The Output Zeroing Problem for General Polynomial Descriptions

by

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Abstract.

The main purpose of this paper is to give a physical interpretaion of invariant zeros and indices in terms of the general zero-output behaviour of a linear dynamical system.

1. Introduction.

Consider a linear, time invariant, multivariable system described by differential and algebraic equations of the following form :

$$\begin{aligned} \Sigma : \quad A(\rho)\beta(t) &= B(\rho)u(t) \\ y(t) &= C(\rho)\beta(t) + D(\rho)u(t) \end{aligned} \tag{1.1}$$

where $\rho := d/dt$ is the differential operator, $A(\rho) \in \Re[\rho]^{n\times n}$ with $\det[A(\rho)] \neq 0$, $B(\rho) \in \Re[\rho]^{n\times m}$, $C(\rho) \in \Re[\rho]^{p\times n}$, $D(\rho) \in \Re[\rho]^{p\times m}$, $\beta(t):(0-,+\infty) \to \Re^n$ is the *pseudostate* of the system, $u(t):(0-,+\infty) \to \Re^m$ is the *input* of the system, and $y(t):(0-,+\infty) \to \Re^p$ is the *output* of the system. Σ may be rewritten as :

$$\begin{pmatrix} A(\rho) & B(\rho) \\ -C(\rho) & D(\rho) \end{pmatrix} \begin{pmatrix} \beta(t) \\ -u(t) \end{pmatrix} = \begin{pmatrix} 0 \\ -I \end{pmatrix} y(t) \tag{1.2}$$

The general output-zero problem for the system Σ may be stated (Karcanias & Hayton 1981) as follows : Find the set of initial conditions or pseudostates and control inputs such that the output is identically zero. Using the system description (1.2), the

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aforementioned problem is reduced to that of studying the structure and properties of the vector space of solutions of the system :

$$\underbrace{\begin{pmatrix} A(\rho) & B(\rho) \\ -C(\rho) & D(\rho) \end{pmatrix}}_{P(\rho)} \underbrace{\begin{pmatrix} \beta(t) \\ -u(t) \end{pmatrix}}_{z(t)} = \mathbf{0}_{(n+p)\times(n+m)} \tag{1.3}$$

where

$$P(\rho) = P_0 + P_1\rho + \cdots + P_{q_1}\rho^{q_1} \in \Re[\rho]^{(n+p)\times(n+m)} \tag{1.4}$$

In the case where the system Σ is in state-space form the zero-output problem has been studied by Karcanias (1975), MacFarlane & Karcanias (1976), Karcanias & Kouvaritakis (1979), Karcanias & Hayton (1981) and the close connection of the aforementioned problem with the finite and infinite invariant zeros of the system was demonstrated. However questions still remains concerning the solution of the general output-zero problem presented above when the system Σ is in the general form (1.1). The above question will be considered in this paper. More specifically in sections 2 and 3 we give a geometric interpretation of the finite and infinite invariant zeros while in section 4 we introduce the notions of left and right invariant indices and we give a geometric interpretation of these indices. These interpretations coincide exactly with those given for conventional state-space systems by MacFarlane&Karcanias (1976), Karcanias & Hayton (1981). In section 5 we give a general solution to the zero-output problem, while we close the paper with an illustrative example.

2. Geometric Interpretation of the Finite Invariant Zeros.

Consider the system Σ in (1.1). Define

$$Z = \left\{ \begin{pmatrix} \beta(t) \\ -u(t) \end{pmatrix} = L^{-1}\begin{pmatrix} \hat{\beta}(s) \\ -\hat{u}(s) \end{pmatrix} \;\middle|\; \begin{pmatrix} \mathrm{A(s)} & \mathrm{B(s)} \\ \text{-}\mathrm{C(s)} & \mathrm{D(s)} \end{pmatrix}\begin{pmatrix} \hat{\beta}(s) \\ -\hat{u}(s) \end{pmatrix} = 0 \right\} \tag{2.1a}$$

$$Z_u = -\pi_u Z \quad \text{and} \quad Z_\beta = \pi_\beta Z \tag{2.1b}$$

$$\begin{aligned} [a] &= \{a + z \text{ where } a \text{ is a solution of (1.3) and } z \in Z\} = \{a\} \oplus Z \\ [a]_u &= \{a + z \text{ where } a \text{ is a solution of (1.3) and } z \in Z_u\} = \pi_u\{a\} \oplus Z_u \\ [a]_\beta &= \{a + z \text{ where } a \text{ is a solution of (1.3) and } z \in Z_\beta\} = \pi_\beta\{a\} \oplus Z_\beta \end{aligned} \tag{2.1c}$$

where $L[z(t)]$ denotes the Laplace transform of the vector $z(t)$, $\pi_\beta:(\beta(t)^T,(-u(t))^T)^T \in Z \to \beta(t)$ and π_u: $(\beta(t)^T,(-u(t))^T)^T \in Z \to -u(t)$. Note also that Z is the solution space of the system (1.3) under zero, input and pseudostate, initial conditions and that its elements represented by the equivalence class [0].

Assume that the Rosenbrock system matrix $P(s) \in \Re[s]^{(n+p)\times(n+m)}$ of (1.3) has

$$U_L(s)P(s)U_R(s) = \text{blockdiag}\left(1,1,\ldots,1, f_\mu(s), f_{\mu+1}(s),\ldots, f_r(s), \mathbf{0}_{n+p-r,n+m-r}\right) \quad (2.2)$$

$1 \le \mu \le r$ as its Smith form of P(s) (in **C**) where $f_i(s) \in \Re[s]$ are the nonunit invariant polynomials of P(s) and $f_i(s)/f_{i+1}(s),\ i = \mu,\mu+1,\ldots,r-1$. Assume that the nonunit invariant polynomials $f_i(s) \in \Re[s]$ have ℓ distinct zeros $\lambda_1,\lambda_2,\ldots,\lambda_\ell$ (where for simplicity of notation we assume that $\lambda_i \in \Re$) with partial multiplicities $\sigma_{1,i},\sigma_{2,i},\ldots,\sigma_{\ell,i}$ $i = \mu,\mu+1,\ldots,r$ where $0 \le \sigma_{i,\mu} \le \sigma_{i,\mu+1} \le \cdots \le \sigma_{i,r}$.

Rosenbrock (1973) was originally first to define the notion of the finite invariant zeros for general systems of the form (1.1). MacFarlane & Karcanias (1976) then utilised the definition of the generalized invariant zero-direction vectors for state space descriptions for the solution of the zeroing output problem in state space systems. An extension of the definition of generalized invariant zero-direction vectors for general systems of the form (1.1) is given in the following :

Definition 1. Let P(s) be the Rosenbrock system matrix of Σ defined in (1.3)-(1.4). We define as *finite invariant zeros* of Σ the finite zeros of P(s). Associated with each invariant zero λ_i, $i \in \ell$ there is a composite vector $z_0^i = \left(\beta_0^{i^T}, -u_0^{i^T}\right)^T$ which lies in the kernel or null space of $P(\lambda_i)$ i.e. $P(\lambda_i)z_0^i = 0$, and does not satisfy the condition $P(s)z_0^i = 0$, and which is called an *invariant zero-direction vector* , and a sequence of vectors $z_q^i = \left(\beta_q^{i^T}, -u_q^{i^T}\right)^T$, $i \in \ell,\ q = 1,2,\ldots,v$ which are called *generalized invariant zero-direction vectors* such that

$$
\begin{aligned}
&P(\lambda_i)z_0^i = 0 \\
&P^{(1)}(\lambda_i)z_0^i + P(\lambda_i)z_1^i = 0 \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
&\frac{1}{\nu!}P^{(\nu)}(\lambda_i)z_0^i + \frac{1}{(\nu-1)!}P^{(\nu-1)}(\lambda_i)z_1^i + \cdots + P^{(1)}(\lambda_i)z_{\nu-1}^i + P(\lambda_i)z_\nu^i = 0
\end{aligned} \tag{2.3}
$$

which do not satisfy the following condition

$$P(s)\left(z_0^i + (s-\lambda_i)z_1^i + \cdots + (s-\lambda_i)^\nu z_\nu^i\right) = 0 \tag{2.4}$$

where $P^{(j)}(s)$ denotes the jth derivative of P(s). •

The existence of the generalized invariant zero-direction vectors is given by the following

Theorem 1. (Existence of the generalized invariant zero-direction vectors)
Consider the Rosenbrock system matrix P(s) of $\sum$ defined in (1.3) and its Smith form in (2.2). Then for any invariant zero $\lambda_i,\ i \in \ell$ of partial multiplicity $\sigma_{i,j},\ j = \mu, \mu+1, \ldots, r$ *there exists* an invariant zero-direction vector $z_{j,0}^i = \left(\beta_{j,0}^{i\ T}, -u_{j,0}^{i\ T}\right)^T$ and a sequence of generalized invariant zero-direction vectors $z_{j,q}^i = \left(\beta_{j,q}^{i\ T}, -u_{j,q}^{i\ T}\right)^T$, $i \in \ell,\ j = \mu, \mu+1, \ldots, r,\ q = 1, 2, \ldots, \sigma_{i,j} - 1$.

