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Structural properties of column (row) reduced MFDs

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Certain Matrix Fraction Descriptions (MFDs) are shown to possess important structural properties in their ability to display the infinite pole (respectively zero) structure of a rational transfer function matrix. Among a number of results, two algorithms are developed, the combination of which produce such an MFD displaying the complete infinite frequency structure of a rational transfer function matrix; and a simply checked necessary and sufficient condition for when the given MFD displays the complete infinite pole-zero structure information is provided.

1. Introduction

If $G(s) \in \mathbb{R}(s)^{m\times l}$ with $\text{rank}_{\mathbb{R}(s)}\, r \le \min(m, l)$ is a transfer function matrix then the simplest internal representations of this input/output behaviour are the matrix fraction descriptions (MFDs)

$$G(s) = N(s)D(s)^{-1} = D_1(s)^{-1}N_1(s) \tag{1}$$

where $N(s) \in \mathbb{R}[s]^{m\times l}, D(s) \in \mathbb{R}[s]^{l\times l}$ (respectively $D_1(s) \in \mathbb{R}[s]^{m\times m}, N_1(s) \in \mathbb{R}[s]^{m\times l}$) are not necessarily right (respectively left) coprime. The extent to which coprime MFDs reflect the finite frequency structure of the transfer function matrix is well catalogued (Wolovich 1974, Pugh and Shelton 1978). For the infinite frequency structure a second factorization is apparently required, this time a factorization of $G(1/w)$. In fact this is not necessary, and Pugh and Ratcliffe (1980) have described how, by taking a specific form of (1), both the finite and infinite frequency structure can be deduced from just one factorization of $G(s)$. The specific forms of (1) which carry the complete pole–zero structure of $G(s)$ in this ready manner are those where

$$M(s) = \begin{bmatrix} D(s) \\ N(s) \end{bmatrix} \in \mathbb{R}[s]^{(l+m)\times l}; \quad M_1(s) = [N_1(s), D_1(s)] \in \mathbb{R}[s]^{m\times(l-m)} \tag{2}$$

are required to be minimal bases (Forney 1975), i.e. where $M(s)$, (respectively $M_1(s)$) is column (respectively row) reduced and has no finite zeros, and such MFDs will be called minimal. Thus, the additional requirement of column or row 'reducedness' of (2) has great implications for the MFD's ability to display the complete pole–zero structure of the rational matrix it represents. It is not surprising therefore that minimal MFDs have proved to be an essential tool in the solution of a number of

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important problems in control theory and design (Kučera and Zagalak 1991, Syrmos and Zagalak 1993). It will be shown, however, in the following, that minimal MFDs are not required for many aspects concerning the infinite frequency information to be detected and that the simple requirement of column or row reducedness of the MFD is sufficient. Such MFDs will hereafter be referred to as column or row reduced matrix fraction descriptions (CRMFDs) or (RRMFDs).

This paper considers ways in which information about the infinite frequency structure of $G(s)$ may be directly determined from CRMFDs (respectively RRMFDs) in a similar manner to that employed in determining whether a rational function has an infinite pole or zero. CRMFDs can be easily computed since any $G(s) \in \mathbb{R}(s)^{m \times l}$ can be written as $G(s) = N(s)[d(s)I_l]^{-1}$ where $d(s)$ is the least common denominator of all the terms of $G(s)$. Then the compound matrix

$$\begin{bmatrix} d(s)I_l \\ N(s) \end{bmatrix} \in \mathbb{R}[s]^{(l+m)\times l} \tag{3}$$

can be made column reduced by postmultiplying (3) by appropriate unimodular matrices via algorithms given in, for example, Vardulakis (1991). Thus, CRMFDs are fairly readily obtained and provide a close analogy to the case of a scalar rational function.

Specifically, the paper establishes that there exist CRMFDs (and therefore RRMFDs) from which the complete infinite frequency structure of $G(s)$ can be detected virtually from inspection. The proofs are algorithmnic, which enables such CRMFDs to be computed from any given CRMFD. The paper also develops necessary and sufficient conditions to detect when a given CRMFD is of this special type, and these conditions are extremely simple to construct and test. When these simple rank conditions are satisfied, the infinite pole and zero structure of $G(s)$ can be determined directly by inspection of the CRMFD and certain relative values of its column degrees. Of course, when the CRMFD is additionally relatively prime then its overall column degrees represent the Forney minimal indices (Forney 1975) of $G(s)$. The results presented here are generalizations of the well-known result (see for example Wolovich 1974) which states that $G(s)$ is (strictly) proper iff certain inequalities are satisfied between the column degrees of the constituent parts of the MFD. In the terms adopted in this paper this is a result which gives an exact characterization of when a MFD has no infinite poles. The paper presents generalizations of this result to include the complete infinite pole/zero information of a non-proper $G(s)$. The results should be seen as complementing those of Vardulakis and Karcanias (1983) in the characterization of bases of polynomial modules.

As regards the structure of the paper, section 2 introduces some preliminary results and section 3 reveals how to obtain the complete infinite pole/zero structure from a CRMFD (respectively RRMFD). Section 4 introduces the essential notion of minimal differences and explores their implications for the infinite frequency structure. Section 5 gives the bounds for the infinite frequency pole and zero structure of $G(s)$, which any CRMFD will give. It then develops the existence results for CRMFDs which display the complete infinite frequency structure, and also the simple necessary and sufficient conditions to enable such desirable CRMFDs to be recognized. In section 6, consideration is given to the feedback invariance of the information given by the minimal differences.

2. Preliminaries

A discrete valuation of a field K is a mapping $V : K \to \mathbb{Z}$ such that for all non-zero elements $x, y \in K$

$$\left.\begin{aligned} V(xy) &= V(x) + V(y) \\ V(x+y) &\geq \min\{V(x), V(y)\} \end{aligned}\right\} \tag{4}$$

In particular, take $g(s) = n(s)/d(s) \in \mathbb{R}(s)$ where $n(s), d(s) \in \mathbb{R}[s], d(s) \not\equiv 0$ and define the map $V_\infty(\cdot) : \mathbb{R}(s) \to \mathbb{Z} \cup \{+\infty\}$ via: $V_\infty(g(s)) := \delta(d(s)) - \delta(n(s))$ if $g(s) \not\equiv 0$, $V_\infty(g(s)) := +\infty$ if $g(s) \equiv 0$, where $\delta(\cdot)$ denotes the degree of the indicated polynomial. It is clear that the map $V_\infty(\cdot)$ satisfies (4) (where $\delta(0) := -\infty$) and therefore serves as a discrete valuation of $\mathbb{R}(s)$. This is usually referred to as the valuation at $s = \infty$ of $g(s) \in \mathbb{R}(s)$. Given now $G(s) \in \mathbb{R}(s)^{m \times l}$, $\text{rank}_{\mathbb{R}(s)} G(s) = r \leq \min(m, l)$ we can define (Vardulakis *et al.* 1982, Vardulakis and Karcanias 1983) its 'valuation at $s = \infty$, by the map $V_\infty(\cdot) : \mathbb{R}(s)^{m \times l} \to \mathbb{Z} \cup \{\infty\}$ via: $V_\infty(G) :=$ min $V_\infty(\cdot)$ among the $V_\infty(\cdot)$ of the rth order minors of $G(s)$ if $r > 0$, $V_\infty(G) := +\infty$ if $r = 0$.

Appropriately, the Smith–MacMillan form at $s = \infty$ can be defined as follows.

Lemma 1 (Vardulakis *et al.* 1982, Vardulakis and Karcanias 1983): *Let $G(s) \in \mathbb{R}(s)^{m \times l}$, rank $_{\mathbb{R}(s)} G(s) = r > 0$. Let $\xi_i(G)$ denote the least $V_\infty(\cdot)$ among the $V_\infty(\cdot)$ of all minors of $G(s)$ of order $i, i = 1, \ldots, r$. Then define $q_1 := -\xi_1(G), q_2 := \xi_1(G) - \xi_2(G), q_3 := \xi_2(G) - \xi_3(G), \ldots, q_r := \xi_{r-1}(G) - \xi_r(G)$. The Smith–MacMillan form at $s = \infty$ of $G(s) \in \mathbb{R}(s)^{m \times l}$ is given by $S^\infty_{T(s)}(s) = \text{diag}\,[s^{q_1}, s^{q_2}, \ldots, s^{q_r}, 0_{m-r, l-r}]$ with $q_1 \geq q_2 \geq \cdots \geq q_k \geq 0 \geq q_{k+1} \geq \cdots \geq q_r$.*

This then leads to the following definition for the infinite poles and zeros of $G(s)$.

Definition 1: If σ is the number of q_is in Lemma 1 satisfying $q_i > 0$ then $G(s)$ has *σ poles at infinity*, each having *degree q_i*. Similarly, if ζ is the number of q_is in Lemma 1 satisfying $q_i < 0$ then $G(s)$ has *ζ zeros at infinity*, each having *degree* $|q_i|$. The number σ (respectively ζ) is known as the *multiplicity* of infinite poles (respectively zeros). □

Let $\mathbb{R}_{\text{pr}}(s)^{p \times m}$ denote the set of $p \times m$ matrices with elements in the ring of proper rational functions in one variable s with coefficients in the field $\mathbb{R}$ of real numbers, i.e. proper rational matrices.

