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Solvability, reachability, controllability and observability of regular PMDs

S. MAHMOOD†, N. P. KARAMPETAKIS‡ and A. C. PUGH†

A recently developed algorithm for obtaining an 'equivalent' generalized state-space representation of a linear multivariable system whose dynamics are expressed by polynomial matrix descriptions (PMDs) is used to present the solution and to analyse and examine certain pseudostate reachability, controllability and observability properties of the associated PMD.

1. Introduction

The study of generalized state space (GSS) systems has been a leading interest for many researchers for the last two decades. The reason for this interest is that such systems extend the range of linear phenomena that can be discussed, beyond those that can be described by the conventional state-space model. Analytic solutions of GSS systems have been given by a number of authors and by a number of different techniques. For example, the solution has been given via the Weierstrass form, the Luenberger 'shuffle' algorithm and the Silverman Structure Algorithm (see for example the survey by Lewis (1986) and the associated references). Structural properties of GSS systems such as controllability, reachability and observability have been studied either through algebraic analysis (Verghese 1978, Yip and Sincovic 1981), or through geometric analysis (see for example Ozcaldiran 1986, Ozcaldiran *et al.* 1992).

GSS and state space systems belong to a class of more general systems called polynomial matrix descriptions (PMDs). In the case of PMDs, the generalized (smooth and impulsive) solution space has been of interest since the initial work of Verghese (1978). The concepts of controllability, reachability and observability have essentially been studied primarily from an algebraic point of view (see Vardulakis 1991, and references). In this paper the solvability, controllability and observability properties of PMDs are studied through the medium of an equivalent generalized state-space description and in this way an analytic study is facilitated. The framework for this development is based on the fact that any PMD can be related to an equivalent generalized state-space representation through certain bijections between the corresponding solution–input spaces, or indeed the corresponding initial condition output spaces (Karampetakis and Vardulakis 1995). Consequently, the particular property of the PMD can be described through the more easily, and analytically, described property of the equivalent generalized state-space system.

Specifically, the generalized solution of a PMD is presented which, in contrast to

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† Department of Mathematical Sciences, The University of Technology, Loughborough, Leicestershire LE11 3TU, UK. E-mail: A.C.Pugh@lboro.ac.uk; Fax: +44 01509 211869.
‡ Department of Mathematics, Aristotle University, Thessaloniki 54006, Greece.

Fragulis and Vardulakis (1995), considers the complete model equations and not simply the dynamical equations. This approach is further able to take into account the role of the non-zero initial conditions which may be present on the input. Subsequently, the controllable, reachable and observable spaces are characterized in a manner that is a transparent extension of the results of Yip and Sincovic (1981) and Cobb (1984). The specific results concerning controllability and observability are equivalent to those given by Fragulis and Vardulakis (1995), but in this presentation emerge in a much more transparent way.

2. Structural properties of generalized state-space systems

Consider the generalized state-space system

$$E\dot{x}(t) = Ax(t) + Bu(t) \tag{1a}$$

$$y(t) = Cx(t) \tag{1b}$$

where $\rho E_n - A \in \mathbb{R}[\rho]^{l\times l}$, $B \in \mathbb{R}^{l\times m}$, $C \in \mathbb{R}^{p\times l}$, $\text{rank}_{\mathbb{R}}\, E < l$ (in which case $n := \deg|\rho E - A| < l$) and where $x(t) : [0-, +\infty) \to \mathbb{R}^l$ is the state vector, $u(t) : [0-, +\infty) \to \mathbb{R}^m$ is the input vector and $y(t) : [0-, +\infty) \to \mathbb{R}^p$ is the output vector of (1). In case $E \in \mathbb{R}^{l\times l}$ is non-singular then (1) is the known state-space representation widely studied by Kailath (1980), Rosenbrock (1970) and Wolovich (1974).

Assuming that the pencil $sE - A \in \mathbb{R}[s]^{l\times l}$ has at least one zero at $s = \infty$ then (1) may also be written in *standard canonical form* (Yip and Sincovec 1981)

$$\dot{x}_1 = E_1 x_1(t) + B_1 u(t) \tag{2a}$$

$$E_2\dot{x}_2(t) = x_2(t) + B_2 u(t) \tag{2b}$$

$$y(t) = [C_1\, C_2]\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \tag{2c}$$

where $E_1 \in \mathbb{R}^{r_1\times r_1}$, $E_2 \in \mathbb{R}^{r_2\times r_2}(E_2^\eta = 0, E_2^{\eta-1} \neq 0)$ with $r_1 + r_2 = l$, $B_1 \in \mathbb{R}^{r_1\times m}$, $B_2 \in \mathbb{R}^{r_2\times m}$ and η is the degree of nilpotency of E_2.

Consider (2) with $u(t) : [0-, \infty) \to \mathbb{R}^l$ then the complete solution $x_{\text{com}}(t)$, of (2) with non-zero initial values $x^{(i)}(0-)$, is given by (Vardulakis 1991)

$$x_{\text{com}}(t) = x_1(t) + x_2(t) \tag{3a}$$

where

$$x_1(t) = e^{E_1 t}x_1(0-) + \int_{0-}^{t} e^{E_1(t-\tau)}B_1 u(\tau)\, d\tau \tag{3b}$$

$$x_2(t) = -\sum_{i=0}^{\eta-1} E_2^i B_2 u^{(i)}(t) + \sum_{i=0}^{\eta-2}\delta^{(i)}(t)E_2^i\left[(-1)E_2 x_2(0-) + \sum_{j=0}^{\eta-2-i} E_2^{j-1}B_2 u^{(j)}(0-)\right] \tag{3c}$$

where $u^{(i)}(t)$ denotes the ith regular derivative of $u(t)$ (see Vardulakis 1991, p. 224). As can be seen from (3c), the complete solution of the generalized state space system (2) exhibits impulsive behaviour and in this regard, we call a $x(0-) = [x_1(0-)^T x_2(0-)^T]^T \in \mathbb{R}^l$ an *impulse free initial value* for (2) if there exists an input $u(t)$ such that the complete solution $x_{\text{com}}(t)$ is continuously differentiable on $[0, T]$ for some $T > 0$ (i.e. $x_{\text{com}}(t)$ is 'impulse free' on $[0, T]$). This then leads to the following.