Proof. The proof of this Theorem is constructive. Let $u_j(s) = u_{j,0}^i + u_{j,1}^i(s-\lambda_i) + \cdots + u_{j,f}^i(s-\lambda_i)^f$, $v_j(s)$ with $j = \mu, \mu+1, \ldots, r$ be the columns of $U_R(s)$ and $U_L(s)^{-1}$ respectively, defined in (2.2) and $u_j^{(q)}(s) := (d^q / ds^q)u_j(s)$, $q = 0, 1, \ldots, (\sigma_{i,j} - 1)$. Then from (2.2) we have that

$$P(s)u_j(s) = f_j(s)v_j(s) \tag{2.5}$$

Thus if $\lambda_i,\ i \in \ell$ is a zero of $f_i(s) \in \Re[s]$ then

$$
\begin{aligned}
&P(\lambda_i)u_j(\lambda_i) = 0 \\
&P^{(1)}(\lambda_i)u_j(\lambda_i) + P(\lambda_i)u_j^{(1)}(\lambda_i) = 0 \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
&P^{(\sigma_{i,j}-1)}(\lambda_i)u_j(\lambda_i) + P^{(\sigma_{i,j}-2)}(\lambda_i)u_j^{(1)}(\lambda_i) + \cdots + P(\lambda_i)u_j^{(\sigma_{i,j}-1)}(\lambda_i) = 0
\end{aligned} \tag{2.6}
$$

and therefore the vectors

$$z^i_{j,q} := \begin{pmatrix} \beta^i_{j,q} \\ -u^i_{j,q} \end{pmatrix} = \frac{1}{q!} u_j^{(q)}(\lambda_i) \equiv u^i_{j,q} \tag{2.7}$$

with $i \in \ell$ and $j = \mu, \mu+1, \ldots, r$ are satisfy the conditions (2.3) of the generalized invariant zero-direction vectors. Note also that in case where these vectors satisfy the condition (2.4) then the polynomial vector

$$\begin{aligned} u(s) &= z^i_{j,0} + z^i_{j,1}(s-\lambda_i) + \cdots + z^i_{j,\sigma_{i,j}-1}(s-\lambda_i)^{\sigma_{i,j}-1} = \\ &\equiv u^i_{j,0} + u^i_{j,1}(s-\lambda_i) + \cdots + u^i_{j,\sigma_{i,j}-1}(s-\lambda_i)^{\sigma_{i,j}-1} \end{aligned} \tag{2.8}$$

belongs to the right null space of P(s) and thus can be written as a linear combination of the columns $u_j(s)$ with $j = r+1, \ldots, n+m$. This condition implies, after some elementary manipulations on the unimodular matrix $U_R(s)$ (Karampetakis 1993), that the $U_R(s)$ possesses a zero at λ_i, $i \in \ell$ which is a contradiction. Thus the vectors defined in (2.7) are generalized invariant zero-direction vectors. •

Theorem 2. Let P(s) be the Rosenbrock system matrix of Σ defined in (1.4) and let λ_i, $i \in \ell$ be an invariant zero of Σ. Then to an input of the form

$$[u^i_q(t)]_u := \sum_{k=0}^{\nu-q} \left[u^i_k \frac{t^{\nu-q-k}}{(\nu-q-k)!} e^{\lambda_i t} \right]_u \quad i \in \ell \text{ and } q = 0,1,\ldots,\nu \tag{2.9a}$$

and with initial conditions of the form :

$$u_q^{i(j)}(0-) = \lambda_i^j u^i_q + j\lambda_i^{j-1} u^i_{q-1} + \frac{j(j-1)}{2!}\lambda_i^{j-2} u^i_{q-2} + \cdots + \frac{j(j-1)\cdots(j-q+1)}{q!}\lambda_i^{j-q} u^i_0 \tag{2.9b}$$

with $j = 0,1,\ldots,q_1 - 1$, corresponds a pseudostate of Σ of the form

$$[\beta^i_q(t)]_\beta := \sum_{k=0}^{\nu-q} \left[\beta^i_k \frac{t^{\nu-q-k}}{(\nu-q-k)!} e^{\lambda_i t} \right]_\beta \quad i \in \ell \text{ and } q = 0,1,\ldots,\nu \tag{2.10a}$$

with initial conditions of the form

$$\beta_q^{i(j)}(0-) = \lambda_i^j \beta^i_q + j\lambda_i^{j-1}\beta^i_{q-1} + \frac{j(j-1)}{2!}\lambda_i^{j-2}\beta^i_{q-2} + \cdots + \frac{j(j-1)\cdots(j-q+1)}{q!}\lambda_i^{j-q}\beta^i_0 \tag{2.10b}$$

with $j = 0,1,\ldots,q_1 - 1$ and consequently to a zero output of Σ i.e.

$$y(t) = 0 \tag{2.11}$$

when the vectors $\left(\beta_k^{i\,T}, -u_k^{i\,T}\right)^T$ are generalized invariant zero-direction vectors in the sense of Definition 1.

Proof. The vectors $\left[\left(\beta_k^i(t)^T, -u_k^i(t)^T\right)^T\right]$ have been defined above, and it is readily verified through the relation (2.3), that these satisfy the relation (1.3) and therefore the proof of the Theorem 2 follows. Note also that although the vectors $\left(\beta_k^i(t)^T, -u_k^i(t)^T\right)^T$ also satisfy the relation (1.3), we use the equivalence classes of these vectors because all the elements of the equivalence class which defines the vector $\left(\beta_k^i(t)^T, -u_k^i(t)^T\right)^T$ also satisfy the relation (1.3) (Karampetakis 1993). •

Corollary 1. From Theorem 1 and 2 we conclude that there exist an input vector space

$$U_0^C = \left\langle \begin{array}{c} [u_{j,q}^i(t)]_u := \sum_{k=0}^{\sigma_{i,j}-1-q} \left[u_{j,k}^i \dfrac{t^{\sigma_{i,j}-q-k-1}}{(\sigma_{i,j}-q-k-1)!} e^{\lambda_i t} \right]_u \\ \\ \text{where } i \in \ell,\ j = \mu, \mu+1, \ldots, r \text{ and } q = 0,1,\ldots,\sigma_{i,j}-1 \end{array} \right\rangle \tag{2.12}$$

which gives rise through the relation (1.1) to the pseudostate vector space

$$B_0^C = \left\langle \begin{array}{c} [\beta_{j,q}^i(t)]_\beta := \sum_{k=0}^{\sigma_{i,j}-1-q} \left[\beta_{j,k}^i \dfrac{t^{\sigma_{i,j}-q-k-1}}{(\sigma_{i,j}-q-k-1)!} e^{\lambda_i t} \right]_\beta \\ \\ \text{where } i \in \ell,\ j = \mu, \mu+1, \ldots, r \text{ and } q = 0,1,\ldots,\sigma_{i,j}-1 \end{array} \right\rangle \tag{2.13}$$

and consequently to the output vector space

$$Y_0 = \{0\} \tag{2.14}$$

where the vectors $\left(\beta_{j,k}^{i\ \ T}, -u_{j,k}^{i\ \ T}\right)^T$ have been defined in the constructive proof of Theorem 2. •