Definition 2 (Vardulakis *et al.* 1982, Vardulakis and Karcanias 1983): $N_1(s) \in \mathbb{R}_{\text{pr}}(s)^{m \times l}, D_1(s) \in \mathbb{R}_{\text{pr}}(s)^{n \times l}$ (respectively $N_2(s) \in \mathbb{R}_{\text{pr}}(s)^{m \times l}, D_2(s) \in \mathbb{R}_{\text{pr}}(s)^{m \times n}$) are *right* (respectively *left*) *coprime at $s = \infty$* if and only if

$$\lim_{s \to \infty} \begin{bmatrix} D_1(s) \\ N_1(s) \end{bmatrix} = E_1 \quad (\text{respectively } \lim_{s \to \infty} [D_2(s) \quad N_2(s)] = E_2) \tag{5}$$

with $\text{rank}_{\mathbb{R}} E_1 \in \mathbb{R}^{(m+n) \times l} = l$ (respectively $\text{rank}_{\mathbb{R}} E_2 \in \mathbb{R}^{m \times (l+n)} = m$). □

Definition 3 (Vardulakis *et al.* 1982, Vardulakis and Karcanias 1983): Let $G(s) \in \mathbb{R}^{m \times l}(s)$, $\text{rank}_{\mathbb{R}(s)}\, r \leq \min(m, l)$. Then

$$G(s) = N_1(s) D_1(s)^{-1} \quad (\text{respectively } G(s) = D_2(s)^{-1} N_2(s)) \tag{6}$$

where $N_1(s) \in \mathbb{R}_{\text{pr}}(s)^{m \times l}, D_1(s) \in \mathbb{R}_{\text{pr}}(s)^{l \times l}$ (respectively $D_2(s) \in \mathbb{R}_{\text{pr}}(s)^{l \times l}, N_2(s) \in$

$\mathbb{R}_{pr}(s)^{m\times l}$) are right (respectively left) coprime at $s=\infty$, is called a *right* (respectively *left*) *proper matrix fraction description* (*PMFD*) of $G(s)$. □

It is well known that any rational function can always be represented as a ratio of coprime at $s=\infty$ proper rational functions. The matrix analogue of the latter leads to the following lemma.

Lemma 2 (Vardulakis *et al.* 1982, Vardulakis and Karcanias 1983): *Let* $G(s)\in\mathbb{R}(s)^{m\times l}$, $\mathrm{rank}_{\mathbb{R}(s)}\, r\le \min(m,l)$ *possess a right or left PMFD of the form* (6) *then the zero* (*respectively pole*) *structure at* $s=\infty$ *of* $G(s)$ *is given by the zero structure at* $s=\infty$ *of* $N_1(s)\in\mathbb{R}_{pr}(s)^{m\times l}$ *or* $N_2(s)\in\mathbb{R}_{pr}(s)^{m\times l}$ (*respectively* $D_1(s)\in\mathbb{R}_{pr}(s)^{l\times l}$ *or* $D_2(s)\in\mathbb{R}_{pr}(s)^{l\times l}$).

Consider now a factorization of $G(s)\in\mathbb{R}(s)^{m\times l}$, $\mathrm{rank}_{\mathbb{R}(s)}\, r\le\min(m,l)$, i.e.

$$G(s)=N(s)D(s)^{-1} \tag{7 a}$$

where $N(s)\in\mathbb{R}[s]^{m\times l}$, $D(s)\in\mathbb{R}[s]^{l\times l}$ (not necessarily coprime) with $m\ge l$ such that

$$M(s)=\begin{bmatrix}D(s)\\N(s)\end{bmatrix}\equiv\underbrace{\begin{bmatrix}D_h\\N_h\end{bmatrix}_h}_{[M]_h}\underbrace{\begin{pmatrix}s^{\delta(m_1)}&0&\cdots&0\\0&s^{\delta(m_2)}&\cdots&0\\\vdots&\vdots&\ddots&\vdots\\0&0&\cdots&s^{\delta(m_l)}\end{pmatrix}}_{\Lambda(s)}+\begin{bmatrix}\bar D(s)\\\bar N(s)\end{bmatrix} \tag{7 b}$$

is column reduced, i.e. the high order (column) coefficient matrix (HOCM) $[M]_h=[D_h^{\mathrm T}\;\;N_h^{\mathrm T}]_h^{\mathrm T}$, has $\mathrm{rank}_{\mathbb{R}}=l$ where $\delta(m_i), i=1,2,\ldots,l$ are the column degrees of the matrix $M(s)\in\mathbb{R}[s]^{(l-m)\times l}$. Such factorizations will be called column reduced matrix fraction descriptions (CRMFDs). It should be noted that in relation to Vardulakis and Karcanias (1983), a CRMFD is a column reduced at $s=\infty$ polynomial basis for the rational vector space spanned by the columns of $[I_l\;\;G^{\mathrm T}(s)]^{\mathrm T}$.

In what follows, analogous statements (for row reducedness) can be made when $m<l$ if rows are substituted for columns. This parallel type of result will not be referred to explicitly in the following.

Theorem 1 (Mahmood *et al.* 1996): *Let* $G(s)\in\mathbb{R}(s)^{m\times l}$, $\mathrm{rank}_{\mathbb{R}(s)}\, r\le\min(m,l)$ *be written as in* (7 *a*) *such that* (7 *b*) *is a CRMFD. Define* $\Lambda(s):=\mathrm{diag}\,[s^{\delta(m_i)}]\in\mathbb{R}[s]^{l\times l}, i=1,\ldots,l$, *as in* (7 *b*), *then*

$$G(s)=[N(s)\Lambda(1/s)][D(s)\Lambda(1/s)]^{-1} \tag{8}$$

is a coprime at $s=\infty$ *PMFD.*

From Lemma 2 and Theorem 1 the following corollaries may be noted.

Corollary 1: *Let* $G(s)\in\mathbb{R}(s)^{m\times l}$, $\mathrm{rank}_{\mathbb{R}(s)}\, r\le\min(m,l)$ *be written as in* (8). *Then the pole* (*respectively zero*) *structure at* $s=\infty$ *of* $G(s)$ *is given by the zero structure at* $s=\infty$ *of* $D(s)\Lambda(1/s)\in\mathbb{R}_{pr}(s)^{l\times l}$ (*respectively* $N(s)\Lambda(1/s)\in\mathbb{R}_{pr}(s)^{m\times l}$).

Corollary 2: *Let* (7) *be a CRMFD of* $G(s)\in\mathbb{R}(s)^{m\times l}$, $\mathrm{rank}_{\mathbb{R}(s)}\, r\le\min(m,l)$ *where* $\Lambda(s)\in\mathbb{R}[s]^{l\times l}$ *is defined in Theorem 1. Then the pole* (*respectively zero*) *structure at* $s=\infty$ *of* $G(s)$ *is isomorphic to the zero structure at* $\omega=0$ *of* $D(1/\omega)\Lambda(\omega)$ (*respectively* $N(1/\omega)\Lambda(\omega)$).

Thus, to obtain the infinite frequency structure of $G(s)$, Corollaries 1–2 have shown that the sole requirement of column reducedness of a polynomial MFD suffices. Such MFDs are readily obtained since there is no requirement for coprimeness. The following section shows how to extract this information from the CRMFD by use of some simple rank calculations.

3. Infinite frequency structure and CRMFDs

Suppose the Laurent expansion at $s = \infty$ of $G(s)$ is of the form (Verghese and Kailath 1981, Pugh *et al.* 1989).

$$G(s) = \sum_{i=-\infty}^{l} G_i s^i = G_l s^l + G_{l-1} s^{l-1} + \cdots + G_0 + G_{-1} s^{-1} + \cdots \tag{9}$$

and the ith *Toeplitz matrix* associated with the Laurent expansion (9), is denoted as $T_i^{\infty}(G)$. Also the *rank indices* at infinity of $G(s) \in \mathbb{R}(s)^{m\times l}$ are defined as

$$\rho_i^{\infty} = \operatorname{rank}_{\mathbb{R}} (T_i^{\infty}(G)) - \operatorname{rank}_{\mathbb{R}} (T_{i-1}^{\infty}(G)), \quad i = -l, -l+1, \ldots \tag{10}$$

where it is assumed that $\operatorname{rank}_{\mathbb{R}} (T^{\infty}_{-l-1}(G)) = 0$. Then the rank indices ρ_i^{∞} determine the infinite frequency structure of $G(s)$ in the following manner.

