Therefore this algorithm works well for textbook-type examples, but if the data is poorly conditioned, the results can be significantly incorrect. However, we have overcome this disadvantage by using symbolic computational packages like Maple [7], where variables can be stored in ≪exact≫ form (i.e., 1/3 as opposed to 0.3333..) resulting in no loss of accuracy during a calculation.

4. Implications of the generalized inverse in linear system theory

One of the important applications of the computation of the generalized inverse is the solution of the equation $AXB = C$, presented in [21]. The preceding problem may be extended to the case of two-variable polynomial matrices, as we shall see later in this section.

Theorem 8. [21] *A necessary and sufficient condition for the matrix equation $AXB = C$ to have a solution is that $AA^+CB^+B = C$, in which case the general solution is*

$$X = A^+CB^+ + Y - A^+AYBB^+ \tag{4.1}$$

where A^+, B^+ are the generalized inverses of A and B, respectively, and Y is arbitrary to within having the dimension of X.

It can be readily seen from [21] that the proof of Theorem 8 is independent of the variable existence in the matrix A, and thus we state the following theorem:

Theorem 9. *A necessary and sufficient condition for the matrix equation*

$$A(z_1, z_2)X(z_1, z_2)B(z_1, z_2) = C(z_1, z_2)$$

to have a solution is that

$$A(z_1, z_2)A(z_1, z_2)^+C(z_1, z_2)B(z_1, z_2)^+B(z_1, z_2) = C(z_1, z_2)\ ,$$

in which case the general solution is

$$\begin{aligned} X(z_1, z_2) =& A(z_1, z_2)^+C(z_1, z_2)B(z_1, z_2)^+ + Y(z_1, z_2) \\ &- A(z_1, z_2)^+A(z_1, z_2)Y(z_1, z_2)B(z_1, z_2)B(z_1, z_2)^+ \end{aligned} \tag{4.2}$$

where $A(z_1,z_2)^+$, $B(z_1,z_2)^+$ are the generalized inverses of $A(z_1,z_2)$ and $B(z_1,z_2)$ respectively, and $Y(z_1,z_2)$ is arbitrary to within having the dimension of $X(z_1,z_2)$.

We have to note here that Theorem 9 remains the same if we substitute the generalized inverses of $A(z_1,z_2)^+$, $B(z_1,z_2)^+$ for the $\{1\}$-inverses of $A(z_1,z_2)$ and $B(z_1,z_2)$.

An interesting application of Theorem 9 is the investigation of the solution space of the Diophantine equation

$$A(z_1,z_2)X(z_2,z_2)+B(z_1,z_2)Y(z_1,z_2)=C(z_1,z_2) \tag{4.3}$$

where $A(z_1,z_2)\in C[z_1,z_2]^{p_1\times n_1}$, $B(z_1,z_2)\in C[z_1,z_2]^{p_1\times n_2}$, and $C(z_1,z_2)\in C[z_1,z_2]^{p_1\times m_1}$ are known polynomial matrices of two variables and $X(z_1,z_2)\in C(z_1,z_2)^{n_1\times m_1}$, $Y(z_1,z_2)\in C(z_1,z_2)^{n_2\times m_1}$ are the unknown matrices.

Theorem 10. *A necessary and sufficient condition for the Diophantine equation (4.3) to have a solution is that*

$$[\,A(z_1,z_2)\quad B(z_1,z_2)\,][\,A(z_1,z_2)\quad B(z_1,z_2)\,]^+C(z_1,z_2)=C(z_1,z_2) \tag{4.4}$$

in which case the general solution is

$$\begin{bmatrix} X(z_1,z_2)\\ Y(z_1,z_2)\end{bmatrix}=[\,A(z_1,z_2)\quad B(z_1,z_2)\,]^+\,C(z_1,z_2)+\begin{bmatrix} X_0(z_1,z_2)\\ Y_0(z_1,z_2)\end{bmatrix}$$
$$-\,[\,A(z_1,z_2)\quad B(z_1,z_2)\,]^+\,[\,A(z_1,z_2)\quad B(z_1,z_2)\,]\begin{bmatrix} X_0(z_1,z_2)\\ Y_0(z_1,z_2)\end{bmatrix} \tag{4.5}$$

where $[\,X_0(z_1,z_2)^T\quad Y_0(z_1,z_2)^T\,]^T$ is an arbitrary rational matrix of two variables and $[\,A(z_1,z_2)\quad B(z_1,z_2)\,]^+$ is the generalized inverse of the compound matrix $[\,A(z_1,z_2)\quad B(z_1,z_2)\,]$.

Proof. The proof is a direct application of Theorem 9 if we take into account that relation (4.3) may be rewritten as

$$[\,A(z_1,z_2)\quad B(z_1,z_2)\,]\times\begin{bmatrix} X(z_1,z_2)\\ Y(z_1,z_2)\end{bmatrix}\times I_{m_1}=C(z_1,z_2)\,. \tag{4.6}$$ □

The investigation of the solution space of (4.3) plays an important role in problems of 1D linear systems such as parametrization of stabilizing controllers, robust stabilization, disturbance rejection, reference tracking, model matching, and H_2-optimal control (see survey paper of Kucera [11]). Here we examine the model matching problem in the 2D case.