3. Geometric Interpretations of the Infinite Invariant Zeros.

Consider the Rosenbrock matrix P(s) in (1.4) and define its "dual" polynomial matrix $\tilde{P}(w)$ (Hayton et. al. 1988) as

$$\tilde{P}(w) := P_{q_1} + P_{q_1-1}w + \cdots + P_1 w^{q_1-1} + P_0 w^{q_1} := w^{q_1} P\left(\frac{1}{w}\right) \tag{3.1}$$

Then there exist unimodular matrices $\tilde{U}_L(w) \in \Re[w]^{(n+p)\times(n+p)}$, $\tilde{U}_R(w) \in \Re[w]^{(n+m)\times(n+m)}$ (Hayton et. al. 1988) such that

$$\tilde{U}_L(w)\tilde{P}(w)\tilde{U}_R(w) =$$

$$= \text{block diag}\left(\tilde{f}_1(w), w^{q_1-q_2}\tilde{f}_2(w), \ldots, w^{q_1-q_k}\tilde{f}_k(w), w^{q_1+\hat{q}_{k+1}}\tilde{f}_{k+1}(w), \ldots, w^{q_1+\hat{q}_r}\tilde{f}_r(w), 0_{n+p-r,n+m-r}\right) \tag{3.2}$$

where $\tilde{f}_i(0) \neq 0$ and $q_1 \geq q_2 \geq \cdots \geq q_k \geq 0$ and $0 < \hat{q}_{k+1} \leq \hat{q}_{k+2} \leq \cdots \leq \hat{q}_r$ are respectively the order of the poles and zeros at $s = \infty$ of P(s).

The notion of the infinite invariant zeros of general systems of the form (1.1) was originally presented by Walker (1988), while the notion of generalized infinite zero-direction vectors for state space systems was utilised by MacFarlane & Karcanias (1976) for the solution of the output zeroing problem in the state space case. In what follows we present an extension of the generalized infinite zero-direction vectors to the case of general systems of the form (1.1).

Definition 2. Let P(s) be the Rosenbrock system matrix of Σ defined in (1.3)-(1.4). We define as *infinite invariant zeros* of Σ the infinite zeros of P(s). Associated with each invariant infinite zero of Σ there is a composite vector $z_0^\infty = \left(\beta_0^{\infty T}, -u_0^{\infty T}\right)^T$ which lies in the kernel or null space of $\tilde{P}(0) \equiv P_{q_1}$ i.e. $\tilde{P}(0)z_0^\infty = 0 \Leftrightarrow P_{q_1}z_0^\infty = 0$, and does not satisfy the relation $P(s)z_0^\infty = 0$, and which is called an *invariant infinite zero-direction vector* , and a sequence of vectors $z_j^\infty = \left(\beta_j^{\infty T}, -u_j^{\infty T}\right)^T$, $j = 1, \ldots, q_1 + q$ which are called *generalized infinite zero-direction vectors* such that

$$
\begin{aligned}
&P_{q_1} z_0^\infty = 0\\
&P_{q_1-1} z_0^\infty + P_{q_1} z_1^\infty = 0\\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\\
&P_0 z_0^\infty + P_1 z_1^\infty + \cdots + P_{q_1-1} z_{q_1-1}^\infty + P_{q_1} z_{q_1}^\infty = 0\\
&P_0 z_1^\infty + P_1 z_2^\infty + \cdots + P_{q_1-1} z_{q_1}^\infty + P_{q_1} z_{q_1+1}^\infty = 0\\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\\
&P_0 z_q^\infty + P_1 z_{q+1}^\infty + \cdots + P_{q_1-1} z_{q_1+q-1}^\infty + P_{q_1} z_{q_1+q}^\infty = 0
\end{aligned}
\tag{3.3}
$$

which do not satisfy the following condition

$$P(s)\left(z_0^\infty s^{q_1+q} + z_1^\infty s^{q_1+q-1} + \cdots + z_{q_1+q}^\infty\right) = 0 \tag{3.4}$$ •

The existence of such a chain of generalized infinite zero-direction vectors is given by the following

Theorem 3. (Existence of the generalized infinite zero-direction vectors)

Consider the Rosenbrock system matrix P(s) of $\sum$ defined in (1.3) and the local Smith form of its dual polynomial matrix at w=0 in (3.2). Then for every infinite invariant zero of order $\hat{q}_j$ with $j = k+1, k+2, \ldots, r$ there *exists* an infinite zero-direction vector $z_{i,0}^\infty = \left(\beta_{i,0}^{\infty\,T}, -u_{i,0}^{\infty\,T}\right)^T$, and a sequence of generalized infinite zero-direction vectors $z_{i,j}^\infty = \left(\beta_{i,j}^{\infty\,T}, -u_{i,j}^{\infty\,T}\right)^T$, $j = 1, \ldots, q_1 + \hat{q}_i - 1$.

Proof. In the same way as the proof of Theorem 2 if $\tilde{u}_j(w) = \tilde{u}_{j,0} + \tilde{u}_{j,1} w + \cdots + \tilde{u}_{j,\tilde{f}} w^{\tilde{f}}, \tilde{v}_j(w) \in \Re[w]^{(n+m)\times 1}$ with $i = k+1, k+2, \ldots, r$ are the columns of the matrices $\tilde{U}_R(w)$ and $\tilde{U}_L(w)^{-1}$ in (3.2) then from (3.2) we have that

$$\tilde{P}(w)\tilde{u}_j(w) = w^{q_1+\hat{q}_j}\tilde{v}_j(w) \tag{3.5}$$

It can then be easily seen that the vectors

$$z_{i,q} := \begin{pmatrix} \beta_{i,q}^\infty \\ -u_{i,q}^\infty \end{pmatrix} = \frac{1}{q!}\tilde{u}_i^{(q)}(0) = \tilde{u}_{i,q} \tag{3.6}$$

where $\tilde{u}_i^{(q)}(w)$, $\tilde{P}^{(q)}(w)$ are the qth derivatives of $\tilde{u}_i(w)$, $\tilde{P}(w)$ with respect to w, and $q = 0, 1, \ldots, q_1 + \hat{q}_i$, i=k+1,k+2,..,r satisfy the relation (3.3). It is again easily shown that in case the generalized infinite zero-direction vectors in (3.6) satisfy the condition

(3.4) then the matrix $\tilde{U}_R(w)$ has a finite zero at w=0 which is a contradiction. Thus the vectors in (3.6) are generalized infinite zero-direction vectors. •

Theorem 4. Let P(s) be the Rosenbrock system matrix P(s) of Σ defined in (1.4). Then to an input of the form

$$[u_q^\infty(t)]_u := \sum_{j=0}^{q} \left[u_j^\infty \delta^{(q-j)}(t)\right]_u \text{ with } u_q^{\infty(i)}(0-) = -u_{j+i+1}^\infty \text{ for } i = 0,1,\ldots,q_1 - 1 \qquad (3.7)$$

corresponds a pseudostate of Σ of the form

$$[\beta_q^\infty(t)]_\beta := \sum_{j=0}^{q} \left[\beta_j^\infty \delta^{(q-j)}(t)\right]_\beta \text{ with } \beta_q^{\infty(i)}(0-) = -\beta_{j+i+1}^\infty \text{ for } i = 0,1,\ldots,q_1 - 1 \qquad (3.8)$$

and a zero output of Σ i.e.

$$y(t) = 0 \qquad (3.9)$$

when the vectors $\left(\beta_j^{\infty T}, -u_j^{\infty T}\right)^T$ are generalized infinite zero-direction vectors.

