On the Division of Polynomial Matrices*

P. Tzekis, N. P. Karampetakis and A. I. Vardulakis

Department of Mathematics, Aristotle University of Thessaloniki

Thessaloniki 54006, Greece

email : Karampetakis@olymp.ccf.auth.gr

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Abstract

The main purpose of this paper is to determine two new algorithms for the division of the polynomial matrix $B(s) \in \mathbf{R}[s]^{p\times q}$ by $A(s) \in \mathbf{R}[s]^{p\times p}$ a) based on the Laurent matrix expansion at $s = \infty$ of the inverse of $A(s)$ i.e. $A(s)^{-1}$ and $b)$ in a similar way to the one presented in [4].

Keywords : division, polynomial matrices, numerical algorithms.

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1. Introduction.

Let $A(s) \in \mathbf{R}[s]^{p\times p}$ $(A(s) \in \mathbf{R}[s]^{q\times q})$ and $B(s) \in \mathbf{R}[s]^{p\times q}$ be regular i.e.

$$B(s) = B_0 s^m + B_1 s^{m-1} + \cdots + B_m \in \mathbf{R}[s]^{p\times q} \text{ with } B_0 \neq 0_{p\times q} \tag{1.1}$$

$$A(s) = A_0 s^n + A_1 s^{n-1} + \cdots + A_n \in \mathbf{R}[s]^{q\times q} (\in \mathbf{R}[s]^{p\times p}) \text{ with } A_0 \neq 0_{p\times p} \text{ and } m \geq n \tag{1.2}$$

Definition 1.1. *[4] The matrix polynomials* $Q(s) \in \mathbf{R}[s]^{p\times q}$ $(\hat{Q}(s) \in \mathbf{R}[s]^{p\times q})$ *and* $R(s) \in \mathbf{R}[s]^{p\times q}$ $(\hat{R}(s) \in \mathbf{R}[s]^{p\times q})$ *are the left (right) quotient and left (right) remainder, respectively, of* $B(s)$ *on the left (right) division by* $A(s)$ *if*

$$B(s) = A(s)Q(s) + R(s) \tag{1.3}$$

$$\left(B(s) = \hat{Q}(s)A(s) + \hat{R}(s)\right)$$

and $A(s)^{-1}R(s)$ $(\hat{R}(s)A(s)^{-1})$ *vanishes at* $s = \infty$ *(if* $A(s)$ *is row (column) reduced then to an equivalent condition is that the degree of the ith row (column) of* $R(s)$ *is less than the respective ith row (column) degree of* $A(s)$*). The above division is always possible and unique.* ■

The problem of the determination of the quotient and the remainder of the above division was the main point of interest in a large number of recent papers [2],[5],[6],[7],[8], because of the large number of its applications in linear system theory. [2],[6],[7],[8] use various forms of the polynomial matrix $A(s)$ useful in the determination of $Q(s)$ and $R(s)$ in a recursive way. The division of a polynomial matrix $B(s)$ by the matrix pencil $(sE-A)$ has been studied by [5] in a completely different form based on the form of the Laurent expansion at $s = \infty$ of the matrix $(sE - A)^{-1}$. An extension of the results presented in [5] and thus of the generalised Bezout theorem for the case where both polynomial matrices are in general form is proposed in Section 2. This result gives rise a) to a relative Cayley Hamilton theorem for polynomial matrices in terms of the fundamental matrix sequence $\{H_k\}$ of $A(s)^{-1}$ and $b)$ to a finite expression for the relative resolvent matrix of $A(s)^{-1}$ in terms of the *Tschirnhausen polynomials*. An extension of the Generalised Bezout theorem for polynomial matrices is also presented in the same section in a quite similar way to the one presented in [4]. Without loss of generality we study in the sequel the left division of polynomial matrices i.e. when the left quotient and the left remainder are to be found.