Lemma 3 (Verghese and Kailath 1981, Pugh *et al.* 1989): *If the Laurent expansion of $G(s) \in \mathbb{R}(s)^{m\times l}$, $\operatorname{rank}_{\mathbb{R}(s)} r \le \min(m, l)$ is given by (9) and $\rho_i^{\infty} - \rho_{i-1}^{\infty} \neq 0$ then $G(s)$ has*

$$\rho_i^{\infty} - \rho_{i-1}^{\infty}, \quad i = -l, -l+1, \ldots \tag{11}$$

zeros (respectively poles) at infinity of degree i (respectively $|i|$) if $i > 0$ (respectively $i < 0$), $i = -l, -l+1, \ldots$

The above result is the infinite frequency anologue of the finite frequency result given by Van Dooren *et al.* (1989). The result itself has an equivalent state-space version through the interpretation of infinite elementary divisors given by the theory of piecewise arithmetic progression sequences (Karcanias and Kalogeropoulos 1986)

The method described above provides the basis of a computational algorithm to obtain the infinite frequency structure of $G(s)$. The following theorem, however, shows how the zero (respectively pole) structure at $s = \infty$ of $G(s)$ can be obtained from the CRMFD and, in particular, from $N(s) \in \mathbb{R}[s]^{m\times l}$ (respectively $D(s) \in \mathbb{R}[s]^{l\times l}$) as in (7 *a*).

Theorem 2 (Mahmood *et al.* 1996): *Let $G(s) \in \mathbb{R}(s)^{m\times l}$, $\operatorname{rank}_{\mathbb{R}(s)} r \le \min(m, l)$ be written as a CRMFD as in (7). Write $N(s) \in \mathbb{R}[s]^{m\times l}$ (respectively $D(s) \in \mathbb{R}[s]^{l\times l}$) in the form*

$$N(s) = N_0 \Lambda(s) + N_1 \Lambda_{-1}(s) + N_2 \Lambda_{-2}(s) + \cdots + N_\sigma \Lambda_{-\sigma}(s) \tag{12 a}$$

$$(\textit{respectively } D(s) = D_0 \Lambda(s) + D_1 \Lambda_{-1}(s) + D_2 \Lambda_{-2}(s) + \cdots + D_\sigma \Lambda_{-\sigma}(s)) \tag{12 b}$$

where $\sigma = \max\{\delta(m_i)\}, i = 1, \ldots, l$ and $\Lambda_{-\ell}(s) \in \mathbb{R}[s]^{l\times l}, \ell \in \sigma$, is the diagonal matrix formed from $\Lambda(s) \in \mathbb{R}[s]^{l\times l}$ by reducing each power of s by the amount $i \in \mathbb{N}$, where it is agreed that $s^j = 1\ \forall j < 0$ and the corresponding element in $N_j \in \mathbb{R}^{m\times l}$ (respectively $D_j \in \mathbb{R}^{l\times l}$ is zero. Then the rank indices ρ_i^{ζ} (respectively ρ_i^{ρ}) of the Toeplitz matrices associated with (12 a) (respectively (12 b)) determine the zero (respectively pole) structure at $s = \infty$ of $G(s)$.

Thus, it follows, for example, that the Toeplitz matrices

$$T_i(N) \triangleq \begin{pmatrix} N_0 & \cdots & \cdots & 0 \\ N_1 & N_0 & \cdots & 0 \\ \vdots & & \ddots & \\ N_i & N_{i-1} & \cdots & N_0 \end{pmatrix}, \quad i = 0, 1, \ldots, \sigma \tag{13}$$

(respectively $T_i(D)$) may be obtained directly from expanding $N(s)$ (respectively $D(s)$) in the form (12 *a*) (respectively (12 *b*)) rather than expanding $N(s)\Lambda(1/s)$ (respectively $D(s)\Lambda(1/s)$). The rank indices $\rho_i^\zeta = \text{rank}_{\mathbb{R}}(T_i(N)) - \text{rank}_{\mathbb{R}}(T_{i-1}(N))$, $i \geq 0$ (respectively ρ_i^ϱ) then determine the infinite zero (respectively pole) structure of $G(s)$.

Theorem 2 shows that the column reducedness and not necessarily coprimeness is all that is required of a MFD to compute the infinite frequency structure of a rational transfer function matrix $G(s)$. Such MFDs are, of course, fairly readily computed.

We next consider the problem of computing the multiplicity of the infinite poles and zeros. It would appear that to determine these multiplicities one would require the complete information of the Smith–MacMillan form at $s = \infty$. However, the following theorem shows how to obtain these multiplicities directly by performing a simple calculation.

Theorem 3 (Mahmood *et al.* 1996): *In the above notation, let N_h (respectively D_h) be the constant matrix extracted from $[M]_h$, the high order coefficient matrix (HOCM) of $M(s)$ derived from the CRMFD* (7). *Then, the multiplicity of the zeros (respectively poles) at $s = \infty$ of $G(s) \in \mathbb{R}^{m\times l}(s)$, $\text{rank}_{\mathbb{R}(s)}\, r \leq \min(m, l)$ given by*

$$\zeta = r - \text{rank}_{\mathbb{R}}(N_h) \quad (\textit{respectively } \varrho = l - \text{rank}_{\mathbb{R}}(D_h)) \tag{14}$$

Theorem 3 immediately gives rise to the following result.

Corollary 3 (Mahmood *et al.* 1996): $G(s) \in \mathbb{R}^{m\times l}(s)$, $\text{rank}_{\mathbb{R}(s)}\, r \leq \min(m, l)$ *possesses no zeros (respectively poles) at $s = \infty$ if and only if* $\text{rank}_{\mathbb{R}}(N_h) = r$ (*respectively* $\text{rank}_{\mathbb{R}}(D_h) = l$).

Pugh *et al.* (1989) gave a test for the absence of infinite zeros of $G(s) \in \mathbb{R}(s)^{m\times l}$ which involved the calculation of the ranks of the corresponding Toeplitz matrices associated with the Laurent expansion at $s = \infty$ of $G(s)$. Corollary 3 simply requires the computation of the rank of the HOCM of the CRMFD $(M(s) \in \mathbb{R}[s]^{(l+m)\times l}$, which is generally easier to calculate.

It has been shown that the simple requirement of column reducedness of a MFD of a given transfer function matrix $G(s) \in \mathbb{R}(s)^{m\times l}$ is all that is necessary for the computation of the infinite frequency structure of $G(s)$ from its separate numerator and denominator parts. In fact, the infinite pole and zero multiplicities can be determined even more simply still by using the component constant matrices obtained from the HOCM of the compound matrix $M(s)$.

4. Infinite frequency structure bounds

In this section we investigate the structure of CRMFDs which reveals the extent to which the infinite frequency structure of $G(s)$ can be deduced from direct inspection of a CRMFD, and without the need for the rank information being derived from the Laurent expansion. The motivation of this section stems from the scalar case where it is known that a rational function $t(s) = n(s)/d(s) \in \mathbb{R}(s) n(s), d(s) \in \mathbb{R}[s]$ possesses an infinite zero of degree $\delta(d(s)) - \delta(n(s))$ in case $\delta(d(s)) > \delta(n(s))$ (in which case $t(s)$ is a strictly proper rational function) or possesses an infinite pole of degree $\delta(n(s)) - \delta(d(s))$ in case $\delta(n(s)) > \delta(d(s))$ (in which case $t(s)$ is a non-proper rational function). Thus, the infinite frequency structure of a rational function can be adducted by inspection. This section generalizes these ideas for the matrix case and details the infinite frequency information which can be seen from inspection. It will be seen that CRMFDs possess many interesting properties from this point of view.

Recall therefore the CRMFD (7) with $G(s) \in \mathbb{R}(s)^{m \times l}$, $\text{rank}_{\mathbb{R}(s)}\, r \leq \min(m, l)$. Unless otherwise stated, it will be assumed that the CRMFD (7) has been transformed by appropriate unimodular transformations to the form

$$M_1(s) = \begin{bmatrix} D_1'(s) & D_2(s) \\ 0_{m,\mu} & N_2(s) \end{bmatrix} \in \mathbb{R}[s]^{(l+m)\times l} \quad \text{where } [M_1]_h := \begin{bmatrix} D_{1h} \\ N_{1h} \end{bmatrix}_h \qquad (15\,a)$$

where $\mu = l - r$ and $\text{rank}_{\mathbb{R}(s)}(N_2(s)) = \text{rank}_{\mathbb{R}(s)}(N(s)) = r$. In this regard

$$M_2(s) = \begin{bmatrix} D_2(s) \\ N_2(s) \end{bmatrix} \in \mathbb{R}[s]^{(l+m)\times r} \quad \text{where } [M_2]_h := \begin{bmatrix} D_{2h} \\ N_{2h} \end{bmatrix}_h \qquad (15\,b)$$

If $G(s) \in \mathbb{R}(s)^{m \times l}$ has $\text{rank}_{\mathbb{R}(s)}\, r = l$ then there is no need for the latter transformations since the given CRMFD will already be of the form (15 *b*). However, if $G(s)$ has $\text{rank}_{\mathbb{R}(s)}\, r < l$ then clearly unimodular transformations may always be performed on any CRMFD to bring it to the form (15 *a*) without destroying the column reducedness of the MFD.

