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# FORWARD, BACKWARD AND SYMMETRIC SOLUTIONS OF DISCRETE ARMA–REPRESENTATIONS

by

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**Abstract.** The main objective of this paper to determine a closed formula for the forward, backward and symmetric solution of a general discrete time AutoRegressive Moving Average (ARMA) Representation. The importance of the above formula is that it is easily implemented in a computer algorithm and gives rise to the solution of analysis, synthesis and design problems.

## 1. Introduction

Consider a nonhomogeneous system of linear difference and algebraic equations described in matrix form by

$$A(\sigma)y(k) = B(\sigma)u(k) \tag{1.1}$$

where $\sigma$ denotes the backwards shift operator i.e. $\sigma^i y(k)=y(k+i)$,

$$A(\sigma)=A_0+A_1\sigma+\cdots+A_q\sigma^q\in\mathbb{R}[\sigma]^{r\times r},\ \mathrm{rank}_{\mathbb{R}(\sigma)}A(\sigma)=r$$
$$B(\sigma)=B_0+B_1\sigma+\cdots+B_q\sigma^q\in\mathbb{R}[\sigma]^{r\times m} \tag{1.2}$$

where at least one of $A_q$, $B_q$ is nonzero, $y(k):\mathbb{Z}^+\longrightarrow\mathbb{R}^r$ be the *output* of the system and $u(k):\mathbb{Z}^+\longrightarrow\mathbb{R}^m$ be the *input* of the system. Following the terminology of Willems (1991) we call the set of equations (1.1) an ARMA representation of B, where B is the solution space

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of equations (1.1) defined by

$$B=\pi_y(B_f) \tag{1.3a}$$

with

$$B_f:=\{ (y(k)\ \ u(k)) : \mathbb{Z}^+ \longrightarrow \mathbb{R}^r \times \mathbb{R}^m \mid \text{(ARMA) is satisfied } \forall k \in \mathbb{Z}^+ \}$$

$$\text{and } \pi_y: \mathbb{R}^r \times \mathbb{R}^m \longrightarrow \mathbb{R}^r \text{ is given by } \pi_y(y(k)\ \ u(k))=y(k) \tag{1.3b}$$

In case where $A(\sigma)=\sigma E-A \in \mathbb{R}[\sigma]^{r\times r}$ and $B(\sigma)=B \in \mathbb{R}^{r\times m}$ then the ARMA representation (1.1) is the known generalised state space representation

$$Ex(k+1) = Ax(k) + Bu(k) \tag{1.4}$$

while in case where $\det[E]\neq 0$, (1.3) is the known state space representation. For a survey of singular systems of the form (1.4) see [8].

ARMA representations of the form (1.1) find numerous applications in analysis of circuits [13], neural networks [2], economics (the Leontieff model, see [10]), power systems [16]. The solution of the ARMA–representation (1.4) has been calculated by many different techniques ([1], [7], [11], [14], [17]) and among them we distinguish [9], [12]. This technique gives a solution of the singular system representation in terms of the fundamental matrix $\phi_k$ and the backward fundamental matrix $\tau_k$ of $(zE-A)^{-1}$. Following similar lines with [9], [12] we produce in section 3 a closed formula for the forward, backward and symmetric solution of the general ARMA–representation (1.1) in terms now of the fundamental matrix $H_k$ and the backward fundamental matrix $V_k$ of $A(s)^{-1}$. A generalized Leverrier technique for computing the forward fundamental matrix $H_k$ is available [3], so that we may assume that this fundamental matrix is given. We shall show in section 2 that the backward fundamental matrix is the forward fundamental matrix of the dual polynomial matrix $\tilde{A}(w)=A_0w^q+A_1w^{q-1}+\cdots+A_q$ of $A(s)$ and thus we may assume that $V_k$ is also given. The whole theory is illustrated via an example in section 4.

## 2. Preliminary Results

We are concerned with the discrete time ARMA–representation

$$A_qy(k+q)+A_{q-1}y(k+q-1)+\cdots+A_0y(k) = B_qu(k+q)+B_{q-1}u(k+q-1)+\cdots+B_0u(k) \tag{2.1}$$

where $y(k)\in\mathbb{R}^r$, $u(k)\in\mathbb{R}^m$ $k\in[0,N]$ and $u(k)$ is nonzero for $k=0,1,\ldots,N-q$. We assume that

$A(z)$ is *regular* i.e. $\det[A(z)]\neq 0$. Given regularity the Laurent series expansion about infinity for the resolvent matrix exists and is given by

$$A(z)^{-1} = H_{\hat{q}_r} z^{\hat{q}_r} + H_{\hat{q}_r-1} z^{\hat{q}_r-1} + \cdots + H_1 z + H_0 + H_{-1} z^{-1} + \cdots \tag{2.2}$$

where $\hat{q}_r$ is the greatest order of the zeros of $A(z)$ at $s=\infty$ and the sequence $H_k$ is known as the *forward fundamental matrix* [3]. The Laurent expansion about zero of the resolvent matrix is

$$A(z)^{-1} = V_{-\ell} z^{-\ell} + V_{-\ell+1} z^{-\ell+1} + \cdots + V_{-1} z^{-1} + V_0 + V_1 z + \cdots \tag{2.3}$$

where the sequence $V_k$ is known [9] as the *backward fundamental matrix.*

The Laurent expansion about zero of $A(z)^{-1}$ given in (2.3) is related with the Laurent expansion about infinity given in (2.2) of the inverse of the dual matrix $\tilde{A}(w)=A_0 w^q + A_1 w^{q-1} + \cdots + A_q$ of $A(z)$ as we can see in the following

**Lemma 2.1** Let the Laurent expansion about infinity of $\tilde{A}(w)^{-1}$ is

$$\tilde{A}(w)^{-1} = \tilde{H}_f w^f + \tilde{H}_{f-1} w^{f-1} + \cdots + \tilde{H}_1 w + \tilde{H}_0 + \tilde{H}_{-1} w^{-1} + \cdots \tag{2.4}$$

and (2.3) is the Laurent expansion about zero of $A(z)^{-1}$. Then

$$q+f=\ell \quad \text{and} \quad V_{-i}=\tilde{H}_{-\ell+f+i} \text{ for } i=\ell,\ell-1,..,1,0,-1,.. \tag{2.5}$$

Proof. We have that

$$A(z) = z^q \tilde{A}\left[\frac{1}{z}\right] \Leftrightarrow A(z)^{-1} = z^{-q} \tilde{A}\left[\frac{1}{z}\right]^{-1} \overset{(2.4)}{\Leftrightarrow}$$

$$\begin{aligned} A(z)^{-1} &= z^{-q}\left[\tilde{H}_f z^{-f} + \tilde{H}_{f-1} z^{-f+1} + \cdots + \tilde{H}_1 z^{-1} + \tilde{H}_0 + \tilde{H}_{-1} z^1 + \cdots\right] \\ &= \tilde{H}_f z^{-q-f} + \tilde{H}_{f-1} z^{-q-f+1} + \cdots + \tilde{H}_1 z^{-q-1} + \tilde{H}_0 z^{-q} + \tilde{H}_{-1} z^{-q+1} + \cdots \end{aligned} \tag{2.6}$$

$$+ \cdots + \tilde{H}_{-q+1} z^{-1} + \tilde{H}_{-q} + \tilde{H}_{-q-1} z + \cdots =$$

$$\equiv V_{-\ell} z^{-\mu} + V_{-\ell+1} z^{-\mu+1} + \cdots + V_{-1} z^{-1} + V_0 + V_1 z + \cdots$$

Equating the coefficients of the powers of z we obtain the proof of Lemma 2.1. □

A direct result from Lemma 2.1 is that the Leverrier algorithm in [3] may be used for the computation both of the forward and backward fundamental matrix.

An interesting result which connects the solutions of the ARMA–representation (2.1) and the ones of the dual discrete time ARMA–representation :

$$A_q\tilde{y}(k)+A_{q-1}\tilde{y}(k+1)+\cdots+A_0\tilde{y}(k+q) = B_q\tilde{u}(k)+B_{q-1}\tilde{u}(k+1)+\cdots+B_0\tilde{u}(k+q) \quad (2.7)$$

in the closed interval [0,N] is given by the following :

**Theorem 2.2**

**(a)** If $\tilde{y}(k)$ is a solution of equation (2.7) for the nonzero input $\tilde{u}(k)$ then the sequence $y(k)=\tilde{y}(N-k)$ is a solution of the dual equation (2.1) for the nonzero input $u(k)=\tilde{u}(N-k)$.