2. Polynomial matrix division and other results

Consider the polynomial matrices $A(s)$, $B(s)$ defined in (1.1,1.2). Then a first way for determining a left quotient $R(s)$ and remainder $R(s)$ is given by the following

Theorem 2.1. *([9], p.37) When the polynomial matrix $B(s)$ is divided on the left by the polynomial matrix $A(s)$, then the left quotient $Q(s)$ and the left remainder*

$R(s)$ are given respectively by

$$Q(s) = \textit{polynomial part of } A(s)^{-1}B(s)$$

$$R(s) = B(s) - A(s)Q(s)$$

Similar results hold for the right division of polynomial matrices. ■

Regularity of the polynomial matrix $A(s)$ implies the existence of the unique Laurent expansion

$$A(s)^{-1} = \sum_{k=-\mu}^{\infty} H_k s^{-k} \tag{2.1}$$

at $s = \infty$, with μ the greatest order of the infinite zeros of $A(s)$ ([9], p.196). A method for finding the *relative fundamental matrix* $\{H_k\}$ in terms of the coefficient matrices of $A(s)$ is given in [3].

Defining

$$B[H_k] = H_k B_0 + H_{k-1} B_1 + \cdots + H_{k-m} B_m \tag{2.2}$$

$$A[H_k] = A_0 H_k + A_1 H_{k-1} + \cdots + A_n H_{k-n} \tag{2.3}$$

$$A^T[H_k] = H_k A_0 + H_{k-1} A_1 + \cdots + H_{k-n} A_n \tag{2.4}$$

$$A[B[H_k]] = A_0 B[H_k] + A_1 B[H_{k-1}] + \cdots + A_n B[H_{k-n}] \tag{2.5}$$

$$B[A[H_k]] = A[H_k] B_0 + A[H_{k-1}] B_1 + \cdots + A[H_{k-m}] B_m \tag{2.6}$$

we have the following :

Lemma 2.2. **a)** $A[B[H_k]] = B[A[H_k]]$

b) $A[H_i] = \left\{ \begin{array}{cc} 0 & i \neq n \\ I_q & i = n \end{array} \right\}$ and $A^T[H_i] = \left\{ \begin{array}{cc} 0 & i \neq n \\ I_q & i = n \end{array} \right\}$

Proof.

a)

$$
\begin{aligned}
A[B[H_k]] &\overset{(2.5)}{=} A_0[H_k B_0 + H_{k-1}B_1 + \cdots + H_{k-m}B_m] + \\
&+ A_1[H_{k-1}B_0 + H_{k-2}B_1 + \cdots + H_{k-m-1}B_m] + \\
&+ \cdots\cdots + \\
&+ A_n[H_{k-n}B_0 + H_{k-n-1}B_1 + \cdots + H_{k-m-n}B_m] = \\
&= [A_0 H_k + A_1 H_{k-1} + \cdots + A_n H_{k-n}]B_0 + \\
&+ [A_0 H_{k-1} + A_1 H_{k-2} + \cdots + A_n H_{k-n-1}]B_1 + \\
&+ \cdots\cdots + \\
&+ [A_0 H_{k-m} + A_1 H_{k-m-1} + \cdots + A_n H_{k-m-n}]B_m = \\
&\overset{(2.3,2.6)}{=} B[A[H_k]]
\end{aligned}
$$

b) The result can be proved by equating the coefficient powers of s of the left and right term in the equation $A(s)A(s)^{-1} = I_q$ $(A(s)^{-1}A(s) = I_q)$. ■

We can thus state the *relative Bezout theorem*

Theorem 2.3. *When a matrix polynomial $B(s)$ is divided on the left by the regular polynomial matrix $A(s)$, the quotient and the remainder are given respectively by*

$$Q(s) = \sum_{i=-\mu}^{m} B[H_i]\, s^{m-i} \tag{2.7}$$

and

$$R(s)=\sum_{i=0}^{n-1}\left(\sum_{j=0}^{i}A_jB\left[H_{m+i+1-j}\right]\right)s^{n-1-i} \tag{2.8}$$