Proof. The vectors $\left[\left(\beta_j^\infty(t)^T, -u_j^\infty(t)^T\right)^T\right]$ as defined above may be readily checked via the relation (3.3) to satisfy the relation (1.4) and therefore the proof of the Theorem 4 follows. Note also that although the vectors $\left(\beta_j^\infty(t)^T, -u_j^\infty(t)^T\right)^T$ also satisfies the relation (1.3), we use (as in Theorem 2) the equivalence classes of these vectors because all the elements of the equivalence class which defines the vector $\left(\beta_j^\infty(t)^T, -u_j^\infty(t)^T\right)^T$ also satisfy the relation (1.3) (Karampetakis 1993). •

Corollary 2. From Theorems 3 and 4 we conclude that there exist an input vector space

$$U_0^\infty = \left\langle \begin{matrix} [u_{i,q}^\infty(t)]_u := \sum_{j=0}^{q} \left[u_{i,j}^\infty \delta^{(q-j)}(t)\right]_u \\ i = k+1, k+2, \ldots, r \text{ and } q = 0,1,\ldots,\hat{q}_i - 1 \end{matrix} \right\rangle \qquad (3.10)$$

which gives rise through the relation (1.1) to the pseudostate vector space

$$B_0^\infty = \left\langle \begin{matrix} [\beta_{i,q}^\infty(t)]_\beta := \sum_{j=0}^{q} \left[\beta_{i,j}^\infty \delta^{(q-j)}(t)\right]_\beta \\ i = k+1, k+2, \ldots, r \text{ and } q = 0,1,\ldots,\hat{q}_i - 1 \end{matrix} \right\rangle \qquad (3.11)$$

and consequently to the output vector space

$$Y_0 = \{0\} \qquad (3.12)$$

where the vectors $\left(\beta_{i,q}^{\infty\,T}, -u_{i,q}^{\infty\,T}\right)^T$ have been defined by the constructive proof of Theorem 3. •

4. Geometric Interpretation of the Invariant Indices.

The Rosenbrock system matrix $P(s) \in \Re[s]^{(n+p)\times(n+m)}$ of the system (1.1) is assumed to have rank $r \le \min(n+p, n+m)$ and therefore the dimension of the right null space of P(s) is equal to n+m-r. Consider a minimal polynomial basis of the right (left) null space of P(s), denoted

$$\begin{gathered}\left[\bar{u}_{r+1}(s),\quad \bar{u}_{r+2}(s),\quad \ldots,\quad \bar{u}_{n+m}(s)\right] \\ \left(\left[v_{r+1}(s),\quad v_{r+2}(s),\quad \cdots,\quad v_{n+p}(s)\right]\right)\end{gathered} \tag{4.1}$$

The greatest degrees of the columns $\bar{u}_i(s),\ i = r+1,\ldots,n+m$, denoted $\{\varepsilon_{r+1}, \varepsilon_{r+2}, \ldots, \varepsilon_{n+m}\}$ are called the *right minimal indices* of P(s), while the greatest degrees of the rows $v_i(s),\ i = r+1,\ldots,n+p$, denoted $\{\eta_{r+1}, \eta_{r+2}, \ldots, \eta_{n+p}\}$ are called the *left minimal indices* of P(s).

Definition 3. Let P(s) be the Rosenbrock system matrix of Σ defined in (1.3)-(1.4). We define the *invariant right (left) indices* of $\sum$, as the right (left) minimal indices of the Rosenbrock system matrix P(s). Associated with each invariant right (left) index there is a sequence of vectors $z_j^{\varepsilon} = \left(\beta_j^{\varepsilon T}, -u_j^{\varepsilon\,T}\right)^T,\ j = 0,1,\ldots,q$, $\left(z_j^{\eta},\ j = 0,1,\ldots,q\right)$ which are called *invariant right (left) index-direction vectors.* such that

$$\begin{aligned}
&P_{q_1} z_0^{\varepsilon} = 0 \\
&P_{q_1-1} z_0^{\varepsilon} + P_{q_1} z_1^{\varepsilon} = 0 \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
&P_0 z_0^{\varepsilon} + P_1 z_1^{\varepsilon} + \cdots + P_{q-1} z_{q-1}^{\varepsilon} + P_q z_q^{\varepsilon} = 0 \quad (\text{if } q < q_1) \\
&(\text{or } P_0 z_0^{\varepsilon} + P_1 z_1^{\varepsilon} + \cdots + P_{q_1-1} z_{q_1-1}^{\varepsilon} + P_{q_1} z_{q_1}^{\varepsilon} = 0 \quad (\text{if } q \ge q_1)) \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
&P_0 z_q^{\varepsilon} = 0
\end{aligned} \tag{4.2a}$$

$$\begin{cases} z_0^\eta P_{q_1} = 0 \\ z_0^\eta P_{q_1-1} + z_1^\eta P_{q_1} = 0 \\ \dots\dots\dots\dots\dots\dots\dots \\ z_0^\eta P_0 + z_1^\eta P_1 + \cdots + z_{q-1}^\eta P_{q-1} + z_q^\eta P_q = 0 \text{ (if } q < q_1) \\ \text{or } z_0^\eta P_0 + z_1^\eta P_1 + \cdots + z_{q_1-1}^\eta P_{q_1-1} + z_{q_1}^\eta P_{q_1} = 0 \text{ (if } q \geq q_1) \\ \dots\dots\dots\dots\dots\dots\dots \\ z_q^\eta P_0 = 0 \end{cases} \tag{4.2b}$$

•

Theorem 5. (Construction of the generalized right (left) index-direction vectors) Consider the Rosenbrock system matrix P(s) of $\sum$ defined in (1.3) and the minimal basis in (4.1) for its right (left) null space. Then for every right (left) index of order ε_i, $i = r+1,..,n+m$, (η_i, $i = r+1,.., p+m$) there exists a sequence of invariant right (left) index-direction vectors $z_{i,j}^\varepsilon = \left(\beta_{i,j}^{\varepsilon\,T}, -u_{i,j}^{\varepsilon\,T}\right)^T$, $j = 0,1,\ldots,\varepsilon_i$, $\left(z_{i,j}^\eta, j = 0,1,\ldots,\eta_i\right)$.

Proof. From the relation

$$\underbrace{\left(P_0 + P_1 s + \cdots + P_q s^q\right)}_{P(s)}\underbrace{\left(\bar{u}_{i,0} + \bar{u}_{i,1} s + \cdots + \bar{u}_{i,\varepsilon_i} s^{\varepsilon_i}\right)}_{\bar{u}_i(s)} = 0$$

$$\left(\underbrace{\left(\bar{v}_{i,0} + \bar{v}_{i,1} s + \cdots + \bar{v}_{i,\eta_i} s^{\eta_i}\right)}_{\bar{v}_i(s)}\underbrace{\left(P_0 + P_1 s + \cdots + P_q s^q\right)}_{P(s)} = 0\right) \tag{4.3}$$

with $i = r+1, r+2,\ldots, n+m$ $(i = r+1, r+2,\ldots, p+m)$ it is easily seen that the vectors

$$z_{i,q}^\varepsilon := \begin{pmatrix} \beta_{i,q}^\varepsilon \\ -u_{i,q}^\varepsilon \end{pmatrix} = \bar{u}_{i,\varepsilon_i - q} \quad \left(z_{i,q}^\eta := \begin{pmatrix} \beta_{i,q}^\eta \\ -u_{i,q}^\eta \end{pmatrix} = \bar{v}_{i,\eta_i - q}\right) \tag{4.4}$$

where $q = 0,1,\ldots,\varepsilon_i(\eta_i)$ and $i = r+1, r+2,\ldots, n+m$ $(i = r+1, r+2,\ldots, p+m)$ satisfies relation (4.2) and thus are invariant right (left) index-direction vectors). •

Theorem 6. Consider the system $\sum$ defined in (1.1) with Rosenbrock system matrix $P(s) \in \Re[s]^{(n+p)\times(n+m)}$ where $\text{rank}_{\Re} P(s) = r \leq \min(n+p, n+m)$. Then to an input of the form

$$[u_k^{\varepsilon}(t)]_u := \sum_{j=0}^{k} \left[u_j^{\varepsilon}\delta^{(k-j)}(t)\right]_u \quad \text{with } u_q^{\varepsilon(i)}(0-) = -u_{j+i+1}^{\varepsilon}$$

$$\text{for } i = 0,1,\ldots,q_1 - 1 \text{ and } k = 0,1,\ldots,q-1 \tag{4.5}$$

corresponds a pseudostate of Σ of the form

$$[\beta_k^{\varepsilon}(t)]_{\beta} := \sum_{j=0}^{k} \left[\beta_j^{\varepsilon}\delta^{(k-j)}(t)\right]_{\beta} \quad \text{with } \beta_q^{\varepsilon(i)}(0-) = -\beta_{j+i+1}^{\varepsilon}$$

$$\text{for } i = 0,1,\ldots,q_1 - 1 \text{ and } k = 0,1,\ldots,q-1 \tag{4.6}$$

and a zero output i.e.

$$y(t) = 0 \tag{4.7}$$

when the vectors $\left(\beta_j^{\varepsilon T}, -u_j^{\varepsilon T}\right)^T$ $j = 0,1,\ldots,q-1$ are invariant right index-direction vectors .