**(b)** If $y(k)$ is a solution of equation (2.1) for the nonzero input $u(k)$ then the sequence $\tilde{y}(k)=y(N-k)$ is a solution of the dual equation (2.7) for the nonzero input $\tilde{u}(k)=u(N-k)$.

**Proof. a)** Let $\tilde{y}(k)$ be a solution of (2.1). This implies that (2.7) is satisfied. Now consider the equation (2.1). If we set $y(k)=\tilde{y}(N-k)$ and $u(k)=\tilde{u}(N-k)$ and take into account that $y(k+j)=\tilde{y}(N-(k+j))$, $u(k+j)=\tilde{u}(N-(k+j))$, $j=0,1,..,q$ we have

$$\begin{aligned} A(\sigma)\tilde{y}(N-k) &= A_q\tilde{y}(N-k-q) + A_{q-1}\tilde{y}(N-k-q+1) + \cdots + A_0\tilde{y}(N-k) \overset{(2.7)}{=} \\ &= B_q\tilde{u}(N-k-q) + B_{q-1}\tilde{u}(N-k-q+1) + \cdots + B_0\tilde{u}(N-k) \overset{u(k)=\tilde{u}(N-k)}{=} \\ &= B(\sigma)\tilde{u}(N-k) \end{aligned} \quad (2.8)$$

which verifies the first part of the Theorem.

b) Following the same way we can show the second part of the Theorem. □

A direct result from the above theorem is that the backward solution of the ARMA representation (1.1) comes directly from the forward solution of the dual ARMA representation (2.7).

# 3. Solutions of ARMA–Representations

There are three different interpretations of the equation (1.1) [9] :

(i) We may consider that the initial conditions $\{y(0), y(1), ..., y(q-1)\}$ are given and that is desired to determine $y(k)$ in a *forward* fashion from the input sequence and the previous values of the output.

(ii) We may consider that the final conditions $\{y(N), y(N-1), ..., y(N-q+1)\}$ are given and that is desired to determine $y(k)$ in a *backward* fashion from the input sequence and the future values of the output.

(iii) We may consider (1.1) as a relationship between the inputs and outputs i.e. economics and thus no causality is assumed. It is desired to determine $y(k)$ for the values of $k\in[q,N-q]$, in terms of the input sequence and the initial and final conditions. We could call this the *symmetric* solution of (1.1).

## 3.1. The Forward Solution of ARMA–Representations

Consider the discrete time ARMA–representation (2.1) where $A(z)$ is *regular* i.e. $\det[A(z)]\neq 0$ and the Laurent series expansion about infinity for the resolvent matrix exists and is given by

$$A(z)^{-1} = H_{\hat{q}_r} z^{\hat{q}_r} + H_{\hat{q}_r-1} z^{\hat{q}_r-1} + \cdots + H_1 z + H_0 + H_{-1} z^{-1} + \cdots \tag{3.1}$$

where $\hat{q}_r$ is the greatest order of the zeros of $A(z)$ at $z=\infty$. The sequence $H_k$ is the *forward fundamental matrix* which is easily implemented [3]. Then we have

**Theorem 3.1** The whole response of the system (2.1) will be :

$$y(k) = [H_{-k-q}, H_{-k-q+1}, \ldots, H_{-k-1}] \begin{bmatrix} A_q & 0 & \cdots & 0 \\ A_{q-1} & A_q & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_1 & A_2 & \cdots & A_q \end{bmatrix} \begin{bmatrix} y(0) \\ y(1) \\ \vdots \\ y(q-1) \end{bmatrix} +$$

$$+ [H_{-k}\ H_{-k+1} \ldots H_0 \ldots H_{\hat{q}_r}] \begin{bmatrix} B_0 & B_1 & \cdots & B_q & & 0 \\ & B_0 & \cdots & B_{q-1} & B_q & \\ 0 & & \ddots & & \ddots & \\ & & B_0 & \cdots & B_{q-1} & B_q \end{bmatrix} \begin{bmatrix} u(0) \\ u(1) \\ \vdots \\ u(k+\hat{q}_r+q) \end{bmatrix} \tag{3.2a}$$

or equivalently

$$y(k) = \sum_{i=1}^{q}\sum_{j=i}^{q} H_{-k-i}\, A_j\, y(j-i) + \sum_{i=0}^{k+\hat{q}_r}\sum_{j=0}^{q} H_{-k+i}\, B_j\, u(i+j) \quad \text{with } k=q,q+1,.. \tag{3.2b}$$

**Proof.** Equating the coefficients of the powers of s in the relation $A(z)\times A(z)^{-1}=I_r$ we have that :

$$\sum_{n=0}^{q} A_n\, H_{i-n} = \delta_i I_r \ \text{ or } \ \sum_{n=0}^{q} H_{i-n}\, A_n = \delta_i I_r \tag{3.3}$$

where $\delta_i=0$ for $i\neq 0$ and $\delta_0=1$. Now substituting y(k) from (3.2b) in (2.1) we have that

$$A(\sigma)\, y(k) = A(\sigma)\left[\sum_{i=1}^{q}\sum_{j=i}^{q} H_{-k-i}\, A_j\, y(j-i) + \sum_{i=0}^{k+\hat{q}_r}\sum_{j=0}^{q} H_{-k+i}\, B_j\, u(i+j)\right] = \tag{3.4}$$

$$= \sum_{n=0}^{q} A_n \sum_{i=1}^{q}\sum_{j=i}^{q} H_{-k-n-i}\, A_j\, y(j-i) + \sum_{n=0}^{q} A_n \sum_{i=0}^{k+n+\hat{q}_r}\sum_{j=0}^{q} H_{-k-n+i}\, B_j\, u(i+j) =$$

$$= \sum_{i=1}^{q}\sum_{j=i}^{q}\sum_{n=0}^{q} (A_n H_{-k-n-i})\, A_j\, y(j-i) +$$

$$+ \sum_{n=0}^{q} A_n\, H_{-k-n} \sum_{j=0}^{q} B_j u(j) + \sum_{n=0}^{q} A_n\, H_{-k-n+1} \sum_{j=0}^{q} B_j\, u(1+j) + \cdots + \sum_{n=0}^{q} A_n\, H_{-n} \sum_{j=0}^{q} B_j\, u(k+j) +$$

$$+ \sum_{n=0}^{q} A_n\, H_{-n+1} \sum_{j=0}^{q} B_j\, u(k+1+j) + \cdots + \sum_{n=0}^{q} A_n\, H_{\hat{q}_r-n} \sum_{j=0}^{q} B_j\, u(k+\hat{q}_r+j) + \cdots +$$

$$+ \sum_{n=1}^{q} A_n\, H_{\hat{q}_r-n+1} \sum_{j=0}^{q} B_j\, u(k+\hat{q}_r+1+j) + \cdots + A_q H_{\hat{q}_r} \sum_{j=0}^{q} B_j\, u(k+\hat{q}_r+q+j) =$$

$$\overset{(3.3)}{=} \sum_{i=1}^{q}\sum_{j=i}^{q} \delta_{-k-i} A_j y(j-i) + \delta_{-k}\sum_{j=0}^{q} B_j u(j) + \delta_{-k+1}\sum_{j=0}^{q} B_j u(1+j) + \cdots + \delta_0 \sum_{j=0}^{q} B_j\, u(k+j) = B(\sigma)\, u(k)$$

which proves the Theorem. □

From the above formula we can see that the discrete time ARMA–representations have no impulsive terms in their responses in contrast to the continuous time ARMA–representations. Another difference is that the discrete time ARMA–representations does not always have a solution. A necessary and sufficient condition such that the ARMA–representation (2.1) has a solution is that the initial conditions $\{y(0),y(1),...,y(q-1)\}$ satisfies the relation (2.1) for $k=0,1,..,q-1$. Therefore we define :