Proof. From Theorem 2.1 we have that the left quotient $Q(s)$ is given by

$$Q(s)=\text{polynomial part}\left\{A(s)^{-1}B(s)\right\}=$$

$$=\text{polynomial part}\left\{\left(\sum_{k=-\mu}^{\infty}H_ks^{-k}\right)\left(\sum_{k=0}^{m}B_ks^{m-k}\right)\right\}$$

$$=\left[H_{-\mu}B_0\right]s^{\mu+m}+\left[H_{-\mu+1}B_0+H_{-\mu}B_1\right]s^{\mu+m-1}+$$

$$+\cdots+$$

$$+\left[H_mB_0+H_{m-1}B_1+\cdots+H_0B_m\right]=$$

$$=B\left[H_{-\mu}\right]s^{\mu+m}+B\left[H_{-\mu+1}\right]s^{\mu+m-1}+\cdots+B\left[H_m\right]=$$

$$=\sum_{i=-\mu}^{m}B\left[H_i\right]s^{m-i}$$

while the left remainder $R(s)$ is given by :

$$R(s)=B(s)-A(s)Q(s)=\left(B_0s^m+B_1s^{m-1}+\cdots+B_m\right)-$$

$$\left(A_0s^n+\cdots+A_n\right)\left(\left[H_{-\mu}B_0\right]s^{\mu+m}+\cdots+\left[H_mB_0+H_{m-1}B_1+\cdots+H_0B_m\right]\right)=$$

$$=\left(B_0s^m+B_1s^{m-1}+\cdot\cdot+B_m\right)-$$

$$\left\{A\left[B\left[H_{-\mu}\right]\right]s^{\mu+m+n}+A\left[B\left[H_{-\mu+1}\right]\right]s^{\mu+m+n-1}+\cdots+A\left[B\left[H_{m+n}\right]\right]-\right.$$

$$\left.-\sum_{i=0}^{n-1}\left(\sum_{j=0}^{i}A_jB\left[H_{m+i+1-j}\right]\right)s^{n-1-i}\right\}\overset{\text{Lemma 2.2 (a)}}{=}\left(B_0s^m+B_1s^{m-1}+\cdot\cdot+B_m\right)-$$

$$\left(B\left[A\left[H_{-\mu}\right]\right]s^{\mu+m+n}+\cdots+B\left[A\left[H_{m+n}\right]\right]-\sum_{i=0}^{n-1}\left(\sum_{j=0}^{i}A_jB\left[H_{m+i+1-j}\right]\right)s^{n-1-i}\right) \tag{2.9}$$

However, from Lemma 2.2 (b), we have that :

$$B\left[A\left[H_{-\mu}\right]\right]=B\left[A\left[H_{-\mu+1}\right]\right]=\cdots=B\left[A\left[H_{n-1}\right]\right]=0$$
$$B\left[A\left[H_{n+i}\right]\right]=B_{m-i} \text{ for } i=0,1,..,m \tag{2.10}$$

Thus from, (2.9) and (2.10)

$$R(s)=\left(B_0 s^m+B_1 s^{m-1}+\cdots+B_m\right)-$$
$$-\left(B_0 s^m+B_1 s^{m-1}+\cdots+B_m-\sum_{i=0}^{n-1}\left(\sum_{j=0}^{i} A_j B\left[H_{m+i+1-j}\right]\right) s^{n-1-i}\right)$$
$$=\sum_{i=0}^{n-1}\left(\sum_{j=0}^{i} A_j B\left[H_{m+i+1-j}\right]\right) s^{n-1-i}$$