Definition 4: Let (15) be a CRMFD and $m_{2,i}, d_{2,i}, n_{2,i}, i = 1, \ldots, r$, denote the ith columns of $M_2(s) \in \mathbb{R}[s]^{(l+m)\times r}, D_2(s) \in \mathbb{R}[s]^{l \times r}, N_2(s) \in \mathbb{R}[s]^{m \times r}$ respectively and $\delta(m_{2,i}), \delta(d_{2,i}), \delta(n_{2,i})$ their respective degrees. The minimal differences of the CRMFD (15) are defined as $\hat{k}_{2,i} = \delta(m_{2,i}) - \delta(n_{2,i}) (\geq 0)$ and $\check{k}_{2i} = \delta(m_{2i}) - \delta(d_{2i}) (\geq 0)$. □

Note that from the above definition it follows that, with regards to the minimal differences, for each $i = 1, \ldots, r$, either $\hat{k}_{2,i}$ or $\check{k}_{2,i}$, or possibly both, are zero.

Reorder the columns of $M_2(s)$ if necessary, so that $\hat{k}_{2,i} > 0, i = 1, \ldots, \theta$, occur in ascending order, $\check{k}_{2,i} > 0, i = r - \phi + 1, \ldots, r$ occur in ascending order and where all other $\hat{k}_{2,i}, \check{k}_{2,i}, i = \theta + 1, \ldots, r - \phi$, are zero. That is, let there be θ columns in which $\hat{k}_{2,i} > 0$ and ϕ columns in which $\check{k}_{2,i} > 0$. Then the following result may be established.

Theorem 4: *Given the CRMFD* (15), *then* $G(s) \in \mathbb{R}(s)^{m \times l}$, $\text{rank}_{\mathbb{R}(s)}\, r \leq \min(m, l)$ *has at least* θ *infinite zeros of degrees* $\nu_1, \ldots, \nu_\theta$ *satisfying* $\nu_i \geq \hat{k}_{2,i}, i = 1, \ldots, \theta$, *and at least* ϕ *infinite poles of degrees* $\gamma_1, \ldots, \gamma_\phi$ *satisfying* $\gamma_i \geq \check{k}_{2,r-\phi+i}, i = 1, \ldots, \phi$.

Proof: By Corollary 2 the infinite zero structure of $G(s)$ of (15) is given by the zero structure of $N_2(1/\omega)\Lambda(\omega)$ at $\omega = 0$. Let $\Delta_i(\omega)$ denote the g.c.d of the $i \times i$ minors of $N_2(1/\omega)\Lambda(\omega)$. It is clear that $\Delta_{r-\theta+1}$ contains the factor $\omega^{\hat{k}_{2,1}}$ since at least one of the θ columns with $\hat{k}_{2,i} > 0$ must be included in all $(r - \theta + 1) \times (r - \theta + 1)$ minors of $N_2(1/\omega)\Lambda(\omega)$. Similarly $\Delta_{r-\theta+2}$ must contain at least the additional factor $\omega^{\hat{k}_{2,2}}$ over $\Delta_{r-\theta+1}$, since at least two of the θ columns with $\hat{k}_{2,i} > 0$ must be included in all $(r - \theta + 2) \times (r - \theta + 2)$ minors of $N_2(1/\omega)\Lambda(\omega)$. It then follows by repeating the argument that Δ_r contains at least the additional factor $\omega^{\hat{k}_{2,r}}$ over Δ_{r-1}. Thus, if $\lambda_i(\omega), i = 1, \ldots, r$ are the invariant polynomials of $N_2(1/\omega)\Lambda(\omega)$, then $\lambda_{r-\theta-j}$ at least contains the factor $\omega^{\hat{k}_{2,j}}, j = 1, \ldots, \theta$ and so by Corollary 2 the result follows for the infinite zeros. The result for the infinite poles follows in a similar way. □

Theorem 4 thus enables some lower bounds on the infinite frequency structure of $G(s)$ which can be determined virtually by inspection. The connection of this theorem with the well-known result (Wolovich 1974, Kailath 1980) characterizing the strict properness of a CRMFD in terms of its relative column degrees is given by the following.

Corollary 4: *Consider the CRMFD* (15). $G(s) \in \mathbb{R}(s)^{m\times l}$ *with* $\text{rank}_{\mathbb{R}(s)}\, r \leq \min(m, l)$ *is strictly proper if and only if each column of* $N_2(s) \in \mathbb{R}[s]^{m\times r}$ *has degree strictly less than the degree of the corresponding column of* $D_2(s) \in \mathbb{R}[s]^{l\times r}$. □

Note the following notation for use in the following. Let

$$M_i(s) = \begin{bmatrix} D_i(s) \\ N_i(s) \end{bmatrix} \in \mathbb{R}[s]^{(l+m)\times l}, \quad i \in \mathbb{Z}^+ \tag{16a}$$

be a CRMFD associated with $G(s) \in \mathbb{R}(s)^{m\times l}$, with $\text{rank}_{\mathbb{R}(s)}\, r \leq \min(m, l)$ then the jth columns of $M_i(s), N_i(s), D_i(s)$ are denoted as $m_{i,j}, n_{i,j}, d_{i,j}$, their respective degrees are denoted as $\delta(m_{i,j}), \delta(n_{i,j}), \delta(d_{i,j})$, the minimal differences of $M_i(s)$ are denoted as $k_{i,j}, \hat{k}_{i,j}$ for some column j, the HOCM of $M_i(s)$ is denoted as

$$[M_i]_h := \begin{bmatrix} D_{ih} \\ N_{ih} \end{bmatrix}_h \tag{16b}$$

and the HOCM of $N_i(s)$ (respectively $D_i(s)$) is denoted as $[N_i]_h$ (respectively $[D_i]_h$). The essential difference between, for example, N_{ih} and $[N_i]_h$ should be noted.

Theorem 4 may now be used to show the structure of the CRMFD (15) in more detail. In particular

$$[M_1]_h = \begin{bmatrix} D_{1h} \\ N_{1h} \end{bmatrix}_h := \begin{bmatrix} D'_{1h} & D''_h & D'_h & 0_{l,\phi} \\ 0_{m,\mu} & 0_{m,\theta} & N'_h & N'''_h \end{bmatrix} \in \mathbb{R}^{(m+l)\times l} \tag{17a}$$

is the HOCM of $M_1(s)$ as in (15a) and thus

$$[M_2]_h = \begin{bmatrix} D_{2h} \\ N_{2h} \end{bmatrix}_h := \begin{bmatrix} D''_h & D'_h & 0_{l,\phi} \\ 0_{m,\theta} & N'_h & N'''_h \end{bmatrix} \in \mathbb{R}^{(m+l)\times r} \tag{17b}$$

is the HOCM of $M_2(s)$ as in (15b). By Theorem 4 (17b). we can say that, by inspection, $M_2(s)$ displays θ infinite zeros of degree $\geq \hat{k}_{2,i}, i = 1, \ldots, \theta$, respectively and ϕ infinite poles of degree $> \bar{k}_{2,i}, i = r - \phi + 1, \ldots, r$, respectively.

5. Determining the complete infinite frequency information

In this section we examine the complete infinite frequency information (as opposed to lower bounds) and consider to what extent this may be adduced by inspection. The following shall be defined.

Definition 5: If $G(s) \in \mathbb{R}(s)^{m\times l}$ has ζ (respectively ϱ) infinite zeros (respectively poles) with degrees $\alpha_1, \ldots, \alpha_\zeta$ (respectively $\beta_1, \ldots, \beta_\varrho$ then the CRMFD (15) (or (17)) is said to display the complete infinite zero (respectively pole) multiplicity information in case $\theta = \zeta$ (respectively $\phi = \varrho$), and if $\hat{k}_{2,i} = \alpha_i, i = 1, \ldots, \zeta$ (respectively

$\hat{k}_{2,l-\varrho+i} = \beta_i, i = 1, \ldots, \varrho$) then it will be said to display the complete infinite zero (respectively pole) degree information. □

Thus, if (17) satisfies Definition 5 then the lower bounds for the infinite poles and zeros proposed in Theorem 4 coincide with the infinite pole-zero multiplicity and degree information.

A natural question concerns the existence of a CRMFD (15), which simultaneously displays the complete infinite zero and pole multiplicity information. There is a positive answer to this question which will be established in an algorithmic manner through the following theorem.

Theorem 5: *Let* $G(s) \in \mathbb{R}(s)^{m \times l}$ *with* $\text{rank}_{\mathbb{R}(s)}\, r \leq \min(m, l)$. *Then there always exists a CRMFD which displays the complete infinite pole and zero multiplicity information.*

Proof: Let $M_1(s)$ be a CRMFD as in (15 *a*) and $[M_1]_h$ its HOCM. Then we shall show that $M_1(s)$ can be transformed into a CRMFD $M_{\varrho\zeta}(s)$, which possesses $\varrho = l - \text{rank}_{\mathbb{R}}(D_{1h})$ columns of non-zero degree satisfying $\check{k}_{1,i} > 0\ \forall i = \{l - (\zeta + \varrho) + 1\}, \ldots, (l - \zeta)$ and $\zeta = r - \text{rank}_{\mathbb{R}}(N_{1h})$ columns of non-zero degree satisfying $\hat{k}_{1,i} > 0\ \forall i = (l - \zeta + 1), \ldots, l$, hence displaying the complete infinite pole–zero multiplicity information (see Theorem 3). The proof will be in the form of an algorithm.

