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# A SPECTRAL CHARACTERIZATION OF THE BEHAVIOR OF DISCRETE TIME AR–REPRESENTATIONS OVER A FINITE TIME INTERVAL

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In this paper we investigate the behavior of the discrete time AR (Auto Regressive) representations over a finite time interval, in terms of the finite and infinite spectral structure of the polynomial matrix involved in the AR-equation. A boundary mapping equation and a closed formula for the determination of the solution, in terms of the boundary conditions, are also gived.

## 1. INTRODUCTION

The class of discrete time descriptor systems has been the subject of several studies in the recent years (see for example [1, 2, 3, 4, 5, 6]). The main reason for this is that descriptor equations naturally represent a very wide class of physical, economic or social systems. One of the most interesting features of discrete time singular systems is without doubt their non causal behavior, while their counterpart in continuous time exhibit impulsive behavior.

However, descriptor systems can be considered as a special – first order case of a more general Auto Regressive Moving Average (ARMA) multivariable model and thus the study of this more general case can be proved to be very important. Such models (known also as polynomial matrix descriptions or PMDs) have been extensively studied in the continuous time case by several authors. In this note we investigate some structural properties of the discrete time autoregressive (AR)-representation, as a first step towards the generalization of the descriptor systems theory to the higher order case.

Consider the discrete time AR equation

$$A_q x_{k+q} + A_{q-1} x_{k+q-1} + \ldots + A_0 x_k = 0 \tag{1.1}$$

where $k = 0, 1, 2, \ldots, N - q$, or equivalently

$$A(\sigma)\, x_k = 0, \quad k = 0, 1, 2, \ldots, N - q \tag{1.2}$$

where $A(\sigma) = A_q \sigma^q + A_{q-1}\sigma^{q-1} + \ldots + A_0 \in \Re^{r\times r}[\sigma]$ is a regular polynomial matrix, i.e. $\det A(\sigma) \neq 0$ for almost every $\sigma$, $x_k \in \Re^r$, $k = 0, 1, 2, \ldots, N$ is a vector sequence and $\sigma$ denotes the forward shift operator $\sigma x_k = x_{k+1}$. Notice that we are interested for the behavior of (1.2) over a specified time interval $k = 0, 1, 2, \ldots, N$ and not over $Z^+$.

The matrix $A_q$ is not in general invertible which means that (1.1) can not be solved by iterating forward, i.e. given $x_0, x_1, \ldots, x_{q-1}$ determine successively $x_q$, $x_{q+1}, \ldots$. This is the main reason why we treat this equation as a boundary condition problem, where both the initial and final conditions should be given. This naturally leads to the restriction of the time domain to a finite interval instead of $Z^+$.The results of the present paper should be compared to [1, 2, 3, 4, 5, 6] where similar problems for systems in descriptor form, are treated in a similar manner.

Finally, following the notation of [12] we define the behavior of (1.2) as

$$\mathcal{B} = \{x_k \mid x_k \in \Re^r, x_k \text{ satisfies } (1.2)\} \tag{1.3}$$

where $k = 0, 1, 2, \ldots, N$.

## 2. PRELIMINARIES — NOTATION

The mathematical background required for this note comes mainly from [7, 8, 9, 10] and [11]. By $\Re^{m\times n}[\sigma]$ we denote the set of $m \times n$ polynomial matrices with real coefficients and indeterminate $\sigma$. A square polynomial $A(\sigma) = A_q\sigma^q + A_{q-1}\sigma^{q-1} + \ldots + A_0 \in \Re^{r\times r}[\sigma]$ matrix is called regular iff $\det A(\sigma) \neq 0$ for almost every $\sigma$. The (finite) eigenvalues of $A(\sigma)$ are defined as the roots of the equation $\det A(\sigma) = 0$. Let

$$S^{\lambda_i}_{A(\sigma)} = \mathrm{diag}\{(\sigma-\lambda_i)^{m_{i1}}, \ldots, (\sigma-\lambda)^{m_{ir}}\}$$

be the local Smith form of $A(\sigma)$ at $\sigma = \lambda_i$ and $\lambda_i$ is an eigenvalue of $A(\sigma)$, where $0 \leq m_{i1} \leq m_{i2} \leq \ldots \leq m_{ir}$. The terms $(\sigma - \lambda_i)^{m_{ij}}$ are called the (finite) elementary divisors of $A(\sigma)$ at $\sigma = \lambda_i$, $m_{ij}$ $j = 1, 2, \ldots, r$ are the partial multiplicities of $\lambda_i$ and $m_i = \sum_{j=1}^{r} m_{ij}$ is the multiplicity of $\lambda_i$.