We can check that

$$\text{polynomial part } \left[A(s)^{-1}R(s)\right] \overset{(2.1,2.8)}{=} H_{-\mu}A_0 B\left[H_{m+1}\right] s^{\mu+n-1}+$$
$$+\left[H_{-\mu}\left[A_0 B\left[H_{m+2}\right]+A_1 B\left[H_{m+1}\right]\right]+H_{-\mu+1}A_0 B\left[H_{m+1}\right]\right] s^{\mu+n-2}+$$
$$+\cdots\cdots+$$
$$+\left[H_0\left[A_0 B\left[H_{m+n}\right]+\cdots+A_{n-1}B\left[H_{m+1}\right]\right]+\cdots+H_{n-1}A_0 B\left[H_{m+1}\right]\right]=$$
$$=\left\{\left[H_{-\mu}A_0\right] B\left[H_{m+1}\right]\right\} s^{\mu+n-1}+\left\{\left[H_{-\mu}A_1+H_{-\mu+1}A_0\right] B\left[H_{m+1}\right]+\right.$$
$$\left.+\left[H_{-\mu}A_0\right] B\left[H_{m+2}\right]\right\} s^{\mu+n-2}+\cdots\cdots+$$
$$+\left\{\left[H_0 A_{n-1}+\cdots+H_{n-1}A_0\right] B\left[H_{m+1}\right]+\cdots+\left[H_0 A_0\right] B\left[H_{m+n}\right]\right\} \overset{(2.4)}{=}$$
$$=A^T\left[H_{-\mu}\right] B\left[H_{m+1}\right] s^{\mu+n-1}+$$
$$\left\{A^T\left[H_{-\mu+1}\right] B\left[H_{m+1}\right]+A^T\left[H_{-\mu}\right] B\left[H_{m+2}\right]\right\} s^{\mu+n-2}+$$
$$+\cdots+\left\{A^T\left[H_{n-1}\right] B\left[H_{m+1}\right]+\cdots+A^T\left[H_0\right] B\left[H_{m+1}\right]\right\} \overset{\text{Lemma 2.2 (b)}}{=} 0$$

and thus $A(s)^{-1}R(s)$ vanishes at $s=\infty$, which proves the Theorem. ■

Corollary 2.4. *In case where* $A(s)=sE-A\equiv A_0 s+A_1$ *with* $\det[E]$ *not necessarily equal to zero the left remainder on the division of* $B(s)$ *by* $A(s)$ *is* *according to Theorem 2.3 equal to*

$$R(s) = A_0 B[H_{m+1}] = EB[H_{m+1}]$$

which coincides with the results in [5]. ■

Corollary 2.5. *The above Theorem is independent of the regularity of the coefficient matrix A_0 i.e. we may have a polynomial matrix $A(s)$ as in (1.2) with $\det[A_0] = 0$. In case, however, where $\det[A_0] \neq 0$ the leading power in the Laurent expansion of $A(s)^{-1}$ is not μ but $-n$ [1] and thus the left quotient and remainder in the above division are given respectively by*

$$Q(s) = [H_n B_0]\, s^{m-n} + [H_{n+1} B_0 + H_n B_1]\, s^{m-n-1} + \cdots +$$

$$+ [H_m B_0 + H_{m-1} B_1 + \cdots + H_n B_{m-n}] =$$

$$\overset{(2.2)}{=} B\,[H_n]\, s^{m-n} + B\,[H_{n+1}]\, s^{m-n-1} + \cdots + B\,[H_m] = \textstyle\sum_{i=n}^{m} B\,[H_i]\, s^{m-i}$$

and

$$R(s) = A_0\,[H_{m+1} B_0 + H_m B_1 + \cdots + H_n B_{m-n+1}]\, s^{n-1} +$$

$$+[A_0\,[H_{m+2} B_0 + H_{m+1} B_1 + \cdots + H_n B_{m-n+2}] +$$

$$+A_1\,[H_{m+1} B_0 + H_m B_1 + \cdots + H_n B_{m-n+1}]] s^{n-2} +$$

$$+ \cdots \cdots +$$

$$+[A_0\,[H_{m+n} B_0 + H_{m+n-1} B_1 + \cdots + H_n B_m] + \cdots +$$

$$+A_{n-1}\,[H_{m+1} B_0 + H_m B_1 + \cdots + H_n B_{m-n+1}]]$$

$$\overset{(2.2)}{=} A_0 B\left[H_{m+1}\right] s^{n-1} + \left[A_0 B\left[H_{m+2}\right] + A_1 B\left[H_{m+1}\right]\right] s^{n-2} + \cdots +$$