Step 1. For simplicity, reorder the columns of $M_2(s)$ (as in (15 *b*)), if necessary, so that

$$[M_2]_h = \begin{bmatrix} D_{2h} \\ N_{2h} \end{bmatrix}_h = \begin{bmatrix} \overbrace{D'_h}^{k} & D''_h & 0_{l,\phi} \\ N'_h & 0_{m,\theta} & N'''_h \end{bmatrix} \tag{18}$$

where obviously $k + \theta + \phi = r$ and (18) is column reduced.

Consider first the infinite pole multiplicity of $G(s) \in \mathbb{R}(s)^{m \times l}$.

Step 2. Find a non-zero $a = (a_1, a_2, \ldots, a_{r-\phi}, 0_{r-\phi+1}, \ldots, 0_r)^{\mathrm{T}}$ such that $D_{2h}a = 0$. Let $\delta_{d_2} = \max\{\delta(d_{2,i})\}$ and $\delta_{n_2} = \max\{\delta(n_{2,i})\}$. Then define

$$a(s) := [a_1 s^{\delta_{d_2} - \delta(d_{2,1})}, a_2 s^{\delta_{d_2} - \delta(d_{2,2})}, \ldots, a_{r-\phi} s^{\delta_{d_2} - \delta(d_{2,r-\phi})}, 0_{r-\phi+1}, \ldots, 0_r]^{\mathrm{T}} \tag{19}$$

Let $a_j, j = 1, \ldots, r - \phi$, be the first non-zero entry in (19). Then replace the column $m_{2,j} = [d_{2,j}^{\mathrm{T}}(s)\ n_{2,j}^{\mathrm{T}}(s)]^{\mathrm{T}}$ of $M_2(s)$ by $m'_{2,j} = [d'^{\mathrm{T}}_{2,j}(s)\ n'^{\mathrm{T}}_{2,j}(s)]^{\mathrm{T}}$ where

$$\begin{aligned} d'_{2,j}(s) := D_2(s)a(s) &= [D_{2h}\,\text{diag}\{s^{\delta(d_{2,i})}\} + \hat{D}(s)]a(s) \\ &= \underbrace{D_{2h}as^{\delta_{d_2}}}_{0} + \hat{D}(s)a(s) \\ &= \hat{D}(s)a(s) \end{aligned} \tag{20}$$

$$n'_{2,j}(s) := N_2(s)a(s) = [N_{2h}\,\mathrm{diag}\,\{s^{\delta(n_{2,i})}\} + \hat{N}(s)]a(s)$$

$$= \underbrace{N_{2h}[a_1 s^{\delta_{d_2}-\delta(d_{2,1})-\delta(n_{2,1})}, a_2 s^{\delta_{d_2}-\delta(d_{2,2})+\delta(n_{2,2})}, \ldots, a_r s^{\delta_{d_2}-\delta(d_{2,r})+\delta(n_{2,r})}]^{\mathrm{T}}}_{\neq 0}$$

$$+ \hat{N}(s)a(s) \tag{21}$$

In (21) note that the indicated term is not zero since there exists no a such that $[M_2]_h a = 0$, since $M_2(s)$ is column reduced. Thus we will have

$$\deg d'_{2,j}(s) < \deg n'_{2,j}(s) \quad \Rightarrow \quad \tilde{k}'_{2,j} > 0 \tag{22}$$

since $D_{2h}a = 0$ but $N_{2h}a \neq 0$. Therefore, by Theorem 4 the new column $m'_{2,j0}$, displays an infinite pole of degree $\geq \tilde{k}'_{2,j0}$.

Step 3. Repeat Steps 1 and 2 until

$$[M_3]_h = \begin{bmatrix} D_{3h} \\ D_{3h} \end{bmatrix}_h := \begin{bmatrix} D'_{3h} & D''_h & 0_{l,\varrho} \\ N'_{3h} & 0_{m,\theta} & N'''_{3h} \end{bmatrix} \tag{23}$$

is the HOCM of

$$M_3(s) = [D_3^{\mathrm{T}}(s) \quad N_3^{\mathrm{T}}(s)]^{\mathrm{T}} \tag{24}$$

where $\mathrm{rank}_{\mathbb{R}}\,(D'_{3h}D''_h) = \mathrm{rank}_{\mathbb{R}}\,(D_{2h})$ and $\varrho = r - \mathrm{rank}_{\mathbb{R}}\,(D_{2h})$. That is, stop if there exists no non-zero a such that $D_{3h}a = 0$. Equation (24) now displays complete infinite pole multiplicity.

Consider now the infinite zero multiplicity of $G(s)$.

Step 4. For simplicity, rearrange the columns of $M_3(s)$ such that

$$[M_3]_h = \begin{bmatrix} D_{3h} \\ N_{3h} \end{bmatrix}_h := \begin{bmatrix} D'_{3h} & 0_{l,\varrho} & D''_h \\ N'_{3h} & N'''_{3h} & 0_{m,\theta} \end{bmatrix} \tag{25}$$

Step 5. Repeat Steps 1 and 2 in respect of

$$[N'_{3h} \quad N'''_{3h}] \tag{26}$$

to obtain

$$[M_4]_h = \begin{bmatrix} D_{4h} \\ N_{4h} \end{bmatrix}_h := \begin{bmatrix} D'_{4h} & 0_{l,\varrho} & D'''_{4h} \\ N'_{4h} & N'''_{3h} & 0_{m,\zeta} \end{bmatrix} \tag{27}$$

as the HOCM of

$$M_4(s) = [D_4^{\mathrm{T}}(s) \quad N_4^{\mathrm{T}}(s)]^{\mathrm{T}} \tag{28}$$

such that

$$\mathrm{rank}_{\mathbb{R}}\,(D'_{4h}\ D'''_{4h}) = \mathrm{rank}_{\mathbb{R}}\,(D_{2h})$$

$$\mathrm{rank}_{\mathbb{R}}\,(N'_{4h}\ N'''_{3h}) = \mathrm{rank}_{\mathbb{R}}\,(N_{2h})$$

and $\zeta = r - \mathrm{rank}_{\mathbb{R}}\,(N_{2h})$.

Step 6. Replace the submatrix $\begin{bmatrix} D'_1(s) \\ 0_{m,\mu} \end{bmatrix}$ from $M_1(s)$ of (15 *a*) into (28) to obtain

$$[M_{\varrho\zeta}]_h = \begin{bmatrix} D_{\varrho\zeta h} \\ N_{\varrho\zeta h} \end{bmatrix}_h := \begin{bmatrix} D'_{1h} & D'_{4h} & 0_{l,\varrho} & D'''_{4h} \\ 0_{m,\mu} & N'_{4h} & N'''_{3h} & 0_{m,\zeta} \end{bmatrix} \tag{29}$$

which is the HOCM of

$$M_{\varrho\zeta}(s) = \begin{bmatrix} D'_1(s) & D_4(s) \\ 0_{m,\mu} & N_4(s) \end{bmatrix} = \begin{bmatrix} D_{\varrho\zeta}(s) \\ N_{\varrho\zeta}(s) \end{bmatrix} \tag{30}$$

where $\mathrm{rank}_{\mathbb{R}}\,(D'_{1h}D'_{4h}D'''_{4h}) = \mathrm{rank}_{\mathbb{R}}\,(D_{1h})$ and $\mathrm{rank}_{\mathbb{R}}\,(N'_{4h}N'''_{3h}) = \mathrm{rank}_{\mathbb{R}}\,(N_{1h})$.

Note that the first μ columns in (29) correspond to the number of zeros in diagonal position (i.e. the rank deficiency) in the Smith–MacMillan form at $s = \infty$. The next $l - (\zeta + \varrho + \mu)$ columns in (29) correspond to the number of unit invariant polynomials in the Smith–MacMillan form at $s = \infty$, while the next ϱ columns correspond to the multiplicity of the infinite poles of $G(s)$. The last ζ columns then correspond to the multiplicity of the infinite zeros. Thus, a CRMFD is obtained in the form (29)–(30) where $\check{k}_{\varrho\zeta,i} > 0\ \forall i = \{l - (\zeta + \varrho) + 1\}, \ldots, l - \zeta$ and $\hat{k}_{\varrho\zeta,i} > 0\ \forall i = (l - \zeta + 1), \ldots, l$. From (29)–(30), $\varrho = l - \mathrm{rank}_{\mathbb{R}}\,(D_{\varrho\zeta h})$ and $\zeta = l - \mu - \mathrm{rank}_{\mathbb{R}}\,(N_{\varrho\zeta h}) = r - \mathrm{rank}_{\mathbb{R}}\,(N_{\varrho\zeta h})$ and so the result holds.

It should be noted that all operations performed in the above algorithm are non-singular transformations on $[M_2]_h$ and thus preserve its full column rank property (i.e. the column reducedness of $[M_2]_h$).