Definition 1: The space of initial values

$$\mathcal{A}_{in} := \left\{x(0-) = \begin{bmatrix} x_1(0-) \\ x_2(0-) \end{bmatrix} : x_1(0-) \in \mathbb{R}^{r_1}, x_2(0-) \in \ker E_2 + \sum_{i=0}^{q-2} E_2^i \operatorname{Im} B_2 \in \mathbb{R}^{r_2} \right\} \tag{4}$$

is called the set of *impulse free initial values* for (2). □

Notice that $x(0-) = 0 \in \mathbb{R}^l$ belongs to $\mathcal{A}_{in}$ because we can take $x_1(0-) = 0 \in \mathbb{R}^{r_1}$ and $u(t)$ such that $u^{(i)}(0-) = 0, i = 0, 1, \ldots, r_2 - 2$. With this background we can now define state reachability, controllability and observability for (2).

Definition 2: Given a point $x_0 = x(0-) \in \mathcal{A}_{in}$, a point $x_T \in \mathbb{R}^l$ is *state reachable* from x_0 if there exists a $u(t)$ and $T > 0-$ such that $x_{\text{com}}(t)$ is continuously differentiable on $[0, T]$ and $x_{\text{com}}(T) = x_T$. If $x_0 = x(0-) \neq 0$ and $x_{\text{com}}(T) = x_T = 0 \in \mathbb{R}^l$, i.e. if the 'origin' $0 \in \mathbb{R}^l$ is state reachable from x_0 then we say that x_0 is *state controllable*. A point $x_T \in \mathbb{R}^l$ is *state observable* if the evolution of $x(T)$ for $T \geq 0-$ may be determined using only knowledge of $u(T)$ and $y(T)$ for $T \geq 0$. □

We define the notation $\langle \cdot | \cdot \rangle$ for an arbitrary matrix pair (E, B), where E is a square matrix and the product EB is well defined

$$\langle E|B \rangle := \operatorname{Im} B + E \operatorname{Im} B + \cdots + E^{j-1} \operatorname{Im} B \tag{5}$$

where j is the degree of nilpotency of E and $\operatorname{Im} B = \{x : x = By, \text{ for all possible } y\}$. Let now $\mathbf{R}$ be the complete set of $x_T \in \mathbb{R}^l$ state reachable from any $x_0 \in \mathcal{A}_{in}$ and $\mathbf{C}$ be the set of $x_0 \in \mathbb{R}^l$ which can reach $x_T = 0$.

Lemma 1 (Vardulakis 1991, Yip and Sincovec 1981):

(a) *The state reachable subspace of* (2) *is given by* $\mathbf{R} = \{\langle E_1|B_1 \rangle \oplus \langle E_2|B_2 \rangle\}$ *and thus every* $x_T \in \mathbb{R}^l$ *is state reachable if and only if*

$$\operatorname{rank}_{\mathbb{R}} \begin{bmatrix} Q_1 & 0_{r_1 \times qm} \\ 0_{r_2 \times r_1 m} & Q_2 \end{bmatrix} \in \mathbb{R}^{(r_1+r_2)\times(r_1+q)m} = l \tag{6}$$

(b) *The state controllable subspace of* (2) *is given by* $\mathbf{C} = \{\langle E_1|B_1 \rangle \oplus [\langle E_2|B_2 \rangle + \ker E_2]\}$ *and thus every* $x_0 \in \mathbb{R}^l$ *is state controllable if and only if*

$$\operatorname{rank}_{\mathbb{R}} \begin{bmatrix} Q_1 & 0_{r_1 \times (qm+\sigma)} \\ 0_{r_2 \times r_1 m} & Q_2, N(E_2) \end{bmatrix} \in \mathbb{R}^{(r_1+r_2)\times[m(r_1-q)+\sigma]} = l \tag{7}$$

(c) *Every* $x_T \in \mathbb{R}^l$ *is state observable if and only if*

$$\operatorname{rank}_{\mathbb{R}} \begin{bmatrix} C_1 \\ C_1 E_1 \\ \vdots \\ C_1 E_1^{r_1-1} \end{bmatrix} = r_1 \tag{8}$$

where $Q_1 := [B_1, E_1 B_1, \ldots, E_1^{r_1-1} B_1] \in \mathbb{R}^{r_1 \times r_1 m}, Q_2 := [B_2, E_2 B_2, \ldots, E_2^{q-1} B_2] \in \mathbb{R}^{r_2 \times qm}$ *are the finite and infinite state reachability matrices respectively of* (2) *and where* $N(E_2) \in \mathbb{R}^{r_2 \times \sigma}, \sigma = r_2 - \epsilon, \epsilon = \operatorname{rank}_{\mathbb{R}} E_2$, *a basis matrix for* $\ker E_2$.

Thus, it is evident from the above that state reachability of (2) implies its state controllability but the converse is not true.