**Definition 3.2.** We define as

$$H_{iu} := \left\{ y(i),\, u(i)\ (i=0,1,\ldots,q-1) : \right.$$

$$\tilde{A}_1 \begin{bmatrix} H_0 & H_1 & \cdots & H_{q-1} \\ H_{-1} & H_0 & \cdots & H_{q-2} \\ \vdots & \vdots & \ddots & \vdots \\ H_{-q+1} & H_{-q+2} & \cdots & H_0 \end{bmatrix} \tilde{A}_1 \begin{bmatrix} y(0) \\ y(1) \\ \\ y(q-1) \end{bmatrix} = \tag{3.5}$$

$$\left. = \tilde{A}_1 \begin{bmatrix} H_0 & \cdots & H_{\hat{q}_r} & 0 & \cdots & 0 \\ H_{-1} & \cdots & H_{\hat{q}_r-1} & H_{\hat{q}_r} & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \\ H_{-q+1} & \cdots & H_{\hat{q}_r-q+1} & H_{\hat{q}_r-q+2} & \cdots & H_{\hat{q}_r} \end{bmatrix} \begin{bmatrix} B_0 & B_1 & \cdots & B_q & & 0 \\ & B_0 & \cdots & B_{q-1} & B_q & \\ 0 & & \ddots & & \ddots & \ddots \\ & & B_0 & \cdots & B_{q-1} & B_q \end{bmatrix} \begin{bmatrix} u(0) \\ u(1) \\ \vdots \\ u(2q+\hat{q}_r-1) \end{bmatrix} \right\}$$

where

$$\tilde{A}_1 = \begin{bmatrix} A_0 & 0 & \cdots & 0 \\ A_1 & A_0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_{q-1} & A_{q-2} & \cdots & A_0 \end{bmatrix} \tag{3.6}$$

the *admissible initial condition space* of (2.1) under nonzero inputs.

**Proof.** Consider the relation (2.1) for k=0,1,...,q–1 and write this in the form

$$\begin{bmatrix} A_q & A_{q-1} & \cdots & A_1 & A_0 & 0 & \cdots & 0 \\ 0 & A_q & \cdots & A_2 & A_1 & A_0 & \cdots & 0 \\ \vdots & & & & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_q & A_{q-1} & A_{q-2} & \cdots & A_0 \end{bmatrix} \begin{bmatrix} y(2q-1) \\ \vdots \\ y(q) \\ \vdots \\ y(0) \end{bmatrix} =$$

$$= \begin{bmatrix} B_q & B_{q-1} & \cdots & B_1 & B_0 & 0 & \cdots & 0 \\ 0 & B_q & \cdots & B_2 & B_1 & B_0 & \cdots & 0 \\ \vdots & & & & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & B_q & B_{q-1} & B_{q-2} & \cdots & B_0 \end{bmatrix} \begin{bmatrix} u(2q-1) \\ \vdots \\ u(q) \\ \vdots \\ u(0) \end{bmatrix} \tag{3.7}$$

Substitution of the values y(q), y(q+1), ..., y(2q–1) with the respective formula of (3.2) and use of (3.3) give us that the initial conditions {y(0),y(1),...,y(q–1)} satisfy the system iff the relation (3.5) is satisfied. □

As we can see in (3.2) the solution of (2.1) is determined in terms of the initial conditions $\{y(0), y(1), \ldots, y(q-1)\}$ and the input sequences of the system. An obvious disadvantage is that for each successive output $y(k)$, specified by $k=q,q+1,\ldots$, the coefficient matrices $H_j$ comprising each specific solution change. Therefore if the solution is required over a comparatively large range, say $[y(q),y(q+1),\ldots,y(100)]$ corresponding to $k=q,q+1,\ldots,100$, we would require the coefficient matrices $H_{-101},H_{-100},\ldots,H_{\hat{q}_r}$. An equivalent forward solution is presented in what follows for the general solution $y(k)$ depends on the *previous* q outputs $\{y(k-1),y(k-2),\ldots,y(k-q)\}$ and not on the q fixed initial conditions $\{y(0),y(1),\ldots,y(q-1)\}$. In this case the coefficient matrices required over a solution range is fixed, (i.e. independent of k), namely $H_{-q},H_{-q+1},\ldots,H_{\hat{q}_r}$.

**Corollary 3.3** Equation (3.2) is equivalent to the following forward recursion :

$$y(k) = -[H_{-1}, H_{-2},\ldots, H_{-q}] \begin{bmatrix} A_0 & 0 & \cdots & 0 \\ A_1 & A_0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_{q-1} & A_{q-2} & \cdots & A_0 \end{bmatrix} \begin{bmatrix} y(k-1) \\ y(k-2) \\ \vdots \\ y(k-q) \end{bmatrix} +$$

$$+ [H_{-q}\, H_{-q+1} \ldots H_0 \ldots H_{\hat{q}_r}] \begin{bmatrix} B_0 & B_1 & \cdots & B_q & & 0 \\ & B_0 & \cdots & B_{q-1} & B_q & \\ 0 & & \ddots & & \ddots & \\ & & B_0 & \cdots & B_{q-1} & B_q \end{bmatrix} \begin{bmatrix} u(k-q) \\ u(k-q+1) \\ \vdots \\ u(k+\hat{q}_r+q) \end{bmatrix} \tag{3.8a}$$

or equivalently

$$y(k) = -\sum_{i=1}^{q}\sum_{j=0}^{i-1} H_{-i}\, A_j\, y(k-i+j) + \sum_{i=0}^{q+\hat{q}_r}\sum_{j=0}^{q} H_{-q+i}\, B_j\, u(k-q+j+i) \tag{3.8b}$$

**Proof.** It is easily seen that the state vector $y(q)$ will be connected with the previous vectors $\{y(0),y(1),\ldots,y(q-1)\}$ according to (3.2) with the following relation

$$y(q) = [H_{-2q}, H_{-2q+1},\ldots, H_{-q-1}] \begin{bmatrix} A_q & 0 & \cdots & 0 \\ A_{q-1} & A_q & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_1 & A_2 & \cdots & A_q \end{bmatrix} \begin{bmatrix} y(0) \\ y(1) \\ \vdots \\ y(q-1) \end{bmatrix} + \tag{3.9}$$

$$+ [H_{-q}\, H_{-q+1} \ldots H_0 \ldots H_{\hat{q}_r}] \begin{bmatrix} B_0 & B_1 & \cdots & B_q & & 0 \\ & B_0 & \cdots & B_{q-1} & B_q & \\ 0 & & \ddots & & \ddots & \ddots \\ & & & B_0 & \cdots & B_{q-1} \; B_q \end{bmatrix} \begin{bmatrix} u(0) \\ u(1) \\ \vdots \\ u(2q+\hat{q}_r) \end{bmatrix} \overset{(3.3)}{=}$$

$$= -[H_{-1}, H_{-2}, \ldots, H_{-q}] \begin{bmatrix} A_0 & 0 & \cdots & 0 \\ A_1 & A_0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_{q-1} & A_{q-2} & \cdots & A_0 \end{bmatrix} \begin{bmatrix} y(q-1) \\ y(q-2) \\ \\ y(0) \end{bmatrix} +$$

$$+ [H_{-q}\, H_{-q+1} \ldots H_0 \ldots H_{\hat{q}_r}] \begin{bmatrix} B_0 & B_1 & \cdots & B_q & & 0 \\ & B_0 & \cdots & B_{q-1} & B_q & \\ 0 & & \ddots & & \ddots & \ddots \\ & & & B_0 & \cdots & B_{q-1} \; B_q \end{bmatrix} \begin{bmatrix} u(0) \\ u(1) \\ \vdots \\ u(2q+\hat{q}_r) \end{bmatrix}$$

The system is time invariant and thus the same relation will connect the output $y(k)$ with the previous vectors $\{y(k-1),y(k-2),\ldots,y(k-q)\}$. Thus if we replace $\{y(q),y(q-1),\ldots,y(0)\}$ with $\{y(k),y(k-1),\ldots,y(k-q)\}$ respectively and $\{u(0),u(1),\ldots,u(2q+\hat{q}_r)\}$ with $\{u(k-q),u(k-q+1),\ldots,u(k+q+\hat{q}_r)\}$ respectively we get the relation (3.8). □

The advantage of the formula (3.8) is, as we have already mentioned, is that it depends only on the $q+\hat{q}_r+1$ Laurent expansion terms $\{H_{-q}, H_{-q+1},\ldots,H_0,\ldots,H_{\hat{q}_r}\}$. The above formula is very usefull when we need to determine $y(k)$ in the closed interval $[q,+\infty]$, because we always have to start to compute from $y(q)$, $y(q+1)$, ... in contrast to the solution formula (3.2) where only the q first initial conditions are enough for the determination of the solution in the interval $[n,+\infty]$ where $n>q$. Another advantage of (3.8) is that the round–of errors for the determination of the $q+\hat{q}_r+1$ Laurent expansion terms $\{H_{-q}, H_{-q+1}, \ldots, H_0, \ldots, H_{\hat{q}_r}\}$ are less than the ones for the determination of $\{H_{-k},\ldots,H_{\hat{q}_r}\}$ in (3.2).