By Theorem 3 we know that the multiplicity of the infinite zeros (respectively poles) is given by (14). Thus, $M_{\varrho\zeta}(s)$ as in (30) is a CRMFD displaying the complete infinite pole and zero multiplicity information. Therefore, using Theorem 4 on $M_{\varrho\zeta}(s)$ we may determine the complete infinite pole–zero multiplicity information of $G(s) \in \mathbb{R}(s)^{m\times l}$ by inspection.

With the use of Corollaries 3 and 4 we have the following.

Corollary 5: *Let* $G(s) \in \mathbb{R}(s)^{m\times l}$ *with* $\mathrm{rank}_{\mathbb{R}(s)}\, r \le \min(m, l)$ *be a strictly proper transfer function matrix, then all CRMFDs of* $G(s)$ *display the complete infinite zero multiplicity information.*

Proof: Let $M_1(s)$ be a CRMFD for a strictly proper $G(s)$. By observing the form (29) we can see that $[M_1]_h$ can only have the form

$$[M_1]_h = \begin{bmatrix} D_{1h} \\ N_{1h} \end{bmatrix}_h = \begin{bmatrix} D'_{1h} & D'''_{4h} \\ 0_{m,\mu} & 0_{m,\zeta} \end{bmatrix} \tag{31}$$

since $G(s)$ only possesses infinite zeros and thus the Smith–MacMillan form at $s = \infty$ of $G(s)$ will possess no unit elements and show no infinite poles. Thus $\hat{k}_{1,i} > 0\ \forall i = l - \zeta - 1, \ldots, l$ and since $\mathrm{rank}_{\mathbb{R}}\,(N_{1h}) = 0$, by Corollary 3, $G(s)$ can be said to possess $r - 0 = \zeta$ infinite zeros. □

Another question addressed in this paper concerns the ability to detect, again in an almost immediate way, when the given CRMFD displays the complete infinite zero (respectively pole) multiplicity information. The following theorem provides a simply checked necessary and sufficient condition for this.

Theorem 6: *Let* $M_1(s)$ *be a CRMFD as defined in* $(15a)$ associated with $G(s) \in \mathbb{R}(s)^{m\times l}$ with $\mathrm{rank}_{\mathbb{R}(s)}\, r \le \min(m, l)$. *Then, with* $M_2(s), D_2(s), N_2(s)$ *defined as in* $(15b)$, $M_1(s)$ *displays the complete infinite pole (respectively zero) multiplicity information if and only if the number of non-zero columns in* D_{2h} *(respectively* N_{2h}*) equals the rank of* D_{2h} *(respectively* N_{2h}*).*

Proof: From (29) it can be seen that since $(D'_{1h}D'_{4h}D'''_{4h})$ (respectively $(N'_{4h}N'''_{3h})$) has full column rank, the result holds. □

The theorem provides readily checkable conditions since the constant matrices D_{2h}, N_{2h} are just the constituent matrices from the HOCM $[M_2]_h$ of $M_2(s)$.

A further question in this vein concerns the existence of a CRMFD which displays the complete infinite zero and pole degree information. In this regard, the following theorem provides a positive answer to this question.

Theorem 7: *Let* $G(s) \in \mathbb{R}(s)^{m\times l}(s)$, $\text{rank}_{\mathbb{R}(s)}\, r \leq \min(m,l)$. *Then there always exists a CRMFD which displays the complete infinite pole and zero degree information.*

Proof: By Theorem 5 we assume that we already have a factorization which displays the complete infinite pole and zero multiplicity information. Thus, recall from the proof of Theorem 5

$$[M_{\varrho\zeta}]_h = \begin{bmatrix} D'_{1h} & D'_{4h} & 0_{l,\varrho} & D'''_{4h} \\ 0_{m,\mu} & N'_{4h} & N'''_{3h} & 0_{m,\zeta} \end{bmatrix} \tag{32}$$

which is the HOCM of a CRMFD displaying the complete infinite pole–zero multiplicity information. Recall also

$$[M_4]_h = \begin{bmatrix} D'_{4h} & 0_{l,\varrho} & D'''_{4h} \\ N'_{4h} & N'''_{3h} & 0_{m,\zeta} \end{bmatrix} \tag{33}$$

which is obtained by deleting the first μ columns of (32).

Consider first the infinite pole degrees of $G(s)$. The idea is to increase $\check{k}_{4,j}, j = r-(\zeta+\varrho)+1,\ldots,(r-\zeta)$, to the appropriate magnitudes via the following algorithm, which has some similarities to that used in Theorem 5.

Step i. Find a non-zero $a = (a_1, a_2, \ldots, a_r)^{\mathrm{T}}$ such that $[D_4]_h a = 0$. Let $\delta_{d_4} = \max\{\delta(d_{4,i})\}$. Then define

$$a(s) := [a_1 s^{\delta_{d_4}-\delta(d_{4,1})}, a_2 s^{\delta_{d_4}-\delta(d_{4,2})}, \ldots, a_r s^{\delta_{d_4}-\delta(d_{4,r})}]^{\mathrm{T}} \in \mathbb{R}[s]^{r\times 1} \tag{34}$$

Let $a_j, j = r-(\zeta+\varrho)+1,\ldots,(r-\zeta)$ be the first non-zero entry in (34) from the range $j = r-(\zeta+\varrho)+1,\ldots,(r-\zeta)$, then replace column $m_{4,j}(s)$ (where $\check{k}_{4,j} > 0$) of $M_4(s)$ by $m'_{4j}(s) = (d'_{4j}(s)\ n'_{4j}(s))$ where

$$\begin{aligned} d'_{4,j}(s) &= D_4(s)a(s) = [[D_4]_h \operatorname{diag}\{s^{\delta(d_{4,i})}\} + \grave{D}(s)]a(s) \\ &= \underbrace{[D_4]_h a s^{\delta_{d_4}}}_{=0} + \grave{D}(s)a(s) \\ &= \grave{D}(s)a(s) \end{aligned} \tag{35}$$

$$\begin{aligned} n'_{4,j}(s) &:= N_4(s)a(s) = [[N_4]_h \operatorname{diag}\{s^{\delta(n_{4,i})}\} + \grave{N}(s)]a(s) \\ &= \underbrace{[N_4]_h [a_1 s^{\delta_{d_4}-\delta(d_{4,1})+\delta(n_{4,1})}, a_2 s^{\delta_{d_4}-\delta(d_{4,2})-\delta(n_{4,2})}, \ldots, a_r s^{\delta_{d_4}-\delta(d_{4,r})-\delta(n_{4,r})}]^{\mathrm{T}}}_{\neq 0} \\ &\quad + \grave{N}(s)a(s) \end{aligned} \tag{36}$$

where $\deg \grave{D}(s) < \deg D_4(s)$ and $\deg \grave{N}(s) < \deg N_4(s)$. Thus we have $\check{k}'_{4,j} > \check{k}_{4,j}$.

Step ii. Repeat Step i until there exists no non-zero a such that $[D_5]_h a = 0$ and

$$[M_5]_h = \begin{bmatrix} D_{5h} \\ N_{5h} \end{bmatrix}_h := \begin{bmatrix} D'_{4h} & 0_{l,\varrho} & D'''_{4h} \\ N'_{4h} & N''_{4h} & 0_{m,\zeta} \end{bmatrix} \tag{37}$$

in which $D_5(s)$ will be column reduced since $[D_5]_h$ is non-singular.

Consider now the infinite zero degrees of $G(s)$. The idea now is to increase $\hat{k}_{5,j}, j = r - \zeta + 1, \ldots, r$, to the appropriate levels and this is done as follows.

Step iii. Perform analogous operations to those of Step ii in respect of $[N_5]_h$ to replace any columns in positions $j = r - \zeta + 1, \ldots, r$ of $M_5(s)$ as necessary and until

$$[M_6]_h = \begin{bmatrix} D_{6h} \\ N_{6h} \end{bmatrix}_h := \begin{bmatrix} D'_{4h} & 0_{l,\varrho} & D'''_{5h} \\ N'_{4h} & N''_{4h} & 0_{m,\zeta} \end{bmatrix} \tag{38}$$

is obtained, in which both $D_6(s)$ and $N_6(s)$ are column reduced since $[D_6]_h$ is non-singular and $[N_6]_h$ has full column rank.

Step iv. Replace the submatrix $\begin{bmatrix} D'_1(s) \\ 0_{m,\mu} \end{bmatrix}$ from $M_1(s)$ to obtain

$$[M'_{\varrho\zeta}]_h = \begin{bmatrix} D'_{\varrho\zeta h} \\ N'_{\varrho\zeta h} \end{bmatrix}_h := \begin{bmatrix} D'_{1h} & D'_{4h} & 0_{l,\varrho} & D'''_{5h} \\ 0_{m,\mu} & N'_{4h} & N''_{4h} & 0_{m,\zeta} \end{bmatrix} \tag{39}$$

which is the HOCM of $M'_{\varrho\zeta}(s)$.