3. Reduction of a PMD to an equivalent generalized state-space system

Consider a linear time invariant multivariable system Σ described by a PMD :

$$A(\rho)\beta(t) = B(\rho)u(t) \tag{9a}$$

$$y(t) = C(\rho)\beta(t) + D(\rho)u(t) \tag{9b}$$

where $(\rho = d/dt)$, $A(\rho) \in \mathbb{R}[\rho]^{l\times l}$ with $|A(\rho)| \neq 0$, $B(\rho) \in \mathbb{R}[\rho]^{l\times m}$, $C(\rho) \in \mathbb{R}[\rho]^{p\times l}$, $D(\rho) \in \mathbb{R}[\rho]^{p\times m}$. $\beta(t) : [0-, \infty) \to \mathbb{R}^l$ is the *pseudo state* of Σ, $u(t) : [0-, \infty) \to \mathbb{R}^m$ is the *control input* and $y(t)$ the *output* of Σ. Σ may be written in the form

$$\underbrace{\begin{bmatrix} A(\rho) & B(\rho) & 0 \\ -C(\rho) & D(\rho) & I_p \\ 0 & I_m & 0 \end{bmatrix}}_{\mathcal{T}(\rho)} \underbrace{\begin{bmatrix} \beta(t) \\ -u(t) \\ y(t) \end{bmatrix}}_{\xi(t)} = \underbrace{\begin{bmatrix} 0 \\ 0 \\ I_m \end{bmatrix}}_{\mathcal{U}} u(t) \tag{10a}$$

$$y(t) = \underbrace{\begin{bmatrix} 0 & 0 & I_p \end{bmatrix}}_{\mathcal{V}} \underbrace{\begin{bmatrix} \beta(t) \\ -u(t) \\ y(t) \end{bmatrix}}_{\xi(t)} \tag{10b}$$

where $\mathcal{T}(\rho) = \mathcal{T}_0 + \mathcal{T}_1\rho + \cdots + \mathcal{T}_q\rho^q \in \mathbb{R}[\rho]^{r\times r}$ with $r = l + p + m = \text{rank } (\mathcal{T}(\rho))$, $\mathcal{U} \in \mathbb{R}^{r\times m}, \mathcal{V} \in \mathbb{R}^{p\times r}$, $\xi(t) : [0-, +\infty) \to \mathbb{R}^r$ is the pseudostate of the *normalized system* (10).

Karampetakis and Vardulakis (1995) proposed an algorithm which reduces a general PMD in normalized form (10) to a generalized state-space system with the same finite and infinite frequency properties. The equivalent generalized state-space system can be found to be of the form

$$\underbrace{\begin{pmatrix} I_n & 0_{n,\mu} \\ 0_{\mu,n} & -J_\infty \end{pmatrix}}_{E} \underbrace{\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix}}_{\dot{x}(t)} = \underbrace{\begin{pmatrix} J & 0_{n,\mu} \\ 0_{\mu,n} & -I_\mu \end{pmatrix}}_{A} \underbrace{\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}}_{x(t)} + \underbrace{\begin{pmatrix} B\mathcal{U} \\ B_\infty\mathcal{U} \end{pmatrix}}_{B} u(t) \tag{11a}$$

$$y(t) = \underbrace{\begin{pmatrix} \mathcal{V}C & \mathcal{V}C_\infty \end{pmatrix}}_{C} \underbrace{\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}}_{x(t)} \tag{11b}$$

where $[C \in \mathbb{R}^{r\times n}, J \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times r}]$ is a minimal realization of $H_{sp}(s)$ (the strictly proper part of $\mathcal{T}(s)^{-1}$) and $[C_\infty \in \mathbb{R}^{r\times\mu}, J_\infty \in \mathbb{R}^{\mu\times\mu}, B_\infty \in \mathbb{R}^{\mu\times r}]$ is a minimal realization of $H_{pol}(s)$ (the polynomial part of $\mathcal{T}(s)^{-1}$), i.e.

$$\mathcal{T}(s)^{-1} = C(sI_n - J)^{-1}B + C_\infty(I_\mu - sJ_\infty)^{-1}B_\infty = (C \ \ C_\infty)(sE - A)^{-1}\begin{pmatrix} B \\ B_\infty \end{pmatrix} \tag{12}$$

The following theorem shows that the PMD (10) and the equivalent generalized state-space system (11) are related via bijective maps between their solution/input pairs.

Theorem 1: *The maps between the solution/input pairs of the PMD* (10) *and the generalized state space system* (11)

$$\begin{pmatrix} \xi(t) \\ -u(t) \end{pmatrix} = \begin{pmatrix} (C \;\; C_\infty) & 0_{r,m} \\ 0_{m,(n-\mu)} & I_m \end{pmatrix} \begin{pmatrix} x(t) \\ -u(t) \end{pmatrix} \tag{13 a}$$

$$\begin{pmatrix} x(t) \\ -u(t) \end{pmatrix} = \begin{pmatrix} (\rho E - A)^{-1} \begin{pmatrix} B \\ B_\infty \end{pmatrix} \mathcal{T}(\rho) & 0_{(n+\mu),m} \\ 0_{m,r} & I_m \end{pmatrix} \begin{pmatrix} \xi(t) \\ -u(t) \end{pmatrix} \tag{13 b}$$

are both bijective.