## 3.2. The Backward Solution of ARMA–Representations

Consider the discrete time ARMA–representation (2.1). The Laurent series expansion about zero for the resolvent matrix is given by

$$A(z)^{-1} = V_{-\ell}z^{-\ell} + V_{-\ell+1}z^{-\ell+1} + \cdots + V_{-1}z^{-1} + V_0 + V_1 z^1 + \cdots \qquad (3.10)$$

where it is easily seen that $\ell$ is the greatest order of the zeros of A(z) at z=0. The sequence $V_k$ is the *backward fundamental matrix* and is easily implemented according to Lemma 2.1 and [3]. Then we have

**Theorem 3.4** The whole response of the system (2.1) will be :

$$y(k) = [V_{N-k}, V_{N-k-1}, \ldots, V_{N-k-q+1}] \begin{bmatrix} A_0 & 0 & \cdots & 0 \\ A_1 & A_0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_{q-1} & A_{q-2} & \cdots & A_0 \end{bmatrix} \begin{bmatrix} y(N) \\ y(N-1) \\ \vdots \\ y(N-q+1) \end{bmatrix} +$$

$$+ [V_{N-k-q}\ V_{N-k-q-1} \cdots V_{-\ell}] \begin{bmatrix} B_q & B_{q-1} & \cdots & B_0 & & 0 \\ & B_q & \cdots & B_1 & B_0 & \\ 0 & & \ddots & & \ddots & \ddots \\ & & B_q & \cdots & B_1 & B_0 \end{bmatrix} \begin{bmatrix} u(N) \\ u(N-1) \\ \vdots \\ u(k-\ell) \end{bmatrix} \qquad (3.11a)$$

or equivalently

$$y(k) = \sum_{i=0}^{q-1} \sum_{j=0}^{i} V_{N-k-i} A_j\, y(N-i+j) + \sum_{i=0}^{q+k-N-\ell} \sum_{j=0}^{q} V_{N-k-q-i} B_j\, u(N+j-i-q) \qquad (3.11b)$$

**Proof.** Consider the dual ARMA–representation (2.7) of (2.1) :

$$\tilde{A}(\sigma)\tilde{y}(k) = \tilde{B}(\sigma)\tilde{u}(k) \qquad (3.12a)$$

where

$$\tilde{A}(\sigma)=A_0\sigma^q + \cdots + A_{q-1}\sigma + A_q \quad \text{and} \quad \tilde{B}(\sigma)=B_0\sigma^q + \cdots + B_{q-1}\sigma + B_q \qquad (3.12b)$$

Let also from (2.4) the Laurent expansion at s=$\infty$ of $\tilde{A}(\sigma)^{-1}$ be

$$\tilde{A}(w)^{-1} = \tilde{H}_f w^f + \tilde{H}_{f-1} w^{f-1} + \cdots + \tilde{H}_1 w + \tilde{H}_0 + \tilde{H}_{-1} w^{-1} + \cdots \qquad (3.13)$$

Then from Theorem 3.1 the solution of (3.12a) will be

$$\tilde{y}(k) = [\tilde{H}_{-k-q}, \tilde{H}_{-k-q+1}, \ldots, \tilde{H}_{-k-1}] \begin{bmatrix} A_0 & 0 & \cdots & 0 \\ A_1 & A_0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_{q-1} & A_{q-2} & \cdots & A_0 \end{bmatrix} \begin{bmatrix} \tilde{y}(0) \\ \tilde{y}(1) \\ \\ \tilde{y}(q-1) \end{bmatrix} + \qquad (3.14)$$

$$+ [\tilde{H}_{-k}\,\tilde{H}_{-k+1}\ldots\tilde{H}_0\ldots\tilde{H}_f]\begin{bmatrix} B_q & B_{q-1} & \cdots & B_0 & & 0 \\ & B_q & \cdots & B_1 & B_0 & \\ 0 & & \ddots & & \ddots & \ddots \\ & & & B_q & \cdots & B_1 \quad B_0 \end{bmatrix}\begin{bmatrix} \tilde{u}(0) \\ \tilde{u}(1) \\ \vdots \\ \tilde{u}(k+f+q) \end{bmatrix} \quad \begin{matrix} (2.5) \\ = \\ V_{-i} = \tilde{H}_{f-\ell+i} \\ q+f=\ell \end{matrix}$$

$$= [V_k, V_{k-1},\ldots, V_{-q+k+1}]\begin{bmatrix} A_0 & 0 & \cdots & 0 \\ A_1 & A_0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_{q-1} & A_{q-2} & \cdots & A_0 \end{bmatrix}\begin{bmatrix} \tilde{y}(0) \\ \tilde{y}(1) \\ \\ \tilde{y}(q-1) \end{bmatrix} +$$

$$+ [V_{-q+k}\,V_{-q+k-1}\ldots V_{-q}\ldots V_{-\ell}]\begin{bmatrix} B_q & B_{q-1} & \cdots & B_0 & & 0 \\ & B_q & \cdots & B_1 & B_0 & \\ 0 & & \ddots & & \ddots & \ddots \\ & & & B_q & \cdots & B_1 \quad B_0 \end{bmatrix}\begin{bmatrix} \tilde{u}(0) \\ \tilde{u}(1) \\ \vdots \\ \tilde{u}(k+f+q) \end{bmatrix}$$

From Theorem 2.2 we have that the solution $y(k)$ of (2.1) for an input $u(k)$ is given by the solution $\tilde{y}(N-k)$ of (2.7) for an input $\tilde{u}(N-k)$ and the converse. Thus we can replace the initial conditions $\tilde{y}(i)$, $\tilde{u}(i)$ of the system (2.7) with the final conditions $y(N-i)$ and $u(N-i)$ of the system (2.1) as well as the solution $\tilde{y}(N-k)$ of the system (2.7) with the solution $y(k)$ of (2.1), which proves the relation (3.11). □

A necessary and sufficient condition such that the ARMA–representation (2.1) has a solution is that the final conditions $\{y(N),y(N-1),\ldots,y(N-q+1)\}$ satisfies the relation (2.1) for $k=N, N-1, \ldots, N-q+1$. Therefore we define :

**Definition 3.5.** We define as

$$\bar{H}_{iu} := \left\{ y(i),\ u(i)\ (i=N,N-1,\ldots,N-q+1) : \right. \tag{3.15}$$

$$\tilde{A}_2\begin{bmatrix} V_{-q} & V_{-q-1} & \cdots & V_{-2q+1} \\ V_{-q+1} & V_{-q} & \cdots & V_{-2q+2} \\ \vdots & \vdots & \ddots & \vdots \\ V_{-1} & V_{-2} & \cdots & V_{-q} \end{bmatrix}\tilde{A}_2\begin{bmatrix} y(N) \\ y(N-1) \\ \vdots \\ y(N-q+1) \end{bmatrix} =$$

$$=\tilde{A}_2\begin{bmatrix} V_{-q} & \cdots & V_{-\ell} & 0 & \cdots & 0 \\ V_{-q+1} & \cdots & V_{-\ell+1} & V_{-\ell} & \cdots & 0 \\ \vdots & & \vdots & & \ddots & \\ V_{-1} & \cdots & V_{-\ell+q-1} & V_{-\ell+q-2} & \cdots & V_{-\ell} \end{bmatrix}\begin{bmatrix} B_q & B_{q-1} & \cdots & B_0 & & 0 \\ & B_q & \cdots & B_1 & B_0 & \\ 0 & & \ddots & & \ddots & \ddots \\ & & B_q & \cdots & B_1 & B_0 \end{bmatrix}\begin{bmatrix} u(N) \\ u(N-1) \\ \vdots \\ u(N-q-\ell+1) \end{bmatrix}\Bigg\}$$

where

$$\tilde{A}_2 = \begin{bmatrix} A_q & 0 & \cdots & 0 \\ A_{q-1} & A_q & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_1 & A_2 & \cdots & A_q \end{bmatrix} \tag{3.16}$$

the *admissible final condition space* of (2.1) under nonzero inputs.