Thus, a CRMFD $M'_{\rho\zeta}(s)$, is obtained which displays the complete infinite pole and zero degree information $G(s)$ (and necessarily the complete infinite pole and zero multiplicity information). □

Theorems 5 and 7 now combine as follows.

Corollary 6: *Let* $G(s) \in \mathbb{R}(s)^{m \times l}$ *with* $\text{rank}_{\mathbb{R}(s)}\, r \le \min(m, l)$. *Then there always exists a CRMFD of the form* (15) *which displays the complete infinite frequency information.*

A further question concerns the ability to detect, again in an almost immediate way, when a CRMFD displays the complete infinite zero (respectively pole) degree information. The following theorem provides a simply checked necessary and sufficient condition for this.

Theorem 8: *Let* $M_1(s)$ *be a CRMFD as defined in* (15 *a*) *associated with* $G(s) \in \mathbb{R}(s)^{m \times l}$ *with* $\text{rank}_{\mathbb{R}(s)}\, r \le \min(m, l)$. *Then, with* $M_2(s), D_2(s), N_2(s)$ *defined as in* (15 *b*), $M_1(s)$ *displays the complete infinite pole (respectively zero) degree information if and only if* $[D_2]_h$ *(respectively* $[N_2]_h$*) has full column rank.*

Proof: From the proof of Theorem 7 it can be seen that $D_6(s)$ (respectively $N_6(s)$) is column reduced and therefore $[D_6]_h$ (respectively $[N_6]_h$) has full column rank which gives rise to the stated conditions. Note that the conditions of Theorem 6 are then automatically satisfied. Then from (39), which is the HOCM of a CRMFD displaying the complete infinite pole and zero degree information, the result holds. □

The condition given in Theorem 8 is again very simple to construct and to check. The matrix $[D_2]_h$ (respectively $[N_2]_h$) is just the HOCM for the matrix $D_2(s)$ (respectively $N_2(s)$) occurring in the factorization (15). The test compares in simplicity to that given in Theorem 6 for the complete infinite pole (respectively zero) multiplicity information where D_{2h} (respectively N_{2h}) is just the matrix taken directly from the HOCM $[M_2]_h$ of $M_2(s)$. The test in Theorem 8 supersedes that in Theorem 6.

Theorem 8 takes on a particularly simple form in the case where $G(s)$ has full column rank.

Corollary 7: *Let $G(s) \in \mathbb{R}(s)^{m\times l}$ have full column rank l. Then the CRMFD $G(s) = D^{-1}(s)N(s)$ displays complete infinite pole and zero degree information if and only if both $D(s)$ and $N(s)$ are column reduced.*

Consider now the following example which will demonstrate the theory proposed in this paper.

Example 1: Consider the following $G(s) \in \mathbb{R}(s)^{4\times 4}$ with $\text{rank}_{\mathbb{R}(s)} = r = 3$

$$G(s) = \begin{pmatrix} s^4 & 0 & 0 & 0 \\ \dfrac{s^5}{s+1} & 1 & 0 & 0 \\ 0 & 0 & \dfrac{1}{s^3} & \dfrac{(s+1)}{s^3(s+3)} \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{40}$$

where a CRMFD, in the notation of (15)–(16), is given by

$$M_1(s) = \left[\begin{array}{cccc} 0 & 0 & 0 & (s+1) \\ 0 & 44s & -22s^3 - 44s^2 + 44s \quad 44 & s^4 \\ s(s+1) & 0 & 22s^3 & 0 \\ s(s+3) & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & s^4(s+1) \\ 0 & 44s & -22s^3 - 44s^2 + 44s - 44 & s^4(s+1) \\ 0 & 0 & 22 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$= \begin{bmatrix} D_1'(s) & D_2(s) \\ 0_{4,1} & N_2(s) \end{bmatrix} \tag{41}$$

where $\mu = l - r = 4 - 3 = 1$, $M_2(s) \in \mathbb{R}[s]^{8\times 3}$ as in (15*b*) is given by the last three columns of $M_1(s)$ and $N_2(s) \in \mathbb{R}[s]^{4\times 3}$ has full column $\text{rank}_{\mathbb{R}(s)}\ r = 3$.

Utilizing the theorems proposed in this paper, the following information may be obtained from (41). From Theorem 3 the multiplicity of the infinite zeros (respectively poles) of $G(s)$ is given by $\zeta = r - \text{rank}_{\mathbb{R}}(N_{1h}) = 3 - 2 = 1$ (respectively $\varrho = l - \text{rank}_{\mathbb{R}}(D_{1h}) = 4 - 3 = 1$). From Theorem 4, $M_1(s)$ displays one infinite pole of degree ≥ 1 (since $\check{k}_{2,3} = 1$). Therefore, from Theorem 6, $M_1(s)$ displays the complete infinite pole multiplicity information but no information regarding the infinite zero. However, by Theorem 5 we can obtain a CRMFD which displays this (complete) infinite zero multiplicity information. Therefore, utilizing the algorithm in the proof of Theorem 5 we have the following.

Steps 1–3. Not necessary (we already have complete infinite pole multiplicity).

Step 4. The columns of $M_2(s)$ are already of the form (24).

Step 5.

$$\left.\begin{aligned} N_{2h}a &= 0_{4,1} \\ \begin{bmatrix} 0 & 0 & 1 \\ 44 & -22 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} &= 0_{4,1} \end{aligned}\right\} \tag{42}$$

and $\delta(n_{2,1}) = 1, \delta(n_{2,2}) = 3, \delta(n_{2,3}) = \delta_{n_2} = 5$ and $a(s) = [s^4, 2s^2, 0]^{\mathrm{T}}$. Now a_1 is the first non-zero entry in $a(s)$ thus choose $m_{2,1}$ and therefore

$$M_2(s) \times \begin{bmatrix} s^4 & 0 & 0 \\ 2s^2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \left[\begin{array}{ccc} 0 & 0 & (s+1) \\ -88s^4 + 88s^3 - 88s^2 & 22s^3 - 44s^2 + 44s - 44 & s^4 \\ 44s^5 & 22s^3 & 0 \\ 0 & 0 & 0 \\ \hline 0 & 0 & s^4(s+1) \\ -88s^4 + 88s^3 - 88s^2 & -22s^3 - 44s^2 + 44s - 44 & s^4(s+1) \\ 44s^2 & 22 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$= M_3(s) = \begin{bmatrix} D_3(s) \\ N_3(s) \end{bmatrix} \tag{43}$$

It is easily seen that there exists no non-zero a such that $N_{3h}a = 0$ and also, since $\hat{k}_{3,1} = 1$, (43) now also displays one infinite zero of degree ≥ 1.

Step 6. Replacing $m_{1,1}$ from (41) we obtain

$$M_{\varrho\zeta}(s) = \left[\begin{array}{cc} d_{1,1} & D_3(s) \\ \hline n_{1,1} & N_3(s) \end{array}\right] = \begin{bmatrix} D_{\varrho\zeta}(s) \\ N_{\varrho\zeta}(s) \end{bmatrix} \tag{44}$$

which shows the complete infinite pole (respectively zero) multiplicity information since, by Theorem 6, the number of non-zero columns in D_{3h} (respectively N_{3h}) equals the rank of D_{3h} (respectively N_{3h}).

With regards to the infinite pole and zero degree information, consider the high order coefficient matrices of $D_3(s), N_3(s)$ respectively which are

$$[D_3]_h = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -22 & 1 \\ 44 & 22 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and } [N_3]_h = \begin{bmatrix} 0 & 0 & 1 \\ -88 & -22 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{45}$$

Now $[D_{\varrho\zeta}]_h$ (respectively $[N_{\varrho\zeta}]_h$) does not have full rank and therefore, by Theorem 8, $M_{\varrho\zeta}(s)$ does not display the complete infinite pole and zero degree information. However, by Theorem 7, we can obtain a CRMFD which displays this (complete) information. Therefore, if we consider the infinite pole degrees of $G(s)$, and utilizing the algorithm in the proof of Theorem 7, we have the following.

Step i.

$$\left.\begin{array}{c} [D_3]_h a = 0_{4,1} \\ \begin{bmatrix} 0 & 0 & 0 \\ 0 & -22 & 1 \\ 44 & 22 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ -44 \end{bmatrix} = 0_{4,1} \end{array}\right\} \tag{46}$$

where $\delta(d_{3,1}) = 5, \delta(d_{3,2}) = 3, \delta(d_{3,3}) = 4, \delta_{d_3} = 5$ and $a(s) = [1, \;\; 2s^2, -44s]^{\mathrm{T}}$. Now a_3 is the first non-zero entry in $a(s)$ for $j = 3$, thus choose $m_{3,3}$ (where $\tilde{k}_{3,3} > 0$). Then

$$M_3(s) \times \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2s^2 \\ 0 & 0 & -44s \end{bmatrix}$$

$$= \left[\begin{array}{ccc} 0 & 0 & -44s(s+1) \\ -88s^4 + 88s^3 - 88s^2 & -22s^3 - 44s^2 + 44s - 44 & 0 \\ 44s^5 & 22s^3 & 0 \\ 0 & 0 & 0 \\ \hline 0 & 0 & -44s^5(s+1) \\ -88s^4 + 88s^3 - 88s^2 & -22s^3 - 44s^2 + 44s - 44 & -44s^6 \\ 44s^2 & 22 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$= M_4(s) = \begin{bmatrix} D_4(s) \\ N_4(s) \end{bmatrix} \tag{47}$$

where $D_4(s)$ is now column reduced, and so automatically satisfies the rank condition on $[D_4]_h$. Therefore Step ii is omitted.