Proof: It is known (Pugh *et al.* 1994) that from the equivalence described by the polynomial transformations

$$\begin{pmatrix} \mathcal{T}(s)(C \;\; C_\infty)(sE - A)^{-1} & 0 \\ 0 & I_p \end{pmatrix} \begin{pmatrix} sE - A & \mathcal{B} \\ -\mathcal{C} & 0 \end{pmatrix} = \begin{pmatrix} \mathcal{T}(s) & \mathcal{U} \\ -\mathcal{V} & 0 \end{pmatrix} \begin{pmatrix} (C \;\; C_\infty) & 0 \\ 0 & I_m \end{pmatrix} \tag{14 a}$$

$$\begin{pmatrix} \begin{pmatrix} B \\ B_\infty \end{pmatrix} & 0 \\ 0 & I_p \end{pmatrix} \begin{pmatrix} \mathcal{T}(s) & \mathcal{U} \\ -\mathcal{V} & 0 \end{pmatrix} = \begin{pmatrix} sE - A & \mathcal{B} \\ -\mathcal{C} & 0 \end{pmatrix} \begin{pmatrix} (sE - A)^{-1} \begin{pmatrix} B \\ B_\infty \end{pmatrix} \mathcal{T}(s) & 0 \\ 0 & I_m \end{pmatrix} \tag{14 b}$$

(or any such transformations) where certain conditions are satisfied, the right transforming polynomial matrices give rise to bijective maps between the solution input spaces of the two systems. Therefore, the result holds. □

Consider now the Laplace transform of (10) given as

$$\hat{\xi}(s) = \mathcal{T}(s)^{-1}(\hat{\alpha}_\mathcal{T}(s) + \mathcal{U}\hat{u}(s)) \tag{15}$$

where $\hat{\alpha}_\mathcal{T}(s)$ is the *initial condition* vector associated with the *initial values* of $\xi(t)$ and its $(q-1)$ derivatives at $t = 0-$, i.e. $\xi(0-), \xi^{(1)}(0-), \ldots, \xi^{(q-1)}(0-)$ given by

$$\hat{\alpha}_\mathcal{T}(s) = [s^{q-1}I, s^{q-2}I, \ldots, I] \begin{bmatrix} \mathcal{T}_q & 0 & \cdots & 0 \\ \mathcal{T}_{q-1} & \mathcal{T}_q & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ \mathcal{T}_1 & \mathcal{T}_2 & \cdots & \mathcal{T}_q \end{bmatrix} \begin{bmatrix} \xi(0-) \\ \xi^{(1)}(0-) \\ \vdots \\ \xi^{(q-1)}(0-) \end{bmatrix}$$

$$=: \mathcal{S}_{q-1} \mathcal{X}_\mathcal{T} \hat{\xi}(0-) \tag{16}$$

Equivalence between the generalized state-space system and the PMD basically requires that the solution spaces of two equivalent systems are isomorphic. The initial conditions represent a section of the solution space taken at time $t = 0-$, and thus represent a set of points from which solutions emanate at this time. It can therefore be expected that the bijection which exists between the solution spaces of equivalent systems would induce a bijection between the corresponding sets of initial conditions. Further, since the initial condition sets are merely sets of 'points' it could be expected that this bijection would be a constant map (see also Pugh *et al.* 1995). These conjectures are confirmed by the following result.

Theorem 2: *There exists bijective maps between the initial conditions* $\mathcal{X}_T\hat{\xi}(0-)$, *of the PMD* (10) *and the initial conditions* $Ex(0-)$, *of the generalized state-space system* (11) *of the form*

$$\mathcal{X}_T\hat{\xi}(0-) = [\mathcal{X}_T\bar{\mathcal{X}}_T] \underbrace{\begin{bmatrix} C & \\ CJ & \\ \vdots & 0_{qr,\mu} \\ CJ^{q-1} & \\ & C_\infty J_\infty^{q-1} \\ 0_{qr,n} & \vdots \\ & C_\infty J_\infty \\ & C_\infty \end{bmatrix}}_{H} Ex(0-),$$

$$\text{where } \mathcal{X}_T := \begin{bmatrix} \mathcal{T}_0 & \mathcal{T}_1 & \cdots & \mathcal{T}_{q-1} \\ 0 & \mathcal{T}_0 & \cdots & \mathcal{T}_{q-2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathcal{T}_0 \end{bmatrix} \tag{17a}$$

$$E\begin{bmatrix} x_1(0-) \\ x_2(0-) \end{bmatrix} = \begin{bmatrix} J^{q-1}B, J^{q-2}B, \ldots, B & 0_{n,q\mu} \\ 0_{\mu,qn} & -J_\infty[B_\infty, J_\infty B_\infty, \ldots, J_\infty^{q-1}B_\infty] \end{bmatrix} \times \begin{bmatrix} \mathcal{X}_T \\ \bar{\mathcal{X}}_T \end{bmatrix}\hat{\xi}(0-) \tag{17b}$$

Proof: For (17 *a*), consider first the bijective map (13 *a*) between the pseudostate of the PMD (10) and the state of the generalized state-space system (11) in the frequency domain as

$$\hat{\xi}(s) = (C \;\; C_\infty)\hat{x}(s) \quad \overset{(15)}{\Rightarrow} \tag{18a}$$

$$\mathcal{T}(s)^{-1}(\mathcal{S}_{q-1}\mathcal{X}_T\hat{\xi}(0-) + \mathcal{U}\hat{u}(s)) = (C \;\; C_\infty)\hat{x}(s) \tag{18b}$$

Taking Laplace transforms in (11 *a*) we have that

$$(sE - A)\hat{x}(s) = Ex(0-) + \begin{pmatrix} B \\ B_\infty \end{pmatrix}\mathcal{U}\hat{u}(s) \tag{19}$$

and thus (18 *b*) can be written as

$$\mathcal{T}(s)^{-1}(\mathcal{S}_{q-1}\mathcal{X}_T\hat{\xi}(0-) + \mathcal{U}\hat{u}(s)) = (C \;\; C_\infty)(sE - A)^{-1}\left[Ex(0-) + \begin{pmatrix} B \\ B_\infty \end{pmatrix}\mathcal{U}\hat{u}(s)\right] \overset{(12)}{\Rightarrow}$$

$$\mathcal{S}_{q-1}\mathcal{X}_T\hat{\xi}(0-) = \mathcal{T}(s)(C \;\; C_\infty)(sE - A)^{-1}Ex(0-) \tag{20}$$