Proof. With vectors $\left[\left(\beta_k^{\varepsilon}(t)^T, -u_k^{\varepsilon}(t)^T\right)^T\right]$ as defined in (4.2), (4.5), (4.6), it is easily checked that relation (1.3) is satisfied and therefore the proof of Theorem 6 follows. Note also that although the vectors $\left(\beta_k^{\varepsilon}(t)^T, -u_k^{\varepsilon}(t)^T\right)^T$ satisfies the relation (1.3) we use its equivalence classes for the same reasons suggested in Theorems 2 and 4. •

Corollary 3. From the above theorem we conclude that there exist an input vector space

$$U_0^{\varepsilon} = \left\langle \begin{array}{c} [u_{i,q}^{\varepsilon}(t)]_u := \sum_{j=0}^{q} \left[u_{i,j}^{\varepsilon}\delta^{(q-j)}(t)\right]_u \\ i = r+1, r+2,\ldots,n+m \text{ and } q = 0,1,\ldots,\varepsilon_i - 1 \end{array} \right\rangle \tag{4.8}$$

which gives rise through the relation (1.1) to the pseudostate vector space

$$B_0^{\varepsilon} = \left\langle \begin{array}{c} [\beta_{i,q}^{\varepsilon}(t)]_{\beta} := \sum_{j=0}^{q} \left[\beta_{i,j}^{\varepsilon}\delta^{(q-j)}(t)\right]_{\beta} \\ i = r+1, r+2,\ldots,n+m \text{ and } q = 0,1,\ldots,\varepsilon_i - 1 \end{array} \right\rangle \tag{4.9}$$

and consequently to the output vector space

$$Y_0 = \{0\} \tag{4.10}$$

where the vectors $\left(\beta_{i,q}^{\varepsilon}(t)^T, -u_{i,q}^{\varepsilon}(t)^T\right)^T$ have been defined in Theorem 5. •

Let now

$$v_i(s) = v_{i,0} + v_{i,1}s + \cdots + v_{i,\eta_i}s^{\eta_i} \quad \left(\stackrel{(4.3)}{=} z^{\eta}_{i,\eta_i} + z^{\eta}_{i,\eta_i-1}s + \cdots + z^{\eta}_{i,0}s^{\eta_i} \right) \quad (4.11)$$

with $i = r+1,\ldots,n+p$, be the vectors of a left minimal polynomial basis of the Rosenbrock system matrix P(s). It has been suggested (Karcanias & Kouvaritakis 1979) that the left minimal basis of the Rosenbrock system matrix P(s) in state space systems plays the same role as the right minimal basis for the "dual" system of (1.1). However this is not particularly the case and we shall show in the next Theorem the precise connection which exists between the invariant left indices and the existence of a solution to the zero-output problem.

Theorem 7. The zero-output problem has a solution iff the following $\eta := \eta_{r+1} + \eta_{r+2} + \cdots + \eta_{n+p}$ conditions between the initial conditions $\left(\beta^{(q)}(0-)^T, -u^{(q)}(0-)^T\right)^T$, $q = 0,1,\ldots,q_1 - 1$ are satisfied

$$\begin{pmatrix} v_{i,\eta_i} & 0 & \cdots & 0 \\ v_{i,\eta_i-1} & v_{i,\eta_i} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ v_{i,\eta_i-q_1+1} & v_{i,\eta_i-q_1+2} & \cdots & v_{i,\eta_i} \\ \vdots & \vdots & \cdots & \vdots \\ v_{i,1} & v_{i,2} & \cdots & v_{i,q_1} \end{pmatrix} \begin{pmatrix} P_0 & P_1 & \cdots & P_{q_1-1} \\ 0 & P_0 & \cdots & P_{q_1-2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & P_0 \end{pmatrix} \begin{pmatrix} \begin{pmatrix} \beta(0-) \\ -u(0-) \end{pmatrix} \\ \begin{pmatrix} \beta^{(1)}(0-) \\ -u^{(1)}(0-) \end{pmatrix} \\ \vdots \\ \begin{pmatrix} \beta^{(q_1-1)}(0-) \\ -u^{(q_1-1)}(0-) \end{pmatrix} \end{pmatrix} = 0_{\eta_i,1} \quad (4.12)$$

for i=r+1,r+2,...,n+p.

Proof. Taking Laplace transforms in equation (1.3) we obtain that

$$\underbrace{\begin{pmatrix} A(s) & B(s) \\ -C(s) & D(s) \end{pmatrix}}_{P(s)=P_0+P_1s+\cdots+P_{q_1}s^{q_1}} \begin{pmatrix} \beta(s) \\ -u(s) \end{pmatrix} = \begin{pmatrix} s^{q_1-1}I & \cdots & sI & I \end{pmatrix} \begin{pmatrix} P_{q_1} & 0 & 0 & 0 \\ P_{q_1-1} & P_{q_1} & 0 & 0 \\ \vdots & \vdots & \ddots & 0 \\ P_1 & P_2 & \cdots & P_{q_1} \end{pmatrix} \begin{pmatrix} \begin{pmatrix} \beta(0-) \\ -u(0-) \end{pmatrix} \\ \begin{pmatrix} \beta^{(1)}(0-) \\ -u^{(1)}(0-) \end{pmatrix} \\ \vdots \\ \begin{pmatrix} \beta^{(q_1-1)}(0-) \\ -u^{(q_1-1)}(0-) \end{pmatrix} \end{pmatrix}$$

(4.13)

Premultiplying now both sides of (4.13) with the vectors $\bar{v}_i(s)$, $i = r+1,\ldots,n+p$ the left side of (4.13) will be zero. Then equating the coefficients of the powers of s with zero we obtain relation (4.13) which must be satisfied between the initial conditions of the inputs and outputs of the system. It can be shown (Karampetakis 1993) that conditions (4.12) are in fact linearly independent. •

5. The Solution Subspace of the Output Zeroing Problem.

Consider the AR-representation (1.3). According to Karampetakis & Vardulakis (1993), Karampetakis (1993) the solution vector space $\hat{B}$ of (1.3) is constituted by equivalence classes and has dimension equal to the total sum of the finite and infinite zeros of P(s) (order accounted for) and the sum of the right minimal indices (order accounted for). More specifically it is known that

$$\hat{B} = \left\langle \begin{array}{c} \left[\left\{ \sum_{k=0}^{\sigma_{i,j}-1-q} \begin{pmatrix} \beta^{i}_{j,k} \\ -u^{i}_{j,k} \end{pmatrix} \frac{t^{\sigma_{i,j}-q-k-1}}{(\sigma_{i,j}-q-k-1)!} \right\} e^{\lambda_i t} \right] \\ i \in \ell,\ j = \mu, \mu+1, \ldots, r \text{ and } q = 0,1,\ldots,\sigma_{i,j}-1 \end{array} \right\rangle \oplus$$

$$\oplus \left\langle \begin{array}{c} \left[\sum_{j=0}^{q} \begin{pmatrix} \beta^{\infty}_{i,j} \\ -u^{\infty}_{i,j} \end{pmatrix} \delta^{(q-j)}(t) \right] \\ i = k+1, k+2, \ldots, r \text{ and } q = 0,1,\ldots,\hat{q}_i - 1 \end{array} \right\rangle \oplus$$

$$\oplus \left\langle \begin{array}{c} \left[\sum_{j=0}^{q} \begin{pmatrix} \beta^{\varepsilon}_{i,j} \\ -u^{\varepsilon}_{i,j} \end{pmatrix} \delta^{(q-j)}(t) \right] \\ i = r+1, r+2, \ldots, n+m \text{ and } q = 0,1,\ldots,\varepsilon_i - 1 \end{array} \right\rangle \tag{5.1}$$

where [a] denotes the equivalence class of the solution a of (1.3) as has been defined in (2.1c) and the generalized finite and infinite zero-direction vectors and the generalized right index-direction vectors are the ones presented in Theorems 2, 4 and 6. If we now denote by $\hat{B}_\beta$, $\hat{B}_u$ the projection of the space $\hat{B}$ to the space of $[\beta(t)]_\beta$, $[-u(t)]_u$ respectively i.e.