$$+\left[A_0 B\left[H_{m+n}\right] + \cdots + A_{n-1} B\left[H_{m+1}\right]\right] =$$

$$= \sum_{i=0}^{n-1} \left(\sum_{j=0}^{i} A_j B[H_{m+i+1-j}]\right) s^{n-1-i}$$

■

Corollary 2.6. *$A(s)$ is a left divisor of $B(s)$ iff the left remainder of the division of $B(s)$ by $A(s)$ is zero or equivalently iff*

$$\begin{aligned}
&R(s) = \sum_{i=0}^{n-1} \left(\sum_{j=0}^{i} A_j B[H_{m+i+1-j}]\right) s^{n-1-i} = \\
&\overset{(2.2)}{=} A_0\left[H_{m+1}B_0 + H_m B_1 + \cdots + H_1 B_m\right] s^{n-1} + \\
&+ [A_0\left[H_{m+2}B_0 + H_{m+1}B_1 + \cdots + H_2 B_m\right] + \\
&+ A_1\left[H_{m+1}B_0 + H_m B_1 + \cdots + H_1 B_m\right]] s^{n-2} + \\
&+ \cdots\cdots + \\
&+ [A_0\left[H_{m+n}B_0 + H_{m+n-1}B_1 + \cdots + H_n B_m\right] + \cdots + \\
&+ A_{n-1}\left[H_{m+1}B_0 + H_m B_1 + \cdots + H_1 B_m\right]] \equiv 0
\end{aligned} \tag{2.11}$$

or equivalently iff the coefficients of the powers of s in (2.11) are equal to zero i.e.

$$A_0\left[H_{m+1}B_0 + H_m B_1 + \cdots + H_1 B_m\right] = 0$$

$$A_0\left[H_{m+2}B_0 + H_{m+1}B_1 + \cdots + H_2 B_m\right] +$$

$$+ A_1\left[H_{m+1}B_0 + H_m B_1 + \cdots + H_1 B_m\right] = 0$$

$$\cdots\cdots$$

$$A_0\left[H_{m+n}B_0 + H_{m+n-1}B_1 + \cdots + H_n B_m\right] + \cdots +$$

$$+A_{n-1}\left[H_{m+1}B_0 + H_m B_1 + \cdots + H_1 B_m\right] = 0$$

or

$$\begin{pmatrix} A_0 & 0 & \cdots & 0 \\ A_1 & A_0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_{n-1} & A_{n-2} & \cdots & A_0 \end{pmatrix} \begin{pmatrix} H_{m+1} & H_m & \cdots & H_1 \\ H_{m+2} & H_{m+1} & \cdots & H_2 \\ \vdots & \vdots & \ddots & \vdots \\ H_{m+n} & H_{m+n-1} & \cdots & H_n \end{pmatrix} \begin{pmatrix} B_0 \\ B_1 \\ \vdots \\ B_m \end{pmatrix} = 0 \quad (2.12)$$

Thus (2.12) is a necessary and sufficient condition for $A(s)$ to be a left divisor of $B(s)$. A similar statement holds for division on the right by $A(s)$. In case where A_0 is nonsingular then the first matrix of the left term in (2.12) is nonsingular and thus (2.12) is equivalent to :

$$\begin{pmatrix} H_{m+1} & H_m & \cdots & H_1 \\ H_{m+2} & H_{m+1} & \cdots & H_2 \\ \vdots & \vdots & \ddots & \vdots \\ H_{m+n} & H_{m+n-1} & \cdots & H_n \end{pmatrix} \begin{pmatrix} B_0 \\ B_1 \\ \vdots \\ B_m \end{pmatrix} = 0$$

■

By letting $B(s) = \Delta(s) \times I_q = \det[A(s)] \times I_q$ we arrive at the *relative Cayley-Hamilton theorem* in terms of the fundamental matrices $\{H_k\}$ defined in (2.1).