Consider now the right-hand side of (20) which, in view of (10)–(12), can be written as

$$(T_0 + T_1 s + \cdots + T_q s^q)(C s^{-1} + CJ s^{-2} + \cdots + C_\infty + C_\infty J_\infty s + \cdots + C_\infty J_\infty^{\eta-1} s^{\eta-1}) E x(0-) \tag{21}$$

where η is the degree of nilpotency of J_∞. Equating now the coefficient matrices of $s^i, i = 0, 1, \ldots, q-1$ in (20)–(21) gives (17 *a*) which is a constant mapping (since $({}_l\mathcal{X}_T \mathcal{X}_T^{\dagger} H)$ is constant, and since it clearly assigns a unique image) between the two sets of initial conditions. Further, it will be a bijection since the map (13 *a*) from which it is derived is a bijection.

Similarly, (17 *b*) may be obtained using the map (13 *b*) or see Vardulakis (1991, p. 208). □

Theorems 1 and 2 now allow the transfer of the solution and all the state reachability, controllability and observability spaces of the easily studied equivalent generalized state-space system (11) to the solution and pseudostate reachability, controllability and observability spaces of the PMD (10).

4. Structural properties of PMDs

The following theorem shows that the complete solution of the PMD (10) may be given using the bijective map (13 *a*) between solutions of the equivalent generalized state-space system (11) and the PMD (10). Here the complete solution is given in terms of the *regular* (or *ordinary*) derivative of $u(t)$ (as opposed to the *distributional* derivative) and can be seen as an extension of Vardulakis (1991), who has provided this for the generalized state-space case (see (3)).

Theorem 3: *The complete solution $\xi_{\text{com}}(t)$, of the PMD (10) is given as*

$$\xi_{\text{com}}(t) = \xi_1(t) + \xi_2(t) \tag{22 a}$$

where

$$\xi_1(t) = C e^{Jt} [J^{q-1}B, J^{q-2}B, \ldots, B] \mathcal{X}_T \hat{\xi}(0-) + \int_{0-}^{t} C e^{J(t-\tau)} B\mathcal{U} u(\tau)\, d\tau$$

$$\xi_2(t) = \sum_{i=0}^{\eta-1} C_\infty J_\infty^i B_\infty \mathcal{U} u^{[i]}(t) + \sum_{i=0}^{\eta-2} \delta^{(i)}(t) C_\infty J_\infty^i [(-1) J_\infty [B_\infty, J_\infty B_\infty, \ldots, J_\infty^{q-1} B_\infty] \bar{\mathcal{X}}_T \hat{\xi}(0-)$$
$$+ \sum_{j=0}^{\eta-2-i} J_\infty^{j+1} B_\infty \mathcal{U} u^{[j]}(0-)] \tag{22 b}$$

and where $u^{[i]}(t)$ denotes the ith regular derivative of $u(t)$ and η is the degree of nilpotency of J_∞.

Proof: Using the standard canonical form (2), the equivalent generalized state-space system (11) may be written in the form

$$\dot{x}_1(t) = J x_1(t) + B\mathcal{U} u(t) \tag{23 a}$$

$$J_\infty \dot{x}_2(t) = x_2(t) + B_\infty \mathcal{U} u(t) \tag{23 b}$$

$$y(t) = \mathcal{V} C x_1(t) + \mathcal{V} C_\infty x_2(t) \tag{23 c}$$

and the complete solution of (23) is given by (3) where $E_1 = J$, $B_1 = B\mathcal{U}$,

$E_2 = J_\infty$, $B_2 = B_\infty \mathcal{U}$, $C_1 = \mathcal{V}C$, $C_2 = \mathcal{V}C_\infty$. By Theorem 1, the solution spaces of the PMD (10) and the equivalent generalized state-space system (23) are related through the bijection (13) and, in particular, $\xi(t)$ may be obtained by applying the bijective map (CC_∞) on $x(t)$. Thus, the complete solution of the PMD (10) is given by

$$\xi_{\text{com}}(t) = [CC_\infty]\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \xi_1(t) + \xi_2(t) \tag{24a}$$

where

$$\xi_1(t) = C\mathrm{e}^{Jt}x_1(0-) + \int_0^t C\mathrm{e}^{J(t-\tau)}B\mathcal{U}u(\tau)\,\mathrm{d}\tau$$

$$\xi_2(t) = \sum_{i=0}^{\eta-1} C_\infty J_\infty^i B_\infty \mathcal{U}u^{[i]}(t) + \sum_{i=0}^{\eta-2} \delta^{(i)}(t) C_\infty J_\infty^i$$

$$\times \left[(-1)J_\infty x_2(0-) + \sum_{j=0}^{\eta-2-i} J_\infty^{j+1} B_\infty \mathcal{U}u^{[j]}(0-)\right] \tag{24b}$$

Substituting for $x_1(0-), x_2(0-)$ (defined in terms of $\hat{\xi}(0-)$ in Theorem 2) and the result holds. □

Therefore, Theorem 3 has shown that the complete solution of a PMD is readily computed by applying the bijective map (13) to the solution space of a generalized state-space system.

The solution (24) of the PMD (10) is in agreement with that established by Mahmood (1996) where it is shown that (24) may be written almost entirely in terms of the coefficient matrices obtained from the Laurent expansion of $T(s)^{-1}$. These coefficient matrices are easily obtained (see Fragulis *et al.* 1991) and therefore the formula for the complete solution of a PMD is computationally attractive. It is also shown in Mahmood (1996) that by writing the closed formula for the solution of a PMD in terms of the regular derivative of $u(t)$ (as opposed to the distributional derivative) one can correctly characterize the impulse free initial values for the PMD (10). This leads to the following.