The dual matrix of $A(\sigma)$ is defined as $\widetilde{A}(\sigma) = \sigma^q A(\sigma^{-1}) = A_0\sigma^q + A_1\sigma^{q-1} + \ldots + A_q$. The infinite elementary divisors of $A(\sigma)$ are the finite elementary divisors of the dual $\widetilde{A}(\sigma)$ at $\sigma = 0$. The total number of elementary divisors (finite and infinite) of $A(\sigma)$ is equal to the product $r \times q$, where $r$ is the dimension and $q$ is the degree of $A(\sigma)$.

A pair of matrices $X_i \in \Re^{r\times m_i}$, $J_i \in \Re^{m_i\times m_i}$, where $J_i$ is in Jordan form and $\lambda_i$ is an eigenvalue of $A(\sigma)$of multiplicity $m_i$ is called an eigenpair of $A(\sigma)$ corresponding to $\lambda_i$ iff

$$\sum_{k=0}^{q} A_k X_i J_i^k = 0, \quad \mathrm{rank}\,\mathrm{col}(X_i J_i^k)_{k=0}^{m_i-1} = m_i \tag{2.1}$$

where

$$\mathrm{col}(X_i J_i^k)_{k=0}^{m_i-1} = \begin{bmatrix} X_i \\ X_i J_i \\ \vdots \\ X_i J_i^{m_i-1} \end{bmatrix}.$$

The matrix $J_i$ consists of Jordan blocks with sizes equal to the partial multiplicities of $\lambda_i$.

Let $\lambda_1, \lambda_2, \ldots, \lambda_p$ be the distinct finite eigenvalues of $A(\sigma)$, and $(X_i, J_i)$ their corresponding eigenpairs. The total number of finite elementary divisors is equal to the determinantal degree of $A(\sigma)$, i.e. $n = \deg(\det A(\sigma)) = \sum_{i=1}^{p} m_i$. The pair of matrices

$$X_F = [X_1, X_2, \ldots, X_p] \in \Re^{r\times n} \tag{2.2}$$
$$J_F = \operatorname{diag}\{J_1, J_2, \ldots, J_p\} \in \Re^{n\times n} \tag{2.3}$$

is defined as a finite spectral pair of $A(\sigma)$ and satisfies the following

$$\sum_{k=0}^{q} A_k X_F J_F^k = 0, \quad \operatorname{rank}\operatorname{col}(X_F J_F^k)_{k=0}^{n-1} = n. \tag{2.4}$$

An eigenpair of the dual matrix $\widetilde{A}(\sigma)$ corresponding to the eigenvalue $\widetilde{\lambda} = 0$ is defined as an infinite spectral pair of $A(\sigma)$, and satisfies the following

$$\sum_{k=0}^{q} A_k X_\infty J_\infty^{q-k} = 0, \quad \operatorname{rank}\operatorname{col}(X_\infty J_\infty^k)_{k=0}^{\mu-1} = \mu \tag{2.5}$$

where $X_\infty \in \Re^{r\times\mu}$, $J_\infty \in \Re^{\mu\times\mu}$.

## 3. MAIN RESULTS

Consider the AR-representation (1.2) and a finite spectral pair $(X_F, J_F)$ of $A(\sigma)$. In the continuous time case, i.e. where $\sigma = \frac{\mathrm{d}}{\mathrm{d}t}$ is the differential operator instead of the forward shift operator, finite spectral pairs give rise to linearly independent solutions. A similar situation occurs in our case. We state the following theorem

**Theorem 1.** If $(X_F, J_F)$ is a finite (If $(X_\infty, J_\infty)$ is an infinite) spectral pair of $A(\sigma)$ of dimensions $r\times n, n\times n$ ($r\times\mu$, $\mu\times\mu$) respectively where $n = \deg|A(\sigma)|$ ($\mu$ is the multiplicity of the eigenvalue at $\sigma = \infty$ of $A(\sigma)$) then the columns of the matrix

$$\Psi_F(k) = X_F J_F^k,\ k = 0, 1, 2, \ldots, N \tag{3.1}$$
$$\left(\Psi_\infty(k) = X_\infty J_\infty^{N-k},\ k = 0, 1, 2, \ldots, N\right) \tag{3.2}$$

are linearly independent solutions of (1.2) for $N \geq n$ ($N \geq \mu$).