Theorem 2.7. *Suppose that* $A(s)$ *is regular with* $\{H_k\}$ *given by (2.1) and*

$$\det[A(s)] = p_0 s^{\ell} + p_1 s^{\ell-1} + \cdots + p_{\ell} \tag{2.13}$$

where $nq \geq \ell \geq n$. *Then*

$$\Delta(H_k) = p_0 H_k + p_1 H_{k-1} + \cdots + p_{\ell} H_{k-\ell} = 0 \text{ for } k > \ell \ \& \ k < \ell - \mu - n \tag{2.14}$$

Proof. We have that

$$A(s)^{-1} = \frac{Adj[A(s)^{-1}]}{\det[A(s)]} \Longleftrightarrow \Delta(s)A(s)^{-1} = Adj[A(s)^{-1}] \tag{2.15}$$

Substituting $A(s)^{-1}$ from (2.1) and equating the coefficients a) of the negative powers of s in (2.15) and b) of the powers of s greater than $\mu + m$ (see (2.16)) we obtain (2.14). ■

It is easily seen from (2.15) that $A(s)$ is a left divisor of $\Delta(s)$ and thus from Corollary 2.6

$$\begin{pmatrix} A_0 & 0 & \cdots & 0 \\ A_1 & A_0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_{n-1} & A_{n-2} & \cdots & A_0 \end{pmatrix} \begin{pmatrix} H_{\ell+1} & H_{\ell} & \cdots & H_1 \\ H_{\ell+2} & H_{\ell+1} & \cdots & H_2 \\ \vdots & \vdots & \ddots & \vdots \\ H_{\ell+n} & H_{\ell+n-1} & \cdots & H_n \end{pmatrix} \begin{pmatrix} p_0 I_q \\ p_1 I_q \\ \vdots \\ p_m I_q \end{pmatrix} = 0$$

$Adj[A(s)^{-1}]$ is the left quotient in the left division of $\Delta(s) \times I_q$ by $A(s)$ and thus it can be written according to Theorem 2.3 as

$$Adj[A(s)^{-1}] = [H_{-\mu}p_0]s^{\mu+n} + [H_{-\mu+1}p_0 + H_{-\mu}p_1]s^{\mu+n-1} +$$
$$+ \cdots + [H_n p_0 + H_{n-1}p_1 + \cdots + H_0 p_n] = \qquad (2.16)$$
$$= \Delta[H_{-\mu}]s^{\mu+n} + \Delta[H_{-\mu+1}]s^{\mu+n-1} + \cdots + \Delta[H_n] = \textstyle\sum_{i=0}^{\mu+n} \Delta[H_{n-i}]s^i$$

or equivalently as

$$Adj[A(s)^{-1}] = H_{-\mu}[p_0 s^{\mu+n} + p_1 s^{\mu+n-1} + \cdots + p_\ell s^{\mu+n-\ell}] +$$
$$+ H_{-\mu+1}[p_0 s^{\mu+n-1} + p_1 s^{\mu+n-2} + \cdots + p_\ell s^{\mu+n-\ell-1}] +$$
$$+ \cdots\cdots +$$
$$+ H_0[p_0 s^n + p_1 s^{n-1} + \cdots + p_n] + \qquad (2.17)$$
$$+ \cdots\cdots +$$
$$+ H_{n-1}[p_0 s + p_1] +$$
$$+ H_n[p_0] \equiv$$
$$\equiv \textstyle\sum_{i=-\mu}^{n} \Delta_i(s) H_i$$

Thus we have derived the following

Theorem 2.8. *The relative resolvent matrix $A(s)$ is expressed in terms of the relative fundamental matrix $\{H_k\}$ by the relation*

$$A(s)^{-1} = \frac{1}{q(s)} \left\{ \sum_{i=-\mu}^{m} \Delta_i(s) H_i \right\}$$

where $q(s)$ is the determinant of $A(s)$ defined in (2.13), and $\Delta_i(s)$, $i = -\mu, -\mu+1, .., m$ are the Tschirnhausen polynomials defined by (2.17). ■