Definition 3: The space of initial values

$$\mathcal{A}_i := \Bigg\{\hat{\xi}(0-) : \Bigg[(-1)J_\infty[B_\infty, J_\infty B_\infty, \ldots, J_\infty^{q-1}B_\infty]\bar{X}_T T\hat{\xi}(0-)$$

$$+ \sum_{j=0}^{\eta-2-i} J_\infty^{j+1} B_\infty \mathcal{U}u^{[j]}(0-)\Bigg] = 0, i = 0, \ldots, \eta-2\Bigg\} \tag{25}$$

for a given $u(t)$, is called *the set of impulse free initial values* for the PMD (10). □

It is easily seen from (24) that under the above set of impulse free initial values, the solution (24) of the PMD (10) will be free of impulses (in case $u(t)$ is smooth).

With this background we can now define pseudostate reachability and controllability and observability for the PMD (10).

Definition 4: Given a point $\xi_0 = \xi(0-) \in \mathcal{A}_i$, a point $\xi_T \in \mathbb{R}^r$ is *pseudostate reachable* from ξ_0 if there exists a $u(t)$ and $T > 0$ such that $\xi_{\text{com}}(t)$ is continuously differentiable on $[0, T]$ and $\xi_{\text{com}}(T) = \xi_T$. If $\xi_0 = \xi(0-) \neq 0$ and $\xi_{\text{com}}(T) = \xi_T = 0 \in \mathbb{R}^r$, i.e. if the 'origin' $0 \in \mathbb{R}^r$ is pseudostate reachable from ξ_0

then we say that ξ_0 is *pseudostate controllable*. A point $\xi_T \in \mathbb{R}^r$ is *pseudostate observable* if the evolution of $\xi(T)$ for $T \geq 0-$ may be determined using only knowledge of $u(T)$ and $y(T)$ for $T \geq 0$. ⌟

Let now $\mathbf{R_p}$ be the complete set of $\xi_T \in \mathbb{R}^r$ pseudostate reachable from $\xi_0 = \xi(0-) \in \mathcal{A}_i$ and let $\mathbf{C_p}$ be the complete set of $\xi_0 \in \mathbb{R}^r$ which are pseudostate controllable. Using now the bijective map (13 *a*) the following pseudostate reachable, controllable and observable criteria of the PMD may be proposed.

Theorem 4:

(*a*) *The pseudostate reachable subspace of* (10) *is given by* $\mathbf{R_p} = [C, C_\infty]\{\langle J|B\mathcal{U}\rangle \oplus \langle J_\infty|B_\infty\mathcal{U}\rangle\}$ *and thus every* $\xi_T \in \mathbb{R}^r$ *is pseudostate reachable if and only if*

$$\operatorname{rank}_{\mathbb{R}} [C, C_\infty]\begin{bmatrix} Q_{1p} & 0_{n,\mu m} \\ 0_{\mu,nm} & Q_{2p} \end{bmatrix} = r \tag{26}$$

(*b*) *The pseudostate controllable subspace is given by* $\mathbf{C_p} = [C, C_\infty]\{\langle J|B\mathcal{U}\rangle \oplus \{\langle J_\infty|B_\infty\mathcal{U}\rangle + \ker J_\infty\}\}$ *and thus every* $\xi_0 \subset \mathbb{R}^r$ *is pseudostate controllable if and only if*

$$\operatorname{rank} [C, C_\infty]\begin{bmatrix} Q_{1p} & 0_{n,(\mu+\zeta)} \\ 0_{\mu,nm} & Q_{2p}, M(J_\infty) \end{bmatrix} = r \tag{27}$$

(*c*) *Every* $\xi_T \in \mathbb{R}^r$ *is pseudostate observable if and only if*

$$\operatorname{rank}_{\mathbb{R}} \begin{bmatrix} \mathcal{V}C \\ \mathcal{V}CJ \\ \vdots \\ \mathcal{V}CJ^{n-1} \end{bmatrix} = n \tag{28}$$

where $Q_{1p} := [B\mathcal{U}, JB\mathcal{U}, \ldots, J^{n-1}B\mathcal{U}] \in \mathbb{R}^{n \times nm}$, $Q_{2p} := [B_\infty\mathcal{U}, J_\infty B_\infty\mathcal{U}, \ldots, J_\infty^{\mu-1}B_\infty\mathcal{U}] \in \mathbb{R}^{\mu \times \mu m}$ *are the finite and infinite pseudostate reachability matrices respectively of the PMD* (10) *and* $M(J_\infty) \in \mathbb{R}^{\mu \times \zeta}$, $\zeta = \mu - \epsilon, \epsilon = \operatorname{rank}_{\mathbb{R}} J_\infty$, *a basis matrix for* $\ker J_\infty$.

Proof: Consider the equivalent generalized state-space system (23). From Lemma 1 the state reachable subspace is given by **R** and is spanned by the columns of (6) and thus, applying the bijection (13 *a*), the pseudostate reachable subspace is given by $\mathbf{R_p}$ and is therefore spanned by the columns of (26), and so the result holds. Similarly for (*b*).