Proof. Let $(X_F, J_F)$ be a finite spectral pair of $A(\sigma)$ . We have

$$\begin{aligned} A(\sigma)\, X_F J_F^k &= \sum_{i=0}^{q} A_i X_F J_F^{k+i} \\ &= \left(\sum_{i=0}^{q} A_i\, X_F\, J_F^i\right) J_F^k \overset{(2.4)}{=} 0 \end{aligned}$$

for $k = 0, 1, 2, \ldots, N-q$ and from the second equation of (2.4) it is obvious that the columns of $X_F J_F^k$ are linearly independent sequences over any interval $k = 0, 1, 2, \ldots, N \geq n$. The proof of (3.2) follows similarly if we take into account (2.5). □

The above theorem proves that we can form solutions of (1.2) as linear combinations of the columns of the matrices $\Psi_F(k)$ and $\Psi_\infty(k)$. It remains to show that the columns of these two matrices are enough to span the entire solution space of the equation over a finite interval $k = 0, 1, 2, \ldots, N$.

Consider equation (1.2) or equivalently the more detailed form (1.1). Then one can write this equation in the following form

$$R_{N+1}(A)\,\overline{x}_{N+1} = 0 \tag{3.3}$$

where $R_{N+1}(A)$ is the resultant matrix of $A(\sigma)$ having $N+1$ block columns

$$R_{N+1}(A) = \begin{bmatrix} A_0 & A_1 & \cdots & A_q & 0 & \cdots & 0 \\ 0 & A_0 & A_1 & \cdots & A_q & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & & \ddots & 0 \\ 0 & \cdots & 0 & A_0 & A_1 & \cdots & A_q \end{bmatrix} \tag{3.4}$$

where $R_{N+1}(A) \in \Re^{r(N-q+1)\times r(N+1)}$ and $\overline{x}_{N+1} = [\, x_0^T \;\; x_1^T \;\; \cdots \;\; \cdots \;\; x_N^T \,]^T \in \Re^{r(N+1)}$. Obviously equations (1.2) and (1.1) are equivalent to (3.3) in the specified time interval $k = 0, 1, 2, \ldots, N$.

With this simple remark and using the theory for the kernels of resultant matrices of a polynomial matrix, we can state the following very important

**Theorem 2.** The behavior of the AR-representation (1.2) over the finite time interval $k = 0, 1, 2, \ldots, N$ is

$$\mathcal{B} = \operatorname{span}[X_F, X_\infty]\{J_F^k \oplus J_\infty^{N-k}\} \tag{3.5}$$

and

$$\dim \mathcal{B} = rq$$

where $r, q$ are respectively the dimension of $A(\sigma)$ and the maximum order of $\sigma$ in $A(\sigma)$, $(X_F, J_F)$ is a finite spectral pair of $A(\sigma)$ and $(X_\infty, J_\infty)$ is an infinite spectral pair of $A(\sigma)$.

Proof. Consider equation (3.3). Obviously the solution space of this equation is the kernel of $R_{N+1}(A)$. Then the behavior of (1.2) is clearly isomorphic to the solution space of (3.3), i.e.

$$\mathcal{B} \simeq \operatorname{Ker} R_{N+1}(A). \tag{3.6}$$

But from Theorem 1.1 in [10] we have as a special case that

$$\operatorname{Ker} R_{N+1}(A) = \operatorname{Im} \operatorname{col}(X_F J_F^i)_{i=0}^N \oplus \operatorname{Im} \operatorname{col}(X_\infty J_\infty^{N-i})_{i=0}^N. \tag{3.7}$$