**Proof.** Consider the relation (2.1) for k=N–q,N–q–1,...,N–2q+1 and write this in the form

$$\begin{bmatrix} A_q & A_{q-1} & \cdots & A_1 & A_0 & 0 & \cdots & 0 \\ 0 & A_q & \cdots & A_2 & A_1 & A_0 & \cdots & 0 \\ \vdots & & & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_q & A_{q-1} & A_{q-2} & \cdots & A_0 \end{bmatrix}\begin{bmatrix} y(N) \\ \vdots \\ y(N-q) \\ \vdots \\ y(N-2q+1) \end{bmatrix} =$$

$$= \begin{bmatrix} B_q & B_{q-1} & \cdots & B_1 & B_0 & 0 & \cdots & 0 \\ 0 & B_q & \cdots & B_2 & B_1 & B_0 & \cdots & 0 \\ \vdots & & & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & B_q & B_{q-1} & B_{q-2} & \cdots & B_0 \end{bmatrix}\begin{bmatrix} u(N) \\ \vdots \\ u(N-q) \\ \vdots \\ u(N-2q+1) \end{bmatrix} \tag{3.17}$$

Substitution of the values y(N–q), y(N–q–1), ..., y(N–2q+1) with the respective formula of (3.11) and use of (3.3) give us that the final conditions {y(N),y(N–1),..., y(N–q+1)} satisfy the system iff the relation (3.16) is satisfied. □

A backward solution formula in terms of the following q terms and the input sequence of the system is provided by the following

**Corollary 3.6** Equation (3.12) is equivalent to the backward recursion :

$$y(k) = [V_q, V_{q-1}, \ldots, V_1]\begin{bmatrix} A_0 & 0 & \cdots & 0 \\ A_1 & A_0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_{q-1} & A_{q-2} & \cdots & A_0 \end{bmatrix}\begin{bmatrix} y(k+q) \\ y(k+q-1) \\ \vdots \\ y(k+1) \end{bmatrix} +$$

$$+ [V_0\, V_{-1} \ldots V_{-\ell}] \begin{bmatrix} B_q & B_{q-1} & \cdots & B_0 & & 0 \\ & B_q & \cdots & B_1 & B_0 & \\ 0 & & \ddots & & \ddots & \ddots \\ & & B_q & \cdots & B_1 & B_0 \end{bmatrix} \begin{bmatrix} u(k+q) \\ u(k+q-1) \\ \vdots \\ u(k-\ell) \end{bmatrix} \tag{3.18}$$

**Proof.** Following similar lines with the proof of Corollary 3.3 we obtain the result. □

The advantage of the formula (3.18) is, that depends only from the $q+\ell+1$ Laurent expansion terms $[V_q\, V_{q-1} \ldots V_0 \ldots V_{-\ell}]$ and thus we don't need the continuous computation of the Laurent expansion terms which gives rise to numerical errors.

## 3.3 The Symmetric Solution

In this section we consider (2.1) as a relation between the y(k) and u(k) over an interval [0,N], with k not necessarily the time index. Such an interpretation is used in economics and elsewhere [8], [10]. Consider the discrete time ARMA– representation (2.1) and the Laurent series expansion about infinity for its resolvent matrix in (3.1). Then

**Lemma 3.7**

(i) A right inverse of the matrix

$$A_N := \begin{bmatrix} A_q & A_{q-1} & \cdots & A_0 & 0 & \cdots & 0 \\ 0 & A_q & \cdots & A_1 & A_0 & \cdots & 0 \\ \vdots & \vdots & \ddots & & & \ddots & \vdots \\ 0 & 0 & \cdots & A_q & A_{q-1} & \cdots & A_0 \end{bmatrix} \in \mathbb{R}^{(N-q+1)p \times (N+1)\ell} \tag{3.19a}$$

is the following

$$A_N^r = \begin{bmatrix} H_{-q} & H_{-q-1} & \cdots & H_{-N} \\ H_{-q+1} & H_{-q} & \cdots & H_{-N+1} \\ \vdots & \vdots & \ddots & \vdots \\ H_{-q+N} & H_{-q+N-1} & \cdots & H_0 \end{bmatrix} \tag{3.19b}$$

(ii) A left inverse of the matrix

$$T := \begin{bmatrix} A_0 & 0 & \cdots & 0 \\ A_1 & A_0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_q & A_{q-1} & \cdots & A_0 \\ 0 & A_q & & A_1 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & A_q \end{bmatrix} \in \mathbb{R}^{(N-q+1)p \times (N-2q+1)\ell} \tag{3.20a}$$

is the following :

$$T^r = \begin{bmatrix} H_0 & H_{-1} & \cdots & H_{-N+q} \\ H_1 & H_0 & \cdots & H_{-N+q+1} \\ \vdots & \vdots & \ddots & \vdots \\ H_{N-2q} & H_{N-2q-1} & \cdots & H_{-q} \end{bmatrix} \tag{3.20b}$$

**Proof.** Using the relation (3.3) we can easily show that $A_N \times A_N^r = I$ and $T^r \times T = I$ which proves the Lemma. □

We can now show the following :

**Theorem 3.8** The solution of the ARMA–representation (2.1) in terms of the initial and final conditions, $\{y(0),y(1),...,y(q-1)\}$ and $\{y(N),y(N-1),...,y(N-q+1)\}$ respectively, is given by the following formula :

$$y(k) = [H_{-k-1}\ H_{-k-2} \cdots H_{-k-q}] \begin{bmatrix} A_q & A_{q-1} & \cdots & A_1 \\ 0 & A_q & \cdots & A_2 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_q \end{bmatrix} \begin{bmatrix} y(q-1) \\ y(q-2) \\ \vdots \\ y(0) \end{bmatrix} +$$

$$+ [H_{N-k}\ H_{N-k-1} \cdots H_{N-k-q+1}] \begin{bmatrix} A_0 & 0 & \cdots & 0 \\ A_1 & A_0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_{q-1} & A_{q-2} & \cdots & A_0 \end{bmatrix} \begin{bmatrix} y(N) \\ y(N-1) \\ \vdots \\ y(N-q+1) \end{bmatrix} + \tag{3.21a}$$

$$+ [H_{N-k-q}\ H_{N-k-q-1} \cdots H_{-k}] \begin{bmatrix} B_q & B_{q-1} & \cdots & B_1 & B_0 & 0 & \cdots & 0 \\ 0 & B_q & \cdots & B_2 & B_1 & B_0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \\ 0 & 0 & \cdots & B_q & B_{q-1} & B_{q-2} & \cdots & B_0 \end{bmatrix} \begin{bmatrix} u(N) \\ u(N-1) \\ \vdots \\ u(0) \end{bmatrix}$$

or equivalently

$$y(k) = \sum_{i=1}^{q} \sum_{j=i}^{q} H_{-k-i} A_j\, y(j-i) + \sum_{i=0}^{q-1} \sum_{j=0}^{i} H_{N-k-i} A_j y(N-i+j) + \tag{3.21b}$$

$$+ \sum_{i=0}^{N-q} \sum_{j=0}^{q} V_{N-k-q-i}\, B_j\, u(N+j-i-q)$$

under the following restrictions between the initial conditions, final conditions and input sequences :

$$\begin{bmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{bmatrix} \begin{bmatrix} X_A \, y_{N-q+1,N} \\ X_{\tilde{A}} \, y_{0,q-1} \end{bmatrix} = \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix} B_N \, u_{0,N} \tag{3.22a}$$

where

$$W_{11} = \begin{bmatrix} H_{-q} & H_{-q-1} & \cdots & H_{-2q+1} \\ H_{-q+1} & H_{-q} & \cdots & H_{-2q+2} \\ \vdots & \vdots & \ddots & \vdots \\ H_{-1} & H_{-2} & \cdots & H_{-q} \end{bmatrix} ; \quad W_{12} = \begin{bmatrix} H_{-N+q-1} & H_{-N+q-2} & \cdots & H_{-N} \\ H_{-N+q} & H_{-N+q-1} & \cdots & H_{-N+1} \\ \vdots & \vdots & \ddots & \vdots \\ H_{-N+2q-2} & H_{-N+2q-3} & \cdots & H_{-N+q-1} \end{bmatrix} \tag{3.22b}$$