$$\hat{B}_\beta := \left\{ [\beta(t)]_\beta \;\middle|\; \exists\, u(t) : \left[\begin{pmatrix} \beta(t) \\ -u(t) \end{pmatrix}\right] \in \hat{B} \right\}$$

$$\hat{B}_u := \left\{ [u(t)]_u \;\middle|\; \exists\, \beta(t) : \left[\begin{pmatrix} \beta(t) \\ -u(t) \end{pmatrix}\right] \in \hat{B} \right\} \tag{5.2}$$

Then we have the following theorem :

Theorem 8. Every input vector $[-u(t)]_u \in \hat{B}_u$ gives rise through the relation (1.1) to a pseudostate vector $[\beta(t)]_\beta \in \hat{B}_\beta$ and subsequently to the zero-output vector space. We observe also that

$$\hat{B}_u \equiv U_0^C + U_0^\infty + U_0^\varepsilon \tag{5.3}$$

and

$$\hat{B}_\beta \equiv B_0^C + B_0^\infty + B_0^\varepsilon \tag{5.4}$$

Proof. The proof is a direct result of the definition of the spaces $\hat{B}_u$ and $\hat{B}_\beta$. It should be noted however that although the solution vectors $\left(\beta(t)^T \quad -u(t)^T\right)^T$ which spans the space $\hat{B}$ are linearly independent, their projections are not and thus the intersection of the spaces $B_0^C, B_0^\infty, B_0^\varepsilon$ $(U_0^C, U_0^\infty, U_0^\varepsilon)$ is not necessarily the empty set. •

Definition 4. The pseudostate vector solutions of the system (1.3) define a subspace of the pseudostate space X of the system (1.1) :

$$B_\beta := \left\{ \beta(t) \;\middle|\; [\beta(t)]_\beta \in \hat{B}_\beta \right\} \tag{5.5}$$

which is called the *general output zeroing subspace*, or the *solution subspace of the output zeroing problem*.. •

Thus the output-zero problem may be described through the following diagram :

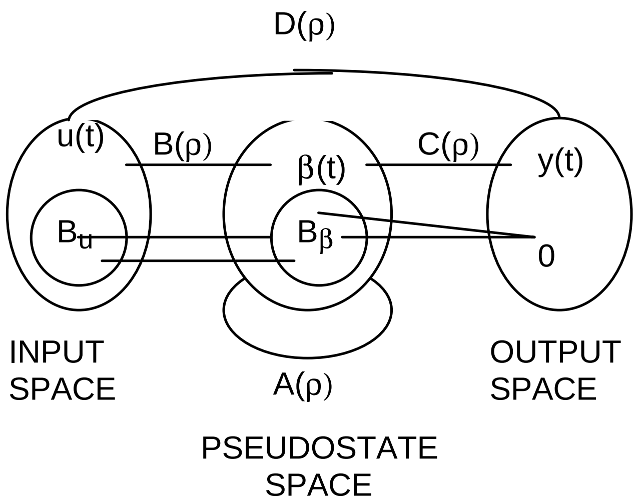

Diagram 1. The zero-output problem.

6. Illustrative Example.

Consider the following system Σ :

$$\begin{pmatrix} 1 & \rho^3 \\ 0 & \rho+1 \end{pmatrix}\begin{pmatrix} \beta_1(t) \\ \beta_2(t) \end{pmatrix} = \begin{pmatrix} \rho+1 \\ 0 \end{pmatrix} u(t)$$

$$y(t) = \begin{pmatrix} -\rho & -\rho^4 \end{pmatrix}\begin{pmatrix} \beta_1(t) \\ \beta_2(t) \end{pmatrix} + (\rho^2+\rho)u(t) \tag{E.1}$$

with Rosenbrock system matrix

$$P(s) = \begin{pmatrix} 1 & s^3 & s+1 \\ 0 & s+1 & 0 \\ s & s^4 & s^2+s \end{pmatrix} \tag{E.2}$$

Define

$$Z = \left\{ \begin{pmatrix} \beta(t) \\ -u(t) \end{pmatrix} = L^{-1}\begin{pmatrix} \hat{\beta}(s) \\ -\hat{u}(s) \end{pmatrix} \;\middle|\; \begin{pmatrix} 1 & s^3 & s+1 \\ 0 & s+1 & 0 \\ s & s^4 & s^2+s \end{pmatrix}\begin{pmatrix} \hat{\beta}(s) \\ -\hat{u}(s) \end{pmatrix} = 0 \right\} =$$

$$= \left\{ \begin{pmatrix} \beta(t) \\ -u(t) \end{pmatrix} = L^{-1}\left\{ \begin{pmatrix} -1-s \\ 0 \\ 1 \end{pmatrix} z(s) \right\} = \begin{pmatrix} \int_{0-}^{t} (-\delta(\tau) - \delta^{(1)}(\tau)) z(t-\tau)\,d\tau \\ 0 \\ \int_{0-}^{t} \delta(\tau) z(t-\tau)\,d\tau \end{pmatrix} \right\} \tag{E.3}$$

$$Z_u = -\pi_u Z = -\int_{0-}^{t} \delta(\tau) z(t-\tau) d\tau$$

$$Z_\beta = \pi_\beta Z = \begin{pmatrix} \int_{0-}^{t} (-\delta(\tau) - \delta^{(1)}(\tau)) z(t-\tau) d\tau \\ 0 \end{pmatrix} \tag{E.4}$$

$$[a] = a \oplus Z, \quad [a]_u = a \oplus Z_u, \quad [a]_\beta = a \oplus Z_\beta \tag{E.5}$$

where z(t) is an arbitrary function.

a) The finite invariant zeros.

There exist unimodular matrices $U_L(s) \in \Re[s]^{3\times 3}$ and $U_R(s) \in \Re[s]^{3\times 3}$ such that

$$\underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -s & 0 & 1 \end{pmatrix}}_{U_L(s)} \underbrace{\begin{pmatrix} 1 & s^3 & s+1 \\ 0 & s+1 & 0 \\ s & s^4 & s^2+s \end{pmatrix}}_{P(s)} \underbrace{\begin{pmatrix} 1 & -s^3 & -1-s \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{U_R(s)} = \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & s+1 & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{S^C_{P(s)}(s)} \tag{E.6}$$

Thus there exist one finite invariant zero $\lambda_1 = -1$ of partial multiplicitiy one i.e. $\sigma_{1,2} = 1$. Let also

$$z^1_{2,0} := \begin{pmatrix} \beta^1_{2,0} \\ -u^1_{2,0} \end{pmatrix} = \frac{1}{0!} u_2^{(0)}(-1) = \left(\begin{array}{c} 1 \\ 1 \\ \hdashline 0 \end{array}\right) \tag{E.7}$$

where $u_i(s)$ denotes the ith column of $U_R(s)$. Define now the vector valued function

$$\begin{pmatrix} \beta^1_{2,0}(t) \\ -u^1_{2,0}(t) \end{pmatrix} := \begin{pmatrix} \beta^1_{2,0} \\ -u^1_{2,0} \end{pmatrix} e^{-t} = \left(\begin{array}{c} 1 \\ 1 \\ \hdashline 0 \end{array}\right) e^{-t} \tag{E.8}$$

Then according to Corollary 1 we have that there exist an input vector space

$$U_0^C := \langle [0]_u \rangle \tag{E.9}$$

which gives rise through the relation (E.1) to the pseudostate vector space

$$B_0^C := \left\langle \left[\begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t} \right]_\beta \right\rangle \tag{E.10}$$

and in the sequel to the output vector space

$$Y_0 = \{0\} \tag{E.11}$$

or equivalently from (2.1c) to any input of the form

$$u(t) = -\int_{0-}^{t} \delta(\tau) z(t-\tau) d\tau \tag{E.12}$$

corresponds a pseudostate of the form

$$\beta(t) = \lambda \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t} + \begin{pmatrix} \int_{0-}^{t} (-\delta(\tau) - \delta^{(1)}(\tau)) z(t-\tau) d\tau \\ 0 \end{pmatrix} \quad \lambda \in \Re \tag{E.13}$$

and a zero output.

b) The infinite elementary divisors.