The dimensions of $X_F$, $J_F$, $X_\infty$, $J_\infty$ are $r \times n$, $n \times n$, $r \times \mu$ and $\mu \times \mu$ respectively, where $n = \deg |A(\sigma)|$ is the total number of finite elementary divisors, $\mu$ is the total number of infinite elementary divisors (multiplicities encountered for in both cases) and $n + \mu = rq$ (see [8]). Furthermore it is known [8] that the columns of $\mathrm{col}(X_F J_F^i, X_\infty J_\infty^{N-i})_{i=0}^N$ are linearly independent. On the other hand from the regularity assumption of $A(\sigma)$ we have

$$\operatorname{rank} R_{N+1}(A) = (N - q + 1)r$$

(this is a well known result, see for example [11] exercise 4.10) and thus

$$\dim \operatorname{Ker} R_{N+1}(A) = rq. \tag{3.8}$$

Obviously (3.6), (3.7) and (3.8) prove that the columns of the matrix

$$[X_F, X_\infty]\{J_F^k \oplus J_\infty^{N-k}\}$$

form indeed a basis of $\mathcal{B}$ and consequently $\dim \mathcal{B} = rq$. □

It is clear that the solution space $\mathcal{B}$ can be decomposed into two subspaces the one corresponding to the finite eigenstructure of the polynomial matrix and the other corresponding to the infinite one, i. e.

$$\mathcal{B} = \mathcal{B}_F \oplus \mathcal{B}_B$$

where $\mathcal{B}_F = \mathrm{Im}\ \mathrm{col}(X_F J_F^i)_{i=0}^N$ and $\mathcal{B}_B = \mathrm{Im}\ \mathrm{col}(X_\infty J_\infty^{N-i})_{i=0}^N$ (the subscripts $F$, $B$ are the initials of the words Forward and Backward)

The first part $\mathcal{B}_F$ gives rise to solutions moving in the forward direction of time and reflects the forward propagation of the initial conditions $x_0, x_1, \ldots, x_{q-1}$, while the second part $\mathcal{B}_B$ gives solutions moving backwards in time, i. e. from $N$ to 0.

This discussion should be compared to that in [1, 2, 3]. Notice that the above decomposition of the solution space into forward and backward subspaces, corresponds to a maximal forward decomposition of the descriptor space in [3].

A very interesting problem is to determine a closed formula for the solution of (1.2) when boundary condition are given. The reason why we have to choose both initial and final conditions is obvious, after the above discussion about the behavior of (1.2).

**Theorem 3.** Given the initial conditions vector $\widehat{x}_I = [\ x_0^T \quad x_1^T \quad \cdots \quad x_{q-1}^T\ ]^T \in \Re^{rq}$ and the final conditions vector $\widehat{x}_F = [\ x_{N-q+1}^T \quad x_{N-q+2}^T \quad \cdots \quad x_N^T\ ]^T \in \Re^{rq}$, (1.2) has the unique solution

$$x_k = \begin{bmatrix} X_F J_F^k M_F & X_\infty J_\infty^{N-k} M_\infty \end{bmatrix} \begin{bmatrix} \widehat{x}_I \\ \widehat{x}_F \end{bmatrix} \tag{3.9}$$

for $k = 0, 1, 2, \ldots, N$, iff the vectors $\widehat{x}_I, \widehat{x}_F$ satisfy the compatibility boundary condition

$$\begin{bmatrix} \widehat{x}_I \\ \widehat{x}_F \end{bmatrix} \in \ker \begin{bmatrix} J_F^{N-q+1} M_F & -M_F \\ -M_\infty & J_\infty^{N-q+1} M_\infty \end{bmatrix} \tag{3.10}$$

where $M_F \in \Re^{n\times rq}$, $M_\infty \in \Re^{\mu\times rq}$ are defined by

$$\begin{bmatrix} M_F \\ M_\infty \end{bmatrix} = (\mathrm{col}(X_F J_F^{i-1}, X_\infty J_\infty^{q-i})_{i=1}^{q})^{-1} \in \Re^{rq\times rq}. \tag{3.11}$$

Proof. Every solution $x_k$, $k = 0,1,2,\ldots,N$ of (1.2) will be a linear combination of the basis of $\mathcal{B}$, i.e. there exists a vector $\zeta \in \Re^{rq}$ such that

$$x_k = \begin{bmatrix} X_F & X_\infty \end{bmatrix} \{J_F^k \oplus J_\infty^{N-k}\}\zeta \tag{3.12}$$

for $k = 0,1,2,\ldots,N$. Our aim is to determine $\zeta$ in terms of the given initial – final conditions. The initial conditions vector will be given by (3.12) for $k = 0,1,2,\ldots,q-1$. Thus