Step iii. Consider now the infinite zero degrees of $G(s)$

$$\left.\begin{array}{c} [N_4]_h a = 0_{4,1} \\ \begin{bmatrix} 0 & 0 & -44 \\ -88 & -22 & -44 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -4 \\ 0 \end{bmatrix} = 0_{4,1} \end{array}\right\} \tag{48}$$

where $\delta(n_{4,1}) = 4, \delta(n_{4,2}) = 3, \delta(n_{4,3}) = 6, \delta_{n_4} = 6$ and $a(s) = [s^2, -4s^3, 0]^{\mathrm{T}}$. Now a_1 is the first non-zero entry in $a(s)$ for $j = 1$, thus choose $m_{4,1}$ (where $\hat{k}_{4,1} > 0$). Then

$$M_4(s) \times \begin{bmatrix} s^2 & 0 & 0 \\ -4s^3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \left[\begin{array}{ccc} 0 & 0 & -44s(s+1) \\ 264s^4(s-1) - 176s^3 & -22s^3 - 44s^2 + 44s - 44 & 0 \\ 44s^6(s-2) & 22s^3 & 0 \\ 0 & 0 & 0 \\ \hline 0 & 0 & -44s^5(s+1) \\ 264s^4(s-1) + 176s^3 & -22s^3 - 44s^2 + 44s - 44 & -44s^6 \\ 44s^3(s-2) & 22 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$= M_5(s) = \begin{bmatrix} D_5(s) \\ N_5(s) \end{bmatrix} \tag{49}$$

where rank $[N_5]_h = 2$ with the number of non-zero columns being 4. Thus we repeat Step iii. Note

$$\left.\begin{array}{c} [N_5]_h a = 0_{4,1} \\ \begin{bmatrix} 0 & 0 & -44 \\ 264 & -22 & -44 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 12 \\ 0 \end{bmatrix} = 0_{4,1} \end{array}\right\} \tag{50}$$

where $\delta(n_{5,1}) = 5, \delta(n_{4,2}) = 3, \delta(n_{4,3}) = 6, \delta_{n_4} = 6$ and $a(s) = [s, 12s^3, 0]^{\mathrm{T}}$. Now a_1 is the first non-zero entry in $a(s)$ for $j = 1$, thus choose $m_{5,1}$ (where $\hat{k}_{5,1} > 0$). Then

$$M_5(s) \times \begin{bmatrix} s & 0 & 0 \\ 12s^3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$
= \left[\begin{array}{ccc}
0 & 0 & -44s(s+1) \\
-792s^5 + 704s^4 - 528s^3 & -22s^3 - 44s^2 + 44s - 44 & 0 \\
44s^7(s-2) + 264s^6 & 22s^3 & 0 \\
0 & 0 & 0 \\
\hline
0 & 0 & -44s^5(s+1) \\
-792s^5 + 704s^4 - 528s^3 & -22s^3 - 44s^2 + 44s - 44 & -44s^6 \\
44s^4(s-2) + 264s^3 & 22 & 0 \\
0 & 0 & 0
\end{array}\right]
$$

$$
M_6(s) = \begin{bmatrix} D_6(s) \\ N_6(s) \end{bmatrix} \tag{51}
$$

where $N_6(s)$ is actually column reduced and so satisfies the required rank condition on $[N_6]_h$.

Step iv. Replacing $m_{\varrho\zeta,1}$ from (44) we obtain

$$
M'_{\varrho\zeta}(s) = \left[\begin{array}{cc} d_{1,1} & D_6(s) \\ \hline n_{1,1} & N_6(s) \end{array}\right] = \begin{bmatrix} D'_{\varrho\zeta}(s) \\ N'_{\varrho\zeta}(s) \end{bmatrix} \tag{52}
$$

which, by Theorem 8 and Corollary 7, shows the complete infinite frequency information. In particular, $G(s)$ has one infinite zero of degree 3 (since $\hat{k}_{6,1} = 3$) and one infinite pole of degree 4 (since $\check{k}_{6,3} = 4$). □

6. Output feedback considerations

Various authors have considered the effect of constant output feedback on the finite poles and zeros of a transfer function matrix, and the results are well-documented. In particular, it is known that the finite zero structure of a given rational transfer function matrix is invariant under constant output feedback (see Rosenbrock 1970), although the finite pole structure does not possess such a property.

Recall that if $G(s) \subset \mathbb{R}(s)^{m \times l}$ is a transfer function matrix, and if constant output feedback, as summarized by the matrix F, is applied in the manner described in figure 1, and if $G_F(s)$ denotes the transfer function matrix of the feedback system so constructed, then

$$
G_F(s) = G(s)(I + FG(s))^{-1} = (I + G(s)F)^{-1}G(s) \tag{53}
$$

where

$$
|I_l + FG(s)| = |I_m + G(s)F| \neq 0 \tag{54}
$$

Now, Pugh and Ratcliffe (1980) considered whether the state of affairs mentioned

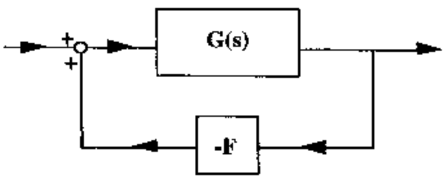

Figure 1. Feedback system.

above with the case of finite poles and zeros persists when one considers the infinite poles and zeros. They proved that if constant output feedback is applied to a minimal MFD of $G(s)$ then $G_F(s)$ is unaffected in its minimality. This result naturally applies to CRMFDs since a minimal MFD is also a CRMFD. Therefore, with regards to the information displayed by CRMFDs the following theorem may be established.

Theorem 9: *Let $M(s)$ be a CRMFD displaying the complete infinite frequency information associated with $G(s) \subset \mathbb{R}(s)^{m\times l}$ with $\operatorname{rank}_{\mathbb{R}(s)} r \leq \min(m, l)$. Then under constant output feedback*

$$G_F(s) = N(s)(D(s) + FN(s))^{-1} \tag{55}$$

is a CRMFD $M'(s)$, displaying the complete infinite zero multiplicity and degree information.

Proof: We know that $M'(s)$ is a CRMFD (see Pugh and Ratcliffe 1980) and so it must now be established that this CRMFD displays the complete infinite zero multiplicity and degree information.

Consider the high order coefficient matrix

$$[M']_h = \begin{bmatrix} I_l & F \\ O & I_m \end{bmatrix} \begin{bmatrix} D'_{1h} & D'_{4h} & 0_{l,\varrho} & D'''_{5h} \\ 0_{m,\mu} & N'_{4h} & N''_{4h} & 0_{m,\zeta} \end{bmatrix} = \begin{bmatrix} D'_{1h} & D'_{5h} & FN''_{4h} & D'''_{5h} \\ 0_{m,\mu} & N'_{4h} & N''_{4h} & 0_{m,\zeta} \end{bmatrix} \tag{56}$$

Let n_i, d'_i denote the ith column of $N(s), (D(s) + FN(s))$ respectively and $\delta(n_i), \delta(d'_i)$ their respective degrees. It is readily seen that, with regards to the infinite zero structure the column property $\delta(d'_i) > \delta(n_i), i = \{l - \zeta + 1\}, \ldots, l$, is still satisfied and the minimal differences $\hat{k}_i, i = \{l - \zeta + 1\}, \ldots, l$, will remain invariant since F is constant. However, with regards to the infinite pole structure it is clear that the column property $\delta(d'_i) < \delta(n_i), i = \{l - (\zeta + \varrho) + 1\}, \ldots, (l - \zeta)$, is destroyed if F has full column rank and, in particular, $\bar{k}_i = 0, i = \{l - (\zeta + \varrho) + 1\}, \ldots, (l - \zeta)$, and so displays no infinite pole information. If F does not have full rank then some or all of the infinite pole information may be destroyed. The infinite pole structure possesses no such invariant property and so the result holds. □

7. Conclusions

This paper has highlighted a form of MFD, the so-called CRMFD which carries a number of interesting properties as regards its ability to display certain aspects of the infinite frequency structure of the associated transfer function matrix. Many of these properties are revealed virtually by inspection and so provide satisfying extensions of the scalar case where again this information is immediately available. In addition, the paper has given a computationally attractive test for the absence of infinite poles and zeros. Further, a structured form of a CRMFD (respectively RRMFD), which conveniently displays the complete infinite frequency structure information, has been developed and a simply checked necessary and sufficient condition has been provided regarding when a CRMFD (respectively RRMFD) displays the complete infinite frequency structure information. Extensions of the results of Kailath (1980) and Pugh and Ratcliffe (1980) have also been provided in an examination of the output feedback invariance of certain of the concepts introduced.

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