$$W_{21} = \begin{bmatrix} H_{N-2q+1} & H_{N-2q} & \cdots & H_{N-3q+2} \\ H_{N-2q+2} & H_{N-2q+1} & \cdots & H_{N-3q+3} \\ \vdots & \vdots & \ddots & \vdots \\ H_{N-q} & H_{N-q-1} & \cdots & H_{N-2q+1} \end{bmatrix} ; \quad W_{22} = \begin{bmatrix} H_0 & H_{-1} & \cdots & H_{-q+1} \\ H_1 & H_0 & \cdots & H_{-q+2} \\ \vdots & \vdots & \ddots & \vdots \\ H_{q-1} & H_{q-2} & \cdots & H_0 \end{bmatrix}$$

$$X_A = \begin{bmatrix} A_q & A_{q-1} & \cdots & A_1 \\ 0 & A_q & \cdots & A_2 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_q \end{bmatrix} ; \quad X_{\tilde{A}} = \begin{bmatrix} A_0 & 0 & \cdots & 0 \\ A_1 & A_0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_{q-1} & A_{q-2} & \cdots & A_0 \end{bmatrix} ; \quad B_N = \begin{bmatrix} B_q & B_{q-1} & \cdots & B_1 & B_0 & 0 & \cdots & 0 \\ 0 & B_q & \cdots & B_2 & B_1 & B_0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \\ 0 & 0 & \cdots & B_q & B_{q-1} & B_{q-2} & \cdots & B_0 \end{bmatrix}$$

$$Z_1 = \begin{bmatrix} H_{-q} & H_{-q-1} & \cdots & H_{-N} \\ H_{-q+1} & H_{-q} & \cdots & H_{-N+1} \\ \vdots & \vdots & \ddots & \vdots \\ H_{-1} & H_{-2} & \cdots & H_{-N+q-1} \end{bmatrix} ; \quad Z_2 = \begin{bmatrix} H_{N-2q+1} & H_{N-2q} & \cdots & H_{-q+1} \\ H_{N-2q+2} & H_{N-2q+1} & \cdots & H_{-q+2} \\ \vdots & \vdots & \ddots & \vdots \\ H_{N-q} & H_{N-q-1} & \cdots & H_0 \end{bmatrix}$$

$$y_{N-q+1,N} = \begin{bmatrix} y(N) \\ y(N-1) \\ \vdots \\ y(N-q+1) \end{bmatrix} ; \quad y_{0,q-1} = \begin{bmatrix} y(q-1) \\ y(q-2) \\ \vdots \\ y(0) \end{bmatrix} ; \quad u_{0,N} = \begin{bmatrix} u(N) \\ u(N-1) \\ \vdots \\ u(0) \end{bmatrix}$$

We call the solution (3.21), the *symmetric solution* of (2.1) and the equations (3.22a) the *boundary mapping equations* of (2.1).

**Proof.** Rewritting (2.1) in the form

$$\underbrace{\begin{bmatrix} A_q & A_{q-1} & \cdots & A_1 & A_0 & 0 & \cdots & 0 \\ 0 & A_q & \cdots & A_2 & A_1 & A_0 & \cdots & 0 \\ \vdots & \vdots & \ddots & & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_q & A_{q-1} & A_{q-2} & \cdots & A_0 \end{bmatrix}}_{A_N} \underbrace{\begin{bmatrix} y(N) \\ y(N-1) \\ \vdots \\ y(0) \end{bmatrix}}_{y_{0,N}} =$$

$$= \underbrace{\begin{bmatrix} B_q & B_{q-1} & \cdots & B_1 & B_0 & 0 & \cdots & 0 \\ 0 & B_q & \cdots & B_2 & B_1 & B_0 & \cdots & 0 \\ \vdots & \vdots & \ddots & & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & B_q & B_{q-1} & B_{q-2} & \cdots & B_0 \end{bmatrix}}_{B_N} \underbrace{\begin{bmatrix} u(N) \\ u(N-1) \\ \vdots \\ u(0) \end{bmatrix}}_{u_{0,N}} \Leftrightarrow \tag{3.23}$$

$$\left[\begin{array}{ccc|ccc|ccc} A_q & A_{q-1} & \cdots & A_0 & 0 & & & & \\ 0 & A_q & \cdots & A_1 & A_0 & & & & \\ \vdots & \vdots & & \vdots & & \ddots & & & \\ \vdots & \vdots & & A_q & A_{q-1} & & & & \\ \vdots & \vdots & \cdots & & A_q & & A_0 & & \\ \vdots & \vdots & & & & \ddots & \vdots & \ddots & \\ 0 & 0 & \cdots & & & A_q & A_{q-1} & \cdots & A_0 \end{array}\right]^{\textstyle T} \left[\begin{array}{c} y(N) \\ \vdots \\ y(N-q+1) \\ \hline y(N-q) \\ \vdots \\ y(q) \\ \hline y(q-1) \\ \vdots \\ y(0) \end{array}\right] =$$

$$\begin{bmatrix} B_q & B_{q-1} & \cdots & B_1 & B_0 & 0 & \cdots & 0 \\ 0 & B_q & \cdots & B_2 & B_1 & B_0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & B_q & B_{q-1} & B_{q-2} & \cdots & B_0 \end{bmatrix} \begin{bmatrix} u(N) \\ u(N-1) \\ \vdots \\ u(0) \end{bmatrix} \Leftrightarrow$$

$$\begin{bmatrix} \begin{bmatrix} A_q & A_{q-1} & \cdots & A_1 \\ 0 & A_q & \cdots & A_2 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_q \end{bmatrix} \begin{bmatrix} y(N) \\ y(N-1) \\ \vdots \\ y(N-q+1) \end{bmatrix} \\ \mathbf{0} \\ \begin{bmatrix} A_0 & 0 & \cdots & 0 \\ A_1 & A_0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_{q-1} & A_{q-2} & \cdots & A_0 \end{bmatrix} \begin{bmatrix} y(q-1) \\ y(q-2) \\ \vdots \\ y(0) \end{bmatrix} \end{bmatrix} = [-T \;\; B_N] \begin{bmatrix} y_{q,N-q} \\ u_{0,N} \end{bmatrix}$$

where $y_{q,N-q}=[y(N-q)^T,\ldots,y(q)^T]^T$. Premultiply both sides of (3.23) by $A_N^r$ we obtain from the first q and the last q equations the relations (3.22a), while from the middle N–2q equations, after the use of Lemma 3.7 we obtain the formula (3.21a). □

A necessary and sufficient condition such that the ARMA–representation (2.1) has a solution is that the initial, final conditions and input sequences satisfies the relation (3.22). Therefore we define :

**Definition 3.9** We define as

$$\tilde{H}_{iu} := \left\{ y_{0,q-1}, y_{N-q+1,N} : \begin{bmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{bmatrix} \begin{bmatrix} X_A y_{N-q+1,N} \\ X_{\tilde{A}} y_{0,q-1} \end{bmatrix} = \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix} B_N u_{0,N} \right\} \quad (3.24)$$

the *symmetric boundary condition space* of (2.1) under nonzero inputs. □

The boundary mapping equation (3.22a) represents the restrictions that the system places on the boundary variables $y_{0,q-1}$, $y_{N-q+1,N}$ in order for the system to be solvable. Addition restrictions on the variables can be applied to the system in the form of an auxiliary equation

$$W_{31}\, y_{N-q+1,N} + W_{32}\, y_{0,q-1} = C \quad (3.25)$$

The combined boundary equation formed from (3.22a) and (3.25)

$$\underbrace{\begin{bmatrix} W_{11}X_A & W_{12}X_{\tilde{A}} \\ W_{21}X_A & W_{22}X_{\tilde{A}} \\ W_{31} & W_{32} \end{bmatrix}}_{Z} \underbrace{\begin{bmatrix} y_{N-q+1,N} \\ y_{0,q-1} \end{bmatrix}}_{Y} = \underbrace{\begin{bmatrix} Z_1 u_{0,N} \\ Z_2 u_{0,N} \\ C \end{bmatrix}}_{C} \quad (3.26)$$

will subsequently define a unique solution iff $ZZ^+C=C$ and Z has full column rank, where $Z^+$ denotes the pseudoinverse of Z i.e. $Y=Z^+C$.