For (*c*), we know that observability of the PMD (10) is connected with observation of $\xi(t)$ or alternatively because of the bijective map (13 *a*), observability of the PMD (10) is connected with observation of $x(t)$. Thus, consider the generalized state-space system (23). Note that $x_2(t)$ can always be computed since $x_2(t) = \sum_{i=0}^{\mu-1} J_\infty^i B_\infty \mathcal{U}u^{[i]}(t)$ for a given $B_\infty\mathcal{U}$ and $u(t)$, thus $x_2(0-) = \sum_{i=0}^{\mu-1} J_\infty^i B_\infty \mathcal{U}u^{[i]}(0-)$. Now the equivalent generalized state-space system (23) is state observable if and only if $x_1(t)$ can be computed from $J, B\mathcal{U}, \mathcal{V}C, y(t) - \mathcal{V}C_\infty x_2(t), u(t)$. Thus, the generalized state-space system (23) is state observable if and only if the state-space system

$$\dot{x}_1(t) = Jx_1(t) + B\mathcal{U}u(t) \tag{29 a}$$

$$y(t) = \mathcal{V}Cx_1(t) + \mathcal{V}C_\infty x_2(t) \tag{29 b}$$

is observable, which gives rise to the known state-space observability criteria where, in case the system is observable, then (Wolovich 1974)

$$x_1(0-) = V^{-1}(0-, T)\int_0^T e^{J^T\tau}(\mathcal{V}C)^T\left[y(t) - \int_0^T (\mathcal{V}C)\, e^{J(T-\tau)} B\mathcal{U}u(\tau)\, d\tau\right] d\tau \quad (30)$$

where $V(0-, T) = \int_0^T e^{J^T\tau}(\mathcal{V}C)^T(\mathcal{V}C)e^{J\tau}\, d\tau$ and the transpose of the matrix J is denoted as J^T. Therefore the result holds. □

Thus, it is evident from the above that pseudostate reachability of the PMD (10) implies its pseudostate controllability, but the converse is not true.

Theorem 4 has shown that the results of state reachability, controllability and observability of the easily studied equivalent generalized state-space system may be used to establish pseudostate reachability, controllability and observability criteria of the PMD.

From Theorem 2 it can be seen that, in the case where the PMD (10) is pseudostate observable, then the map between the respective observable initial conditions is given by

$$\mathcal{X}_T\xi(0-) = [\mathcal{X}_T\mathcal{X}_T]\bar{H}\begin{bmatrix} x_1(0-) \\ -J_\infty x_2(0-) \end{bmatrix} \quad (31)$$

where $[\mathcal{X}_T\mathcal{X}_T]H, x_1(0-), -J_\infty x_2(0-)$ are defined in (17).

5. Illustrative example

Consider the PMD

$$\left.\begin{aligned} (\rho+2)(\rho+3)\beta(t) &= (\rho-1)u(t) \\ y(t) &= (5-2\rho)\beta(t) + (3\rho-2)u(t) \end{aligned}\right\} \quad (32)$$

where the normalized system is given by

$$\underbrace{\begin{bmatrix} (\rho+2)(\rho+3) & \rho-1 & 0 \\ 2\rho-5 & 3\rho+2 & 1 \\ 0 & -1 & 0 \end{bmatrix}}_{T(\rho)=T_2\rho^2+T_1\rho+T_0}\xi(t) = \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{U} u(t)\, y(t) = \underbrace{[0 \;\; 0 \;\; 1]}_{V}\xi(t) \quad (33)$$

Let

$$\mathcal{T}(s)^{-1} = \begin{bmatrix} \dfrac{1}{(s+2)(s+3)} & 0 & \dfrac{s+1}{(s+2)(s+3)} \\ 0 & 0 & -1 \\ \dfrac{5-2s}{(s+2)(s+3)} & 1 & \dfrac{3s^3+15s^2+31s+17}{(s+2)(s+3)} \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{1}{(s+2)(s+3)} & 0 & \dfrac{s+1}{(s+2)(s+3)} \\ 0 & 0 & 0 \\ \dfrac{5-2s}{(s+2)(s+3)} & 0 & \dfrac{13s+17}{(s+2)(s+3)} \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 3s \end{bmatrix} = H_{sp}(s) - H_{pol}(s)$$

(34)

Let also

$$C = \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 9 & 11 \end{bmatrix}, \quad J = \begin{bmatrix} -2 & 0 \\ 0 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 0 & 2 \end{bmatrix} \tag{35}$$

be a minimal realization of $H_{sp}(s)$ and

$$C_\infty = \begin{bmatrix} 0 & 0 \\ 0 & -1 \\ 3 & -3 \end{bmatrix}, \quad J_\infty = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad B_\infty = \begin{bmatrix} 0 & 1/3 & 1 \\ 0 & 0 & 1 \end{bmatrix} \tag{36}$$

be a minimal realization of $H_{pol}(s)$. Then the equivalent generalized state-space system can be found to be (Karampetakis and Vardulakis 1995)

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{E} \dot{x}(t) = \underbrace{\begin{bmatrix} -2 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{A} x(t) + \underbrace{\begin{bmatrix} -1 \\ 2 \\ 1 \\ 1 \end{bmatrix}}_{B} u(t)$$

$$y(t) = \underbrace{[9 \quad 11 \quad 3 \quad -3]}_{C} x(t) \tag{37}$$

By Theorem 1 the bijective map between the solution input pairs of the PMD (33) and the equivalent generalized state space system (37) is given by

$$\begin{bmatrix} \xi(t) \\ -u(t) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 9 & 11 & 3 & -3 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x(t) \\ -u(t) \end{bmatrix} \tag{38}$$

and by Theorem 2 the bijective map between the initial conditions of the PMD (33) and the initial conditions of the equivalent generalized state-space system (37) is given by

$$\mathcal{X}_T \dot{\xi}(0-) = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 3 & 2 & 0 & -1 \\ 2 & 2 & 3 & -5 \\ 0 & 0 & 0 & 1 \end{bmatrix} Ex(0-) \tag{39}$$