$$\widehat{x}_I = \mathrm{col}\left(X_F J^i, X_\infty J_\infty^{N-i}\right)_{i=0}^{q-1} \zeta$$

or equivalently

$$\widehat{x}_I = Q \begin{bmatrix} I_n & 0 \\ 0 & J_\infty^{N-q+1} \end{bmatrix} \zeta \tag{3.13}$$

where the matrix $Q = \mathrm{col}(X_F J_F^{i-1}, X_\infty J_\infty^{q-i})_{i=1}^{q}$ in the above equation is invertible (see decomposable pairs in [8]). At this point it would be useful to partition $\zeta = \begin{bmatrix} \zeta_F^T & \zeta_\infty^T \end{bmatrix}^T$, where $\zeta_F$ and $\zeta_\infty$ have appropriate dimensions. Now from (3.13) using the definition of $M_F$, $M_\infty$ in (3.11) we obtain

$$\begin{aligned} \zeta_F &= M_F \widehat{x}_I \\ J_\infty^{N-q+1}\zeta_\infty &= M_\infty \widehat{x}_I \end{aligned} \tag{3.14}$$

Notice that (3.14) determines $\zeta_F$ but not $\zeta_\infty$. Following similar lines for the final conditions vector we obtain

$$\begin{aligned} \zeta_\infty &= M_\infty \widehat{x}_F \\ J_F^{N-q+1}\zeta_F &= M_F \widehat{x}_F \end{aligned} \tag{3.15}$$

Similarly (3.15) determines $\zeta_\infty$ but not $\zeta_F$. Now using the first equations in (3.14) and (3.15) we obtain

$$\zeta = \begin{bmatrix} \zeta_F \\ \zeta_\infty \end{bmatrix} = \begin{bmatrix} M_F & 0 \\ 0 & M_\infty \end{bmatrix} \begin{bmatrix} \widehat{x}_I \\ \widehat{x}_F \end{bmatrix}$$

which in view of (3.12) gives the solution formula (3.9), and combining the second equations in (3.14) and (3.15) we obtain the boundary compatibility condition

$$\begin{bmatrix} J_F^{N-q+1} M_F & -M_F \\ -M_\infty & J_\infty^{N-q+1} M_\infty \end{bmatrix} \begin{bmatrix} \widehat{x}_I \\ \widehat{x}_F \end{bmatrix} = 0$$

which is obviously identical to (3.10). □

Notice that equation (3.10) plays the role of the boundary mapping equation in [2] and it can be considered as a direct generalization of it. Equation (3.10) summarizes the restrictions posed at both end points of the time interval by the system and with an appropriate choice of boundary conditions by (3.9) we can determine uniquely all the intermediate values of $x_k$.

For simplicity of notation and following similar lines with [2], we set

$$Z(0,N) = \begin{bmatrix} J_F^{N-q+1}M_F & -M_F \\ -M_\infty & J_\infty^{N-q+1}M_\infty \end{bmatrix} \in \Re^{rq\times 2rq}$$

and we prove the following

**Theorem 4.** $\operatorname{rank} Z(0,N) = rq$.

Proof. We set

$$N = \operatorname{col}(X_F J_F^{i-1}, X_\infty J_\infty^{q-i})_{i=1}^{q}$$

then from (3.11) we have

$$\begin{bmatrix} M_F \\ M_\infty \end{bmatrix} N = \begin{bmatrix} I_n & 0 \\ 0 & I_\mu \end{bmatrix}. \tag{3.16}$$

Now, post-multiply $Z(0,N)$ by $\operatorname{diag}\{N,N\}$ which has obviously full rank and use (3.16). We have

$$Z(0,N)\begin{bmatrix} N & 0 \\ 0 & N \end{bmatrix} = \begin{bmatrix} J_F^{N-q+1} & 0 & -I_n & 0 \\ 0 & -I_\mu & 0 & J_\infty^{N-q+1} \end{bmatrix}.$$

Obviously the matrix on the right hand side of the above equation has full row rank and hence

$$\operatorname{rank} Z(0,N) = rq. \tag{3.17}$$

□

This result should be compared to Theorem 1 in [2], where it is proved that a boundary mapping matrix of full rank exists if and only if the corresponding descriptor system is solvable and conditionable. In our case the system is obviously solvable, since we have already determined a solution and conditionable since we have proved that the boundary conditions satisfying (3.10) characterize uniquely the solution. However solvability and conditionability of (1.2) can be easily checked using rank tests as in [1], but in our case this would be trivial due the regularity of $A(\sigma)$.