Example 2.9. *Let*

$$B(s) = \begin{pmatrix} s^2+1 & s \\ s & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} s^2 + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} s + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \equiv B_0 s^2 + B_1 s + B_2$$

and

$$A(s) = \begin{pmatrix} s & 1 \\ 0 & s \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} s + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \equiv A_0 s + A_1$$

where $\det[A_0] = 2 \neq 0$. *Then we have that*

$$A(s)^{-1} = \begin{pmatrix} \frac{1}{s} & -\frac{1}{s^2} \\ 0 & \frac{1}{s} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \frac{1}{s} + \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} \frac{1}{s^2} \equiv H_1 s^{-1} + H_2 s^{-2}$$

Then the left quotient of the division of $B(s)$ *by* $A(s)$ *is given according to Corollary 2.4 by*

$$Q(s) = B[H_1]s + B[H_2] = [H_1 B_0]\, s + [H_2 B_0 + H_1 B_1] = \begin{pmatrix} s & 1 \\ 1 & 0 \end{pmatrix}$$

while the left remainder is given by

$$R(s) = A_0 B[H_3] = A_0[H_3 B_0 + H_2 B_1 + H_1 B_2] = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

■

Example 2.10. *Let*

$$B(s) = \begin{pmatrix} s^2 & s \\ 0 & s \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} s^2 + \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} s + \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \equiv B_0 s^2 + B_1 s + B_2$$

and

$$A(s)=\begin{pmatrix}1 & s\\ 0 & 1\end{pmatrix}=\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}s+\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\equiv A_0 s+A_1$$

where $\det[A_0]=0$. *Then we have that*

$$A(s)^{-1}=\begin{pmatrix}1 & -s\\ 0 & 1\end{pmatrix}=\begin{pmatrix}0 & -1\\ 0 & 0\end{pmatrix}s+\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\equiv H_{-1}s+H_0$$

Then the left quotient of the division of $B(s)$ *by* $A(s)$ *is given according to Theorem 2.3 by*

$$\begin{aligned}Q(s)&=B[H_{-1}]s^3+B[H_0]s^2+B[H_1]s+B[H_2]=\\ &=[H_{-1}B_0]\,s^3+[H_0B_0+H_{-1}B_1]\,s^2+[H_1B_0+H_0B_1+H_{-1}B_2]\,s+\\ &\quad+[H_2B_0+H_1B_1+H_0B_2]=\begin{pmatrix}s^2 & -s^2+s\\ 0 & s\end{pmatrix}\end{aligned}\tag{2.18}$$

while the left remainder is given by

$$R(s)=A_0B[H_3]=A_0[H_3B_0+H_2B_1+H_1B_2]=\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}\tag{2.19}$$

However, we can easily see that

$$\hat{Q}(s)=\begin{pmatrix}s^2+sL_{2,1}+L_{1,1} & -s^2+s\,(1+L_{2,2})+L_{1,2}\\ -L_{2,1} & s-L_{2,2}\end{pmatrix}$$

$$\hat{R}(s)=\begin{pmatrix}-L_{1,1} & -L_{1,2}\\ L_{2,1} & L_{2,2}\end{pmatrix}$$

where $L_{i,j}$ are arbitrary constant numbers, also satisfy the equation $B(s) = A(s)\hat{Q}(s) + \hat{R}(s)$. However, we can check that $A(s)^{-1}\hat{R}(s)$ vanishes at $s = \infty$ iff $L_{i,j} = 0$ and thus $Q(s)$ and $R(s)$ are of the form (2.18) and (2.19). This example shows that condition (1.3) does not guarantee by itself the uniqueness of $Q(s)$ and $R(s)$ but the further condition of strict properness of $A(s)^{-1}R(s)$ must be satisfied. ■