Alternative forms of the solution formula (3.21) are given by the following

**Corollary 3.10** The symmetric solution (3.21) can be written in the alternative forms

FORWARD-SYMMETRIC

$$y(k) = -[H_{-1}\ H_{-2} \cdots H_{-q}] \begin{bmatrix} A_0 & 0 & \cdots & 0 \\ A_1 & A_0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_{q-1} & A_{q-2} & \cdots & A_0 \end{bmatrix} \begin{bmatrix} y(k-1) \\ y(k-2) \\ \vdots \\ y(k-q) \end{bmatrix} +$$

$$+ [H_{N-k}\ H_{N-k-1} \cdots H_{N-k-q+1}] \begin{bmatrix} A_0 & 0 & \cdots & 0 \\ A_1 & A_0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_{q-1} & A_{q-2} & \cdots & A_0 \end{bmatrix} \begin{bmatrix} y(N) \\ y(N-1) \\ \vdots \\ y(N-q+1) \end{bmatrix} + \tag{3.27a}$$

$$+ [H_{N-k-q}\ H_{N-k-q-1} \cdots H_{-q}] \begin{bmatrix} B_q & B_{q-1} & \cdots & B_1 & B_0 & 0 & \cdots & 0 \\ 0 & B_q & \cdots & B_2 & B_1 & B_0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \\ 0 & 0 & \cdots & B_q & B_{q-1} & B_{q-2} & \cdots & B_0 \end{bmatrix} \begin{bmatrix} u(N) \\ u(N-1) \\ \vdots \\ u(k-q) \end{bmatrix}$$

or

$$y(k) = -\sum_{i=1}^{q} \sum_{j=0}^{i-1} H_{-i} A_j\, y(k-j-i) + \sum_{i=0}^{q-1} \sum_{j=0}^{i} H_{N-k-i} A_j y(N-i+j) + \\ + \sum_{i=0}^{N-k} \sum_{j=0}^{q} H_{N-k-q-i}\, B_j\, u(N+j-i-q) \tag{3.27b}$$

BACKWARD-SYMMETRIC

$$y(k) = [H_{-k-1}\ H_{-k-2} \cdots H_{-k-q}] \begin{bmatrix} A_q & A_{q-1} & \cdots & A_1 \\ 0 & A_q & \cdots & A_2 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_q \end{bmatrix} \begin{bmatrix} y(q-1) \\ y(q-2) \\ \vdots \\ y(0) \end{bmatrix} +$$

$$- [H_0\ H_{-1} \cdots H_{-q+1}] \begin{bmatrix} A_q & A_{q-1} & \cdots & A_1 \\ 0 & A_q & \cdots & A_2 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_q \end{bmatrix} \begin{bmatrix} y(k+q) \\ y(k+q-1) \\ \vdots \\ y(k+1) \end{bmatrix} + \tag{3.27c}$$

$$+ [H_0\ H_{-1} \cdots H_{-k}] \begin{bmatrix} B_q & B_{q-1} & \cdots & B_1 & B_0 & 0 & \cdots & 0 \\ 0 & B_q & \cdots & B_2 & B_1 & B_0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \\ 0 & 0 & \cdots & B_q & B_{q-1} & B_{q-2} & \cdots & B_0 \end{bmatrix} \begin{bmatrix} u(k+q) \\ u(k-1) \\ \vdots \\ u(0) \end{bmatrix}$$

or

$$y(k) = \sum_{i=1}^{q} \sum_{j=i}^{q} H_{-k-i} A_j\, y(j-i) - \sum_{i=0}^{q-1} \sum_{j=i+1}^{q} H_{-i} A_j y(k+j-i) + \tag{3.27d}$$

$$+ \sum_{i=0}^{k} \sum_{j=0}^{q} H_{-i} B_j\, u(k+j-i)$$

**Proof**. Taking the solution formula (3.21a) and using the following three tasks

**(i)** Assume that $k=\nu q+v$ $(N-k=\nu q+v)$.

**(ii)** Do the following replacement :

$$[H_{-s}\ H_{-s+1} \cdots H_{-s+q-1}] \begin{bmatrix} A_q & 0 \cdots 0 \\ A_{q-1} & A_q \cdots 0 \\ \vdots & \vdots \ddots \vdots \\ A_1 & A_2 \cdots A_q \end{bmatrix} = \tag{3.28a}$$

$$= -[H_{-s+q}\ H_{-s+q+1} \cdots H_{-s+2q-1}] \begin{bmatrix} A_0 & A_1 & \cdots & A_{q-1} \\ 0 & A_0 & \cdots & A_{q-2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_0 \end{bmatrix} \text{ for } s \neq q$$

$$\left\{ [H_{N-k-sq}\ H_{N-k-sq-1} \cdots H_{N-k-(s+1)q+1}] \begin{bmatrix} A_0 & 0 & \cdots & 0 \\ A_1 & A_0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_{q-1} & A_{q-2} & \cdots & A_0 \end{bmatrix} = \right. \tag{3.28b}$$

$$\left. = -[H_{N-k-(s+1)q}\ H_{N-k-(s+1)q-1} \cdots H_{N-k-(s+2)q+1}] \begin{bmatrix} A_q & A_{q-1} & \cdots & A_1 \\ 0 & A_q & \cdots & A_2 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_q \end{bmatrix} \text{ for } s \neq q \right\}$$

which is based on (3.3).

**(iii)** Do the following replacement (using (2.1))

$$\begin{bmatrix} A_0 & A_1 & \cdots & A_{q-1} \\ 0 & A_0 & \cdots & A_{q-2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_0 \end{bmatrix} \begin{bmatrix} y(k+sq) \\ y(k+sq+1) \\ \vdots \\ y(k+(s+1)q-1 \end{bmatrix} = - \begin{bmatrix} A_q & 0 \cdots 0 \\ A_{q-1} & A_q \cdots 0 \\ \vdots & \vdots \ddots \vdots \\ A_1 & A_2 \cdots A_q \end{bmatrix} \begin{bmatrix} y(k+(s+1)q) \\ y(k+(s+1)q+1) \\ \vdots \\ y(k+(s+2)q-1) \end{bmatrix} + \tag{3.29a}$$

$$+\begin{bmatrix} B_0 & B_1 & \cdots & B_{q-1} & B_q & 0 & \cdots & 0 \\ 0 & B_0 & \cdots & B_{q-2} & B_{q-1} & B_q & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \\ 0 & 0 & \cdots & B_0 & B_1 & B_2 & \cdots & B_q \end{bmatrix} \begin{bmatrix} u(k+sq) \\ u(k+sq+1) \\ \vdots \\ u(k+(s+2)q-1 \end{bmatrix}$$

$$\left\{ \begin{bmatrix} A_q & A_{q-1} & \cdots & A_1 \\ 0 & A_q & \cdots & A_2 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_q \end{bmatrix} \begin{bmatrix} y(N-sq) \\ y(N-sq-1) \\ \vdots \\ y(N-(s+1)q+1) \end{bmatrix} = -\begin{bmatrix} A_0 & 0 & \cdots & 0 \\ A_1 & A_0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_{q-1} & A_{q-2} & \cdots & A_0 \end{bmatrix} \begin{bmatrix} y(N-(s+1)q) \\ y(N-(s+1)q-1) \\ \vdots \\ y(N-(s+2)q+1) \end{bmatrix} + \right. \tag{3.29b}$$

$$\left. +\begin{bmatrix} B_0 & B_1 & \cdots & B_{q-1} & B_q & 0 & \cdots & 0 \\ 0 & B_0 & \cdots & B_{q-2} & B_{q-1} & B_q & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \\ 0 & 0 & \cdots & B_0 & B_1 & B_2 & \cdots & B_q \end{bmatrix} \begin{bmatrix} u(N-sq) \\ u(N-sq-1) \\ \vdots \\ u(N-(s+2)q+1) \end{bmatrix} \right\}$$

which is based on (2.1) we get the solution formula (3.27). □

In the *Forward–Symmetric* case we still solve within the region [0,N] but now the solution depends on the q final conditions {y(N),y(N–1),...,y(N–q+1)} and the *previous* q outputs {y(k–1),y(k–2),...,y(k–q)} and no longer on the q fixed initial conditions {y(0),y(1),...,y(q–1)}. Therefore we solve *forwards* in the interval.