Consider an open loop system S_1 (see Figure 1) with transfer function matrix $G(z_1,z_2)\in\Re(z_1,z_2)^{n\times m}$. We want to find out when there is an output feedback of the form

$$u(z_1,z_2)=-F(z_1,z_2)y(z_1,z_2)+v(z_1,z_2)\ \text{ with }\ F(z_1,z_2)\in\Re(z_1,z_2)^{m\times n} \tag{4.7}$$

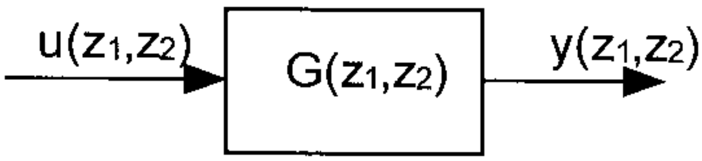

Figure 1. Open loop system.

such that the closed loop system of Figure 2 has transfer function $H(z_1, z_2) \in \Re(z_1, z_2)^{n\times m}$. Therefore, we want to find the rational matrix

$$F(z_1, z_2) \in \Re(z_1, z_2)^{m\times n}$$

that satisfies the following equation:

$$\begin{aligned} &H(z_1, z_2) = (I_n + G(z_1, z_2)F(z_1, z_2))^{-1}G(z_1, z_2) \\ \Leftrightarrow &G(z_1, z_2)F(z_1, z_2)H(z_1, z_2) = G(z_1, z_2) - H(z_1, z_2)\,. \end{aligned} \tag{4.8}$$

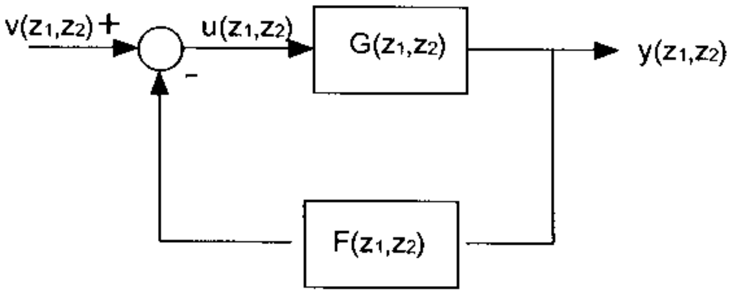

Figure 2. Closed loop system.

Let $G(z_1, z_2) = \frac{\tilde{G}(z_1,z_2)}{g(z_1,z_2)}$ where $\tilde{G}(z_1, z_2) \in \Re[z_1, z_2]^{n\times m}$ and $g(z_1, z_2)$ is the least common multiple of all the denominators of the matrix $G(z_1, z_2)$. In the same way, let $H(z_1, z_2) = \frac{\tilde{H}(z_1,z_2)}{h(z_1,z_2)}$ where $\tilde{H}(z_1, z_2) \in \Re[z_1, z_2]^{n\times m}$ and $h(z_1, z_2)$ is the least common multiple of all the denominators of the matrix $H(z_1, z_2)$. Then equations (4.8) may be rewritten as

$$\begin{aligned} &\frac{\tilde{G}(z_1, z_2)}{g(z_1, z_2)}F(z_1, z_2)\frac{\tilde{H}(z_1, z_2)}{h(z_1, z_2)} = \frac{\tilde{G}(z_1, z_2)}{g(z_1, z_2)} - \frac{\tilde{H}(z_1, z_2)}{h(z_1, z_2)} \\ \Leftrightarrow &\tilde{G}(z_1, z_2)F(z_1, z_2)\tilde{H}(z_1, z_2) = \tilde{G}(z_1, z_2)h(z_1, z_2) - \tilde{H}(z_1, z_2)g(z_1, z_2)\,. \end{aligned} \tag{4.9}$$

In the light of Theorem 9 we can easily obtain a necessary and sufficient condition for the existence of a solution of the equation (4.9) by the following:

Theorem 11. *A necessary and sufficient condition for equation (4.9) to have a solution is that*

$$\tilde{G}(z_1,z_2)\tilde{G}(z_1,z_2)^+\left[\tilde{G}(z_1,z_2)h(z_1,z_2)-\tilde{H}(z_1,z_2)g(z_1,z_2)\right]\tilde{H}(z_1,z_2)^+\tilde{H}(z_1,z_2)$$
$$=\tilde{G}(z_1,z_2)h(z_1,z_2)-\tilde{H}(z_1,z_2)g(z_1,z_2) \qquad (4.10)$$

in which case the general solution is

$$F(z_1,z_2)=\tilde{G}(z_1,z_2)^+\left[\tilde{G}(z_1,z_2)h(z_1,z_2)-\tilde{H}(z_1,z_2)g(z_1,z_2)\right]\tilde{H}(z_1,z_2)^+$$
$$+Y(z_1,z_2)-\tilde{G}(z_1,z_2)^+\tilde{G}(z_1,z_2)Y(z_1,z_2)\tilde{H}(z_1,z_2)\tilde{H}(z_1,z_2)^+ \qquad (4.11)$$

where $Y(z_1,z_2)$ is arbitrary to within having the dimension of $F(z_1,z_2)$ and $\tilde{G}(z_1,z_2)^+$, $\tilde{H}(z_1,z_2)^+$ are the generalized inverses of $\tilde{G}(z_1,z_2)$ and $\tilde{H}(z_1,z_2)$, respectively.