Assume now that A_0 is a regular matrix. Without loss of generality we may assume that $A_0 = I_q$, otherwise, instead of making the left division of $B(s)$ by $A(s)$ we can make the division of $A_0^{-1}B(s)$ by $A_0^{-1}A(s)$ i.e.

$$A_0^{-1}B(s) = A_0^{-1}A(s)\hat{Q}(s) + \hat{R}(s)$$

and thus the left quotient and remainder of the division of $B(s)$ by $A(s)$ will be respectively $Q(s) = \hat{Q}(s)$ and $R(s) = A_0\hat{R}(s)$ respectively. Then we can state the *relative Bezout theorem.*

Theorem 2.11. *If $A_0 = I_q$ then the left quotient and remainder of the division of $B(s)$ by $A(s)$ are respectively :*

$$Q(s) = Y_0 s^{m-n} + Y_1 s^{m-n-1} + \cdots + Y_{m-n}$$

$$R(s) = Y_{m-n} s^{n-1} + Y_{m-n+1} s^{n-2} + \cdots + Y_m$$

where Y_i are defined according to the following recursive way

$$Y_0 = B_0$$
$$Y_j = B_j - \sum_{i=max(0,j-n)}^{min(j-1,m-n)} A_{j-i} Y_i \tag{2.20}$$

Proof. To determine the left remainder we use the usual division scheme

$$B(s) = A(s)B_0 s^{m-n} + (B_1 - A_1 B_0)\, s^{m-1} +$$
$$+ (B_2 - A_2 B_0)\, s^{m-2} + \cdots + B_m =$$
$$= A(s)\left[B_0 s^{m-n} + [B_1 - A_1 B_0]\, s^{m-n-1}\right] +$$
$$+ \left[B_2 - A_2 B_0 - A_1 [B_1 - A_1 B_0]\right] s^{m-2} + \cdots + B_m =$$
$$= \cdots\cdots =$$
$$= A(s)\left[Y_0 s^{m-n} + Y_1 s^{m-n-1} + \cdots + Y_{m-n}\right] +$$
$$+ \left[Y_{m-n+1} s^{n-1} + Y_{m-n+2} s^{n-2} + \cdots + Y_m\right]$$

However, $\det[A_0] \neq 0$. Thus, $A(s)$ is column (row) reduced and its ith row degree (n) is greater than the ith row degree of $R(s)$ which is at most $n-1$. Therefore, $A(s)^{-1}R(s)$ vanishes at $s = \infty$ according to Definition 1.1 which proves the Theorem. ■

Example 2.12. *Let $B(s)$ and $A(s)$ as in Example 2.9. Then*

$$Y_0 = B_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$Y_1 = B_1 - A_1 Y_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$Y_2 = B_2 - A_2Y_0 - A_1Y_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

Thus according to Theorem 2.11

$$Q(s) = Y_1 s + Y_2 = \begin{pmatrix} s & 1 \\ 1 & 0 \end{pmatrix}$$

$$R(s) = Y_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

■

Further assuming that $B(s) = \Delta(s) = \det[A(s)] = p_0 s^{qn} + p_1 s^{qn-1} + \cdots + p_{qn}$ and applying Theorem 2.11 we have that

$$R(s) = Y_{m-n+1}s^{n-1} + Y_{m-n+2}s^{n-2} + \cdots + Y_m \equiv 0$$

or equivalently

$$Y_j = 0 \text{ for } \ j = m - n + 1,\, m - n + 2, ..., m$$

Substituting B_i for $p_i I_q$ and Y_j from relations (2.20) we have an alternative form of the generalised Cayley-Hamilton Theorem.

3. Conclusions

The division of two polynomial matrices has been studied through two different algorithms. The first algorithm was an extension of the already known algorithm of Lewis [5], while the second one was an extension of the general Bezout theorem presented in Gantmacher [4]. Some interesting applications of the presented algorithm, such as a) the relative Cayley Hamilton Theorem and *b*) the expression of the relative adjoint matrix in terms of the Tschirnhausen polynomials have also been presented.

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