In the *Backward–Symmetric* case we again still solve within the region [0,N] but now the solution depends on the q initial conditions {y(0),y(1),...,y(q–1)} and the *future* q outputs {y(k+1),y(k+2),...,y(k+q)} and no longer on the q fixed final conditions {y(N),y(N–1),...,y(N–q+1)}. Therefore we solve *backwards* in the interval.

## 4. Illustrative Example

Consider the following discrete time ARMA–representation :

$$\underbrace{\begin{bmatrix} \sigma^2+5\sigma+6 & \sigma+1 & 0 \\ 2\sigma-5 & 3\sigma+2 & 1 \\ 0 & -1 & 0 \end{bmatrix}}_{A(\sigma)\equiv A_0+A_1\sigma+A_2\sigma^2} \underbrace{\begin{bmatrix} y_1(k) \\ y_2(k) \\ y_3(k) \end{bmatrix}}_{y(k)} = \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{B(\sigma)} u(k) \tag{4.1}$$

Let also the Laurent expansion of $A(\sigma)^{-1}$ at $s=\infty$ :

$$A(\sigma)^{-1}=\underbrace{\begin{bmatrix}0&0&0\\0&0&0\\0&0&3\end{bmatrix}}_{H_1}s+\underbrace{\begin{bmatrix}0&0&0\\0&0&-1\\0&1&0\end{bmatrix}}_{H_0}+\underbrace{\begin{bmatrix}0&0&1\\0&0&0\\-2&0&13\end{bmatrix}}_{H_{-1}}s^{-1}+$$

$$+\underbrace{\begin{bmatrix}1&0&-4\\0&0&0\\15&0&-48\end{bmatrix}}_{H_{-2}}s^{-2}+\underbrace{\begin{bmatrix}-5&0&14\\0&0&0\\-63&0&162\end{bmatrix}}_{H_{-3}}s^{-3}+\cdots \tag{4.2}$$

and the Laurent expansion at s=0 of $A(\sigma)^{-1}$ :

$$A(\sigma)^{-1}=\underbrace{\begin{bmatrix}1/6&0&1/6\\0&0&-1\\5/6&1&17/6\end{bmatrix}}_{V_0}+\underbrace{\begin{bmatrix}-5/36&0&1/36\\0&0&0\\-37/36&0&101/36\end{bmatrix}}_{V_1}s+$$

$$+\begin{bmatrix}19/216&0&-11/216\\0&0&0\\155/216&0&-67/216\end{bmatrix}s^2+\cdots \tag{4.3}$$

A *forward recursive representation* of (4.1) is given according to Corollary 3.3 by

$$y(k)=-(H_{-1}\ H_{-2})\begin{bmatrix}A_0&0\\A_1&A_0\end{bmatrix}\begin{bmatrix}y(k-1)\\y(k-2)\end{bmatrix}+(H_{-2}\,H_{-1}\,H_0\,H_1)\begin{bmatrix}B_0&0&0&0\\0&B_0&0&0\\0&0&B_0&0\\0&0&0&B_0\end{bmatrix}\begin{bmatrix}u(k-2)\\u(k-1)\\u(k)\\u(k+1)\end{bmatrix} \tag{4.4}$$

Substitution of $H_{-1}$ and $H_{-2}$ from (4.2) and $A_0$, $A_1$ from (4.1) and making the operations in (4.4) we obtain that

$$y(k)\equiv\begin{bmatrix}y_1(k)\\y_2(k)\\y_3(k)\end{bmatrix}=\begin{bmatrix}-5y_1(k-1)-6y_1(k-2)-5y_2(k-2)-4u(k-2)+u(k-1)\\-u(k)\\-63y_1(k-1)-90y_1(k-2)-63y_2(k-2)-48u(k-2)+13u(k-1)+3u(k+1)\end{bmatrix} \tag{4.5}$$

The *admissible initial condition space* $H_{iu}$ of (4.1) under nonzero inputs is given from (4.7) as follows

$$H_{iu}:=\Bigg\{\,y(i),\ u(i)\ (i=0,1):$$

$$\begin{bmatrix}A_0&0\\A_1&A_0\end{bmatrix}\begin{bmatrix}H_0&H_1\\H_{-1}&H_0\end{bmatrix}\begin{bmatrix}A_0&0\\A_1&A_0\end{bmatrix}\begin{bmatrix}y(0)\\y(1)\end{bmatrix}=\begin{bmatrix}A_0&0\\A_1&A_0\end{bmatrix}\begin{bmatrix}H_0&H_1&0\\H_{-1}&H_0&H_1\end{bmatrix}\begin{bmatrix}B_0&0&0\\0&B_0&0\\0&0&B_0\end{bmatrix}\begin{bmatrix}u(0)\\u(1)\\u(2)\end{bmatrix}\Bigg\} \tag{4.6}$$

or equivalently from (4.1), (4.2)

$$H_{iu} := \left\{ y(i),\ u(i)\ (i=0,1) : \right. \tag{4.7}$$

$$\left. \begin{bmatrix} 0 & -4 & 0 & 0 & 1 & 0 \\ -5 & 2 & 1 & 2 & 3 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & -6 & 0 & 0 & 1 & 0 \\ -12 & -10 & 0 & -15 & 2 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} y_1(0) \\ y_2(0) \\ y_3(0) \\ y_1(1) \\ y_2(1) \\ y_3(1) \end{bmatrix} = \begin{bmatrix} 4 & -1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 6 & -1 & 0 \\ 8 & -2 & 3 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} u(0) \\ u(1) \\ u(2) \end{bmatrix} \right\}$$

A *backward recursive representation* of (4.1) is given from Corollary 3.6 by :

$$y(k) = (V_2 \ \ V_1) \begin{bmatrix} A_0 & 0 \\ A_1 & A_0 \end{bmatrix} \begin{bmatrix} y(k+2) \\ y(k+1) \end{bmatrix} + V_0 B_0 u(k) \tag{4.8}$$

Substitution of $V_2$ and $V_1$ from (4.3) and $A_0$, $A_1$ from (4.1) and making the operations in (4.8) we obtain that

$$y(k) \equiv \begin{bmatrix} y_1(k) \\ y_2(k) \\ y_3(k) \end{bmatrix} = \begin{bmatrix} -(1/6)y_1(k+2)-(5/6)y_1(k+1)-(1/6)y_2(k+1)+(1/6)u(k) \\ -u(k) \\ -(5/6)y_1(k+2)-(37/6)y_1(k+1)-(23/6)y_2(k+1)+(17/6)u(k) \end{bmatrix} \tag{4.9}$$

The *admissible final condition space* $\bar{H}_{iu}$ of (4.1) under nonzero inputs is given from (3.15) as follows

$$\bar{H}_{iu} := \left\{ y(i),\ u(i)\ (i=0,1) : \right.$$

$$\left. \begin{bmatrix} A_2 & 0 \\ A_1 & A_2 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} A_2 & 0 \\ A_1 & A_2 \end{bmatrix} \begin{bmatrix} y(N) \\ y(N-1) \end{bmatrix} = \begin{bmatrix} A_2 & 0 \\ A_1 & A_2 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} B_0 u(N) \right\} \equiv$$

$$\equiv \{y(i),\ u(i)\ (i=0,1) \text{ arbitrary}\} \tag{4.10}$$

In the same way we can use relation (3.21) to find the symmetric solution of (4.1) under the restrictions between the final and initial conditions described by (3.22).

## 5. Conclusion.

In the case of regular discrete time ARMA–representations exact solutions where proposed in three different forms : a) forward solutions, b) backward solutions and c) symmetric solutions. It is easily seen that the proposed solutions are extensions of the ones proposed by Lewis & Mertzios (1990) for discrete time generalised state space systems. The solution formula presented in this work has been implemented via MAPLE in a recent publication (Jones et. al. 1996). Certain controllability, reachability and observability criteria based on the proposed solutions are being studied and will be discussed in a future publication.

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