By Theorem 3 the complete solution $\xi_{com}(t)$, of the PMD (33) is given by (46). Thus

$$\xi_{com}(t) = [\delta(t)]C_\infty J_\infty B_\infty \left\{ \mathcal{U}u(0) - [\mathcal{T}_0 \mathcal{T}_1]\hat{\xi}(0-) \right\} + Ce^{Jt}[JB, B] \begin{bmatrix} \mathcal{T}_2 & 0 \\ \mathcal{T}_1 & \mathcal{T}_2 \end{bmatrix} \hat{\xi}(0-)$$

$$+ \sum_{i=0}^{1} C_\infty J_\infty^i B_\infty \mathcal{U} u^{[i]}(t) + \int_0^t Ce^{J(t-\tau)} B\mathcal{U} u(\tau) \, d\tau \tag{40}$$

Thus, from (40) the solution of the PMD (33) will be the following

$$\xi_{\mathrm{com}}(t) = [\delta(t)]\left\{\begin{bmatrix} 0 \\ 0 \\ 3u(0)+3\xi_2(0-) \end{bmatrix}\right\}$$
$$+\begin{bmatrix} (3e^{-2t}-2e^{-3t})\xi_1(0-)+(e^{-2t}-e^{-3t})\xi_2(0-)+(e^{-2t}-e^{-3t})\xi_1^{(1)}(0-) \\ 0 \\ (27e^{-2t}-22e^{-3t})\xi_1(0-)+(9e^{-2t}-11e^{-3t})\xi_2(0-)+(9e^{-2t}-11e^{-3t})\xi_1^{(1)}(0-) \end{bmatrix}$$
$$+\begin{bmatrix} \int_0^t (-e^{-2(t-\tau)}+2e^{-3(t-\tau)})u(\tau)\,d\tau \\ -u(t) \\ 3u^{[1]}(t)+\int_0^t(-9e^{-2(t-\tau)}+11e^{-3(t-\tau)})u(\tau)\,d\tau \end{bmatrix} \quad (41)$$

From Definition 3 and (41) it can easily be seen that the set of impulse free initial values for the PMD (33) is the following

$$A_i := \left\{\hat{\xi}(0-) : \xi_2(0-)+u(0)=0\right\} \quad (42)$$

Also from (26), the pseudostate reachable subspace $\mathbf{R_p}$, is spanned by the columns of the matrix

$$[CB\mathcal{U}, CJB\mathcal{U}, C_\infty B_\infty \mathcal{U}, C_\infty J_\infty B_\infty \mathcal{U}] = \begin{bmatrix} 1 & -4 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 13 & -48 & 0 & 3 \end{bmatrix} \quad (43)$$

which has rank $= 3(=r)$, i.e. every $\xi_T \in \mathbb{R}^3$ is pseudostate reachable.

With regards to pseudostate controllability, let a basis matrix for ker J_∞ be $M(J_\infty) = [1\ 0]^T$. Then from (27) the pseudostate controllable subspace $\mathbf{C_p}$, is spanned by the columns of the matrix

$$[CB\mathcal{U}, CJB\mathcal{U}, C_\infty B_\infty \mathcal{U}, C_\infty J_\infty B_\infty \mathcal{U}, C_\infty M(J_\infty)] = \begin{bmatrix} 1 & -4 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 13 & -48 & 0 & 3 & 3 \end{bmatrix} \quad (44)$$

which has rank $= 3(=r)$, i.e. every $\xi_0 \in \mathbb{R}^3$ is pseudostate controllable.

From (28) and (35)

$$\begin{bmatrix} \mathcal{V}C \\ \mathcal{V}CJ \end{bmatrix} = \begin{bmatrix} 9 & 11 \\ -18 & -33 \end{bmatrix} \quad (45)$$

has rank $= 2(=n)$, therefore every $\xi_T \in \mathbb{R}^3$ is pseudostate observable.

6. Conclusions

It has been shown that an 'equivalent' generalized state-space representation of a linear multivariate system, whose dynamics are expressed via a PMD, may be used to represent the solution of the PMD in closed form. As such, this provides an extension of the known generalized state-space results of Yip and Sincovic (1981) and Callier and Desoer (1982). The map between the state of this equivalent system and the pseudo-state of the original system is a constant bijection, and enables certain 'state' referenced notions (Yip and Sincovic 1981, Cobb 1984, Oscaldiran 1986) such as reachability, controllability and observability, to be defined for the PMD. In this way, satisfying extensions to the results concerning the form of the solution in terms of the regular derivative of $u(t)$, state reachability, controllability and observability of generalized state space systems, have been developed for the general PMD case. The approach has also enabled a characterization of the impulse free initial conditions for a general PMD to be made analogously to that given for generalized state space systems (Vardulakis 1991).

There are several potential advantages of the theory proposed in this paper. As noted above, the expression given in Theorem 3 for the solution of the PMD may be written almost entirely in terms of coefficient matrices, which may be obtained from the Laurent expansion of $T(s)^{-1}$. Similarly (26) (respectively (27)) may be written entirely (almost entirely) in terms of these same coefficient matrices (see for example, Mahmood 1996), which are readily obtained via a recursive algorithm such as that established by Fragulis *et al.* (1991) or by simply using a symbolic computer algebra program such as Maple V. In addition, it can be seen that controllability of $\xi(t)$ implies controllability of both $\beta(t)$ and $y(t)$. In the special case where $C(\rho) = I_{p,l}$, $D(\rho) = 0_{p,m}$ in (9 *b*), controllability of $\xi(t)$ implies controllability of $\beta(t)$. One particular deficiency of what is proposed here, however, is that a condition for the existence of output controllability alone is not immediately apparent. The subsequent aims of this work would be the incorporation of the results into a computer algebra system such as Maple V, in that way revising procedures as described in Jones *et al.* (1997), and the extension in a theoretical way of the results to the case of non-regular PMDs, where the left and right minimal indices will also feature in the description of the solution space.

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