Consider the dual polynomial matrix of the Rosenbrock system matrix P(s) :

$$\tilde{P}(w) := \begin{pmatrix} w^4 & w & w^4 + w^3 \\ 0 & w^4 + w^3 & 0 \\ w^3 & 1 & w^3 + w^2 \end{pmatrix} \tag{E.14}$$

Then there exist unimodular matrices $\tilde{U}_L(w) \in \Re[w]^{3\times3}$ and $\tilde{U}_R(w) \in \Re[w]^{3\times3}$ such that

$$\underbrace{\begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & w^3 \\ -1 & 0 & w \end{pmatrix}}_{\tilde{U}_L(w)} \underbrace{\begin{pmatrix} w^4 & w & w^4 + w^3 \\ 0 & w^4 + w^3 & 0 \\ w^3 & 1 & w^3 + w^2 \end{pmatrix}}_{\tilde{P}(w)} \underbrace{\begin{pmatrix} 0 & -1 & 1+w \\ 1 & -w^2 & 0 \\ 0 & 1 & -w \end{pmatrix}}_{\tilde{U}_R(w)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & w^5(w+1) & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{E.15}$$

Because $5 = q_1 + \hat{q}_2 = 4 + \hat{q}_2$, we have that $\hat{q}_2 = 1$ and thus P(s) has one invariant infinite zero of order 1. Let also

$$z_{2,0}^{\infty} := \begin{pmatrix} \beta_{2,0}^{\infty} \\ -u_{2,0}^{\infty} \end{pmatrix} = \frac{1}{0!} \tilde{u}_2^{(0)}(0) = \begin{pmatrix} -1 \\ 0 \\ \hline 1 \end{pmatrix} \tag{E.16}$$

where $\tilde{u}_2(w)$ denotes the second column of $\tilde{U}_R(w)$. Define now the vector valued function

$$\begin{pmatrix} \beta_{2,0}^{\infty}(t) \\ -u_{2,0}^{\infty}(t) \end{pmatrix} := \begin{pmatrix} \beta_{2,0}^{\infty} \\ -u_{2,0}^{\infty} \end{pmatrix} \delta(t) = \begin{pmatrix} -1 \\ 0 \\ \hline 1 \end{pmatrix} \delta(t) \tag{E.17}$$

Then according to Corollary 2 we have that there exist an input vector space

$$U_0^{\infty} := \left\langle [-\delta(t)]_t \right\rangle \tag{E.18}$$

which gives rise through the relation (E.1) to the pseudostate vector space

$$B_0^{\infty} := \left\langle \left[\begin{pmatrix} -1 \\ 0 \end{pmatrix} \delta(t) \right]_{\beta} \right\rangle \tag{E.19}$$

and in the sequel to the output vector space

$$Y_0 = \{0\} \tag{E.20}$$

or equivalently from (2.1c) to any input of the form

$$u(t) = -\lambda\delta(t) - \int_{0-}^{t} \delta(\tau) z(t-\tau) d\tau \quad \lambda \in \Re \tag{E.21}$$

corresponds a pseudostate of the form

$$\beta(t) = \lambda \begin{pmatrix} -1 \\ 0 \end{pmatrix} \delta(t) + \begin{pmatrix} \int_{0-}^{t} (-\delta(\tau) - \delta^{(1)}(\tau)) z(t-\tau) d\tau \\ 0 \end{pmatrix} \quad \lambda \in \Re \tag{E.22}$$

and a zero output.

c) The right and left invariant indices.

It is easily seen that the vector

$$u_3(s) = \begin{pmatrix} -1-s \\ 0 \\ 1 \end{pmatrix} \tag{E.23}$$

constitutes a minimal polynomial basis for the right null space of P(s). Therefore Σ has one right invariant index of order 1 i.e. the degree of $u_3(s)$. Let also

$$z_{3,0}^{\varepsilon} := \begin{pmatrix} \beta_{3,0}^{\varepsilon} \\ -u_{3,0}^{\varepsilon} \end{pmatrix} = \frac{1}{0!} \bar{u}_3^{(0)}(0) = \begin{pmatrix} -1 \\ 0 \\ --- \\ 0 \end{pmatrix} \tag{E.24}$$

where $\bar{u}_3(s)$ denotes the"dual" polynomial of $u_3(s)$. Define now the vector valued function

$$\begin{pmatrix} \beta_{3,0}^{\varepsilon}(t) \\ -u_{3,0}^{\varepsilon}(t) \end{pmatrix} := \begin{pmatrix} \beta_{3,0}^{\varepsilon} \\ -u_{3,0}^{\varepsilon} \end{pmatrix} \delta(t) = \begin{pmatrix} -1 \\ 0 \\ --- \\ 0 \end{pmatrix} \delta(t) \tag{E.25}$$

Then according to Corollary 3 we have that there exist an input vector space

$$U_0^{\varepsilon} := \langle [0]_u \rangle \tag{E.26}$$

which gives rise through the relation (E.1) to the pseudostate vector space

$$B_0^{\varepsilon} := \left\langle \left[\begin{pmatrix} -1 \\ 0 \end{pmatrix} \delta(t) \right]_{\beta} \right\rangle \tag{E.27}$$

and in the sequel to the output vector space

$$Y_0 = \{0\} \tag{E.28}$$

or equivalently from (2.1c) to any input of the form

$$u(t) = \int_{0-}^{t} \delta(\tau) z(t-\tau) d\tau \tag{E.29}$$

corresponds a pseudostate of the form

$$\beta(t) = \lambda \begin{pmatrix} -1 \\ 0 \end{pmatrix} \delta(t) + \begin{pmatrix} \int_{0-}^{t} (-\delta(\tau) - \delta^{(1)}(\tau)) z(t-\tau) d\tau \\ 0 \end{pmatrix} \quad \lambda \in \Re \tag{E.30}$$

and a zero output.

Note also that the vector

$$v_3(s) = \begin{pmatrix} -s & 0 & 1 \end{pmatrix} =: v_{3,0} + v_{3,1} s \tag{E.31}$$

constitutes a minimal polynomial basis for the left kernel of the Rosenbrock system matrix P(s). Therefore Σ has one left invariant index of order $\eta_3 = 1$ i.e. the degree of $v_3(s)$. The zero-output problem has a solution, according to Theorem 4, if and only if the following $\eta_3 = 1$ condition between the initial conditions $\left(\beta^{(q)}(0-)^T, -u^{(q)}(0-)^T\right)^T$, q=0,1,2,3 is satisfied

$$\begin{pmatrix} v_{3,1} & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} P_0 & P_1 & P_2 & P_3 \\ 0 & P_0 & P_1 & P_2 \\ 0 & 0 & P_0 & P_1 \\ 0 & 0 & 0 & P_0 \end{pmatrix} \begin{pmatrix} \begin{pmatrix} \beta(0-) \\ -u(0-) \end{pmatrix} \\ \begin{pmatrix} \beta^{(1)}(0-) \\ -u^{(1)}(0-) \end{pmatrix} \\ \begin{pmatrix} \beta^{(2)}(0-) \\ -u^{(2)}(0-) \end{pmatrix} \\ \begin{pmatrix} \beta^{(3)}(0-) \\ -u^{(3)}(0-) \end{pmatrix} \end{pmatrix} = 0 \Leftrightarrow$$

$$-\beta_1(0-) - \beta_2^{(3)}(0-) - u(0-) - u^{(1)}(0-) = 0 \tag{E.32}$$

d) The Solution Subspace of the Output Zeroing Problem.