It is important to notice here that (3.17) implies

$$\dim\ker Z(0,N) = rq = \dim \mathcal{B}$$

which means that the initial and final conditions vectors are chosen from a $rq$–dimensional vector space. Thus $rq$ is the total number of arbitrarily assigned values, distributed at both end points of the time interval. The connection between $\dim\ker Z(0,N)$ and $\dim\mathcal{B}$ is obvious.

## 4. EXAMPLE

In order to illustrate the above results we shall give an example which exhibits only backward behavior. This is done for brevity reasons, while it is well known that the finite eigenstructure of $A(\sigma)$ gives rise to forward linearly independent solutions (see for example [8]). Consider the unimodular polynomial matrix

$$A(\sigma) = \begin{bmatrix} 1 & \sigma^2 \\ 0 & 1 \end{bmatrix}.$$

Obviously there are no finite elementary divisors and thus no finite spectral pairs, since $\det A(\sigma) = 1$. Consider also the AR equation

$$A(\sigma)\, x_k = 0$$

for $k = 0, 1, 2, \dots, N - q$. According to the notation used earlier we have $q = 2$ and $r = 2$ and thus we have to expect

$$\dim \mathcal{B} = rq = 4.$$

Indeed, consider an infinite spectral pair of $A(\sigma)$

$$X_\infty = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix}, \quad J_\infty = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Then according to Theorem 3, a basis of $\mathcal{B} = \mathcal{B}_B$ is formed by the columns of the matrix

$$\Psi(k) = X_\infty J_\infty^{N-k} = \begin{bmatrix} \delta_{N-k} & \delta_{N-k-1} & \delta_{N-k-2} & \delta_{N-k-3} \\ 0 & 0 & -\delta_{N-k} & -\delta_{N-k-1} \end{bmatrix}$$

where $\delta_i = 0$ for $i \neq 0$ and $\delta_0 = 1$. The boundary mapping equation will be

$$Z(0, N) = [-M_\infty, J_\infty^{N-q+1} M_\infty]$$

since there is no finite spectral pair. Now we can see that

$$M_\infty = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & -1 & 0 & 0 \end{bmatrix}$$

and thus for $N > 4$, we have $J_\infty^{N-q+1} = 0$

$$Z(0, N) = \begin{bmatrix} 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

which has full row rank. Obviously by (3.10) the final conditions can be freely assigned while for $N > 4$ the initial conditions must be zero. This is natural because, the infinite spectral pair of $A(\sigma)$ gives rise to reversed in time deadbeat modes, which after four steps of backward propagation of the final conditions, become zero. The intermediate solution formula can be obtained by (3.9)

$$x_k = X_\infty J_\infty^{N-k} M_\infty \widehat{x}_F = \begin{bmatrix} \delta_{N-k-1} & -\delta_{N-k-3} & \delta_{N-k} & -\delta_{N-k-2} \\ 0 & \delta_{N-k-1} & 0 & \delta_{N-k} \end{bmatrix} \widehat{x}_F,$$

where no initial conditions are involved because there is no finite spectral pair of $A(\sigma)$.

## 5. CONCLUSIONS

In this note we have determined the solution space or the behavior $\mathcal{B}$ of the discrete time Auto-Regressive representations having the form $A(\sigma)\, x_k = 0$, where the matrix $A(\sigma)$ is a square regular polynomial matrix and $x_k$ is a vector sequence over a finite time interval $k = 0, 1, 2, \dots, N$. The solution space $\mathcal{B}$ is proved to be a linear vector space, of dimension equal to the product of the dimension $r$ of the matrix $A(\sigma)$ and the highest degree of $\sigma$ occurring in the polynomial matrix.

It is also shown that the behavior can be decomposed into a direct sum of the forward and backward subspace, which corresponds to a maximal F/B decomposition of the descriptor space in [3]. We have also determined a basis for the solution space, using a construction based on both the finite and infinite spectral structure of $A(\sigma)$.

We introduce the notion of the dual AR representation which is simply the same system but with reversed time direction. Finally, a generalization of the boundary mapping defined for first order systems in [2], to the higher order case is given and it is shown that such a boundary mapping can be obtained in terms of the spectral pairs of $A(\sigma)$.

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