Proof. Let $A(z_1,z_2)=\tilde{G}(z_1,z_2)$, $B(z_1,z_2)=\tilde{H}(z_1,z_2)$, $F(z_1,z_2)=X(z_1,z_2)$, and $C(z_1,z_2)=\tilde{G}(z_1,z_2)h(z_1,z_2)-\tilde{H}(z_1,z_2)g(z_1,z_2)$ in Theorem 9. Then the proof of Theorem 11 follows. □

5. Illustrative example

Suppose that we are interested in finding the solution of the Diophantine equation

$$\begin{pmatrix}1 & z_1\\ 0 & 1\end{pmatrix}X(z_1,z_2)+\begin{pmatrix}0\\ z_2\end{pmatrix}Y(z_1,z_2)=I_2 \qquad (5.1)$$

where $X(z_1,z_2)\in C(z_1,z_2)^{2\times 2}$ and $Y(z_1,z_2)\in C(z_1,z_2)^{1\times 2}$. From Example 6 we have that

$$A(z_1,z_2)^+=\frac{1}{1+|z_2|^2+|z_1|^2|z_2|^2}\begin{pmatrix}1+|z_2|^2 & -z_1\\ \bar{z}_1|z_2|^2 & 1\\ -\bar{z}_1\bar{z}_2 & \bar{z}_2\left(1+|z_1|^2\right)\end{pmatrix}. \qquad (5.2)$$

A necessary and sufficient condition, according to Theorem 10, so that the Diophantine equation (5.1) can have a solution is that

$$\begin{pmatrix}1 & z_1 & 0\\ 0 & 1 & z_2\end{pmatrix}\begin{pmatrix}1 & z_1 & 0\\ 0 & 1 & z_2\end{pmatrix}^+ I_2=I_2$$
$$\Leftrightarrow\begin{pmatrix}1 & z_1 & 0\\ 0 & 1 & z_2\end{pmatrix}\times\left\{\frac{1}{1+|z_2|^2+|z_1|^2|z_2|^2}\begin{pmatrix}1+|z_2|^2 & -z_1\\ \bar{z}_1|z_2|^2 & 1\\ -\bar{z}_1\bar{z}_2 & \bar{z}_2(1+|z_1|^2)\end{pmatrix}\right\}=I_2\,. \qquad (5.3)$$

We can easily check that this is satisfied $\forall(z_1, z_2) \in C^2$. Thus the solution of the Diophantine equation (5.1) will be, from (4.5),

$$\begin{pmatrix} X(z_1,z_2) \\ Y(z_1,z_2) \end{pmatrix} = \frac{1}{1+|z_2|^2+|z_1|^2|z_2|^2} \begin{pmatrix} 1+|z_2|^2 & -z_1 \\ \bar{z}_1|z_2|^2 & 1 \\ -\bar{z}_1\bar{z}_2 & \bar{z}_2(1+|z_1|^2) \end{pmatrix} + \begin{pmatrix} X_0(z_1,z_2) \\ Y_0(z_1,z_2) \end{pmatrix}$$
$$- \begin{pmatrix} 1+|z_2|^2 & z_1|z_2|^2 & -z_1z_2 \\ \bar{z}_1|z_2|^2 & |z_1|^2|z_2|^2+1 & z_2 \\ -\bar{z}_1\bar{z}_2 & \bar{z}_2 & |z_2|^2(1+|z_1|^2) \end{pmatrix} \begin{pmatrix} X_0(z_1,z_2) \\ Y_0(z_1,z_2) \end{pmatrix} \tag{5.4}$$

where $(X_0(z_1, z_2)^T \quad Y_0(z_1, z_2)^T)^T$ is an arbitrary rational matrix of two variables having the dimension of $(X(z_1, z_2)^T \quad Y(z_1, z_2)^T)^T$.

6. Conclusions

A five-dimensional algorithm was determined for the computation of the generalized inverse of a two-variable polynomial matrix $A(z_1, z_2) = \sum_{i=0}^{q_1} \sum_{j=0}^{q_2} A_{i,j} z_1^i z_2^j \in C[z_1, z_2]^{n\times m}$ in terms of the coefficient matrices $A_{i,j} \in C^{n\times m}$, $i = 0, 1, \ldots, q_1$ and $j = 0, 1, \ldots, q_2$. The algorithm has already been implemented in the symbolic computational package Maple [7] and can be obtained by contacting the author directly. The whole theory has been illustrated by examples from multidimensional systems theory: solution of Diophantine equations and and the model matching problem.

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