The solution vector space $\hat{B}$ of the AR-representation (E.1) is spanned by the equivalence classes which define the vectors in (E.8), (E.17) and (E.25) :

$$\hat{B} = \left\langle \left[\begin{pmatrix} 1 \\ 1 \\ --- \\ 0 \end{pmatrix} e^{-t}\right], \left[\begin{pmatrix} -1 \\ 0 \\ --- \\ 1 \end{pmatrix} \delta(t)\right], \left[\begin{pmatrix} -1 \\ 0 \\ --- \\ 0 \end{pmatrix} \delta(t)\right] \right\rangle \qquad \text{(E.33)}$$

Denote by $\hat{B}_\beta$, $\hat{B}_u$ the projection of the space $\hat{B}$ to the spaces of $[\beta(t)]$, $[-u(t)]$ respectively

$$\hat{B}_\beta = \left\langle \left[\begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t}\right]_\beta, \left[\begin{pmatrix} -1 \\ 0 \end{pmatrix} \delta(t)\right]_\beta \right\rangle$$

(E.34)

$$\hat{B}_u = \langle [\delta(t)]_u \rangle$$

Then every input vector $[-u(t)] \in \hat{B}_u$ gives rise through the relation (E.1) to a pseudostate vector $[\beta(t)] \in \hat{B}_\beta$ and in the sequel to the zero-output vector or equivalently every input vector of the form

$$u(t) = \lambda_2 \delta(\mathrm{t}) - \int_{0-}^{t} \delta(\tau) z(t-\tau) d\tau \quad \lambda_2 \in \Re \qquad \text{(E.35)}$$

gives rise through the relation (E.1) to a pseudostate $\beta(t)$ of the form

$$\beta(t) = \lambda_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t} + (\lambda_2 + \lambda_3) \begin{pmatrix} -1 \\ 0 \end{pmatrix} \delta(t) + \begin{pmatrix} \int_{0-}^{t} (-\delta(\tau) - \delta^{(1)}(\tau)) z(t-\tau) d\tau \\ 0 \end{pmatrix} \quad \lambda_1, \lambda_2, \lambda_3 \in \Re$$

(E.36)

and consequently to the zero-output. The space B_β :

$$B_\beta = \left\{ \beta(t) \;\middle|\; \beta(t) = \lambda_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t} + (\lambda_2 + \lambda_3) \begin{pmatrix} -1 \\ 0 \end{pmatrix} \delta(t) + \begin{pmatrix} \int_{0-}^{t} (-\delta(\tau) - \delta^{(1)}(\tau)) z(t-\tau) d\tau \\ 0 \end{pmatrix} \quad \lambda_1, \lambda_2, \lambda_3 \in \Re \right\}$$

(E.37)

is called the *general output zeroing subspace*, or the *solution subspace of the output zeroing problem*. •

7. Conclusions.

A geometric interpretation of the finite and infinite invariant zeros and the right and left minimal indices of a system has been given in terms of the solution of the zero-output problem. More specifically it has been shown that while the first three prementioned characteristics of the system give rise to the solution space of the zero-output problem the fourth one gives rise to conditions for the existence of solution to the above problem. The whole theory has been illustrated via an example.

REFERENCES

[1] Hayton G.E., Pugh A.C. and Fretwell P., 1988, Infinite elementary divisors of a matrix polynomial and implications., *Int. Journal of Control*, **47**, 53-64.

[2] Karampetakis N. P., 1993, Notions of Equivalence for Linear Time Invariant Multivariable Systems., Ph.D. Thesis, Department of Mathematics, Aristotle University of Thessaloniki, Thessaloniki, Greece.

[3] Karampetakis N. and Vardulakis A.I.G., 1993, On the solution space of continuous time AR-representations., *Proc. of the Second European Control Conference*, pp.1784-1789, June 28-July 1, Groningen, The Netherlands.

[4] Karcanias N., 1975, Geometric theory of zeros and its use in feedback analysis., Ph.D. Thesis, University of Manchester, Elec. Eng. Dept., U.K..

[5] Karcanias N. and Kouvaritakis B., 1979, The output zeroing problem and its relationship to the invariant zero structure., *Int. J. Control*, **30**, pp.395-415.

[6] Karcanias N. and Hayton G.E., 1981, State-space and transfer function invariant zeros : A unified approach., *Joint Automatic Control Conference TA4C*, Charlottesville, V.A..

[7] MacFarlane A.G.J. and Karcanias N., 1976, Poles and zeros of linear multivariable systems : a survey of the algebraic, geometric and complex-variable theory., *Int. J. Control*, **24**, pp.33-74.

[8] Rosenbrock H.H., 1973, The zeros of a system., *Int. J. Control*, **18**, pp.297.

[9] Walker B.W., 1988, Equivalence transformations for linear systems., Ph.D. Thesis, Hull University, U.K..

Define now the vector valued functions

$$\hat{B} = \left\langle \begin{matrix} \left[\left\{ \sum_{k=0}^{\sigma_{i,j}-1-q} \begin{pmatrix} \beta_{j,k}^{i} \\ -u_{j,k}^{i} \end{pmatrix} \frac{t^{\sigma_{i,j}-q-k-1}}{(\sigma_{i,j}-q-k-1)!} \right\} e^{\lambda_i t} \right] \\ i \in \ell,\ j=\mu,\mu+1,\ldots,r \text{ and } q=0,1,\ldots,\sigma_{i,j}-1 \end{matrix} \right\rangle \oplus$$

$$\oplus \left\langle \begin{matrix} \left[\sum_{j=0}^{q} \begin{pmatrix} \beta_{i,j}^{\infty} \\ -u_{i,j}^{\infty} \end{pmatrix} \delta^{(q-j)}(t) \right] \\ i=k+1,k+2,\ldots,r \text{ and } q=0,1,\ldots,\hat{q}_i-1 \end{matrix} \right\rangle \oplus$$

$$\oplus \left\langle \begin{matrix} \left[\sum_{j=0}^{q} \begin{pmatrix} \beta_{i,j}^{\varepsilon} \\ -u_{i,j}^{\varepsilon} \end{pmatrix} \delta^{(q-j)}(t) \right] \\ i=r+1,r+2,\ldots,n+m \text{ and } q=0,1,\ldots,\varepsilon_i-1 \end{matrix} \right\rangle$$

$$\begin{pmatrix} \beta_{j,q}^{i}(t) \\ -u_{j,q}^{i}(t) \end{pmatrix} := \left\{ \sum_{k=0}^{\sigma_{i,j}-1-q} \begin{pmatrix} \beta_{j,k}^{i} \\ -u_{j,k}^{i} \end{pmatrix} \frac{t^{\sigma_{i,j}-q-k-1}}{(\sigma_{i,j}-q-k-1)!} \right\} e^{\lambda_i t} \tag{2.5}$$

$$i \in \ell,\ j=\mu,\mu+1,\ldots,r \text{ and } q=0,1,\ldots,\sigma_{i,j}-1$$

Then for initial conditions :

$$\begin{pmatrix} \beta^{(i)}(0-) \\ -u^{(i)}(0-) \end{pmatrix} = -\begin{pmatrix} \beta_{j,q+i+1}^{\infty} \\ -u_{j,q+i+1}^{\infty} \end{pmatrix} \tag{3.5}$$

$$i=0,1,\ldots,q_1 \quad \text{j=k+1,...,r} \quad \text{and} \quad q=0,1,\ldots,\hat{q}_j-1$$

we obtain (Karampetakis & Vardulakis 1993) respectively the linearly independent solutions

$$\begin{pmatrix} \beta_{j,q}^{\infty}(t) \\ -u_{j,q}^{\infty}(t) \end{pmatrix} = \sum_{i=0}^{q} \begin{pmatrix} \beta_{j,i}^{\infty} \\ -u_{j,i}^{\infty} \end{pmatrix} \delta^{(q-i)}(t) \tag{3.6}$$

j=k+1,...,r and $q=0,1,\ldots,\hat{q}_j-1$

Then for initial conditions :

$$\begin{pmatrix} \beta^{(i)}(0-) \\ -u^{(i)}(0-) \end{pmatrix} = -\begin{pmatrix} \beta_{j,q+i+1}^{\varepsilon} \\ -u_{j,q+i+1}^{\varepsilon} \end{pmatrix} \tag{4.4}$$

$$i=0,1,\ldots,q_1 \quad \text{j=r+1,...,n+m} \quad \text{and} \quad q=0,1,\ldots,\varepsilon_j-1$$

we obtain (Karampetakis & Vardulakis 1993) respectively the linearly independent solutions

$$\begin{pmatrix} \beta_{j,q}^{\varepsilon}(t) \\ -u_{j,q}^{\varepsilon}(t) \end{pmatrix} = \sum_{i=0}^{q} \begin{pmatrix} \beta_{j,i}^{\varepsilon} \\ -u_{j,i}^{\varepsilon} \end{pmatrix} \delta^{(q-i)}(t) \tag{4.5}$$

j=r+1,...,n+m and $q = 0,1,\ldots,\varepsilon_j - 1$