

On the solution and impulsive behaviour of polynomial matrix descriptions of free linear multivariable systems

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In this note we examine the solution and the impulsive behaviour of autonomous linear multivariable systems whose pseudo-state $\beta(t)$ obeys a linear matrix differential equation $A(\rho)\beta(t) = 0$ where $A(\rho)$ is a polynomial matrix in the differential operator $\rho := d/dt$. We thus generalize to the general polynomial matrix case some results obtained by Verghese and colleagues which regard the impulsive behaviour of the generalized state vector $x(t)$ of input free *generalized state space* systems.

1. Introduction

Consider a free system whose dynamics are described by the linear homogenous matrix differential equation

$$A(\rho)\beta(t) = 0 \quad t \geq 0 \quad (1.1)$$

where

$$A(\rho) = A_q\rho^q + A_{q-1}\rho^{q-1} + \dots + A_1\rho + A_0 \in \mathbb{R}[\rho]^{r \times r} \quad (1.2)$$

is a polynomial matrix in $\rho = d/dt$, $A_i \in \mathbb{R}^{r \times r}$, $i = 0, 1, 2, \dots, q > 0$, $\text{rank}_{\mathbb{R}(\rho)} A(\rho) = r$ and $\beta(t) : [0, \infty) \rightarrow \mathbb{R}^r$ is what is known as the *pseudo-state* of the system.

In this note we firstly review the fact that if $A(s)^{-1}$ is a non-proper rational matrix then depending on the choice of the initial values $\beta(0-)$, $\beta^{(1)}(0-), \dots, \beta^{(q-1)}(0-)$, where $\beta^{(i)}(t) := (d^i \beta(t)/dt^i)$, the solution $\beta(t)$ of (1.1) might exhibit an ‘*impulsive behaviour*’ at $t = 0$ which consists of a combination of the Dirac impulse $\delta(t)$ and its $(\hat{q}_r - 1)$ th order distributional derivatives (where \hat{q}_r is the maximum order of the zero at $s = \infty$ of $A(s)$, see below). Due to the fact that $A(s)^{-1}$ is a non-proper rational matrix if and only if $A(s)$ has zeros at $s = \infty$, the impulsive behaviour of $\beta(t)$ at $t = 0$ for appropriate initial values can be seen as being associated to the *zero structure* at $s = \infty$ of $A(s)$, i.e. due to the fact that the natural modes of (1.1), defined as values of s where $A(s)$ loses rank, include also the point at $s = \infty$. Based on these facts and assuming that $A(s)$ has zeros at $s = \infty$ we then characterize the set of initial values $\beta^{(i)}(0-)$, $i = 0, 1, 2, \dots, q-1$ that are such so that $\beta(t)$ has no impulsive behaviour at $t = 0$. Furthermore we characterize the set of initial values that are such that not only $\beta(t)$ but also its derivatives $\beta^{(i)}(t)$ up to a certain order $i = 1, 2, \dots, j \leq q-1$ are continuous at $t = 0$ so that $\beta^{(i)}(0-) = \beta^{(i)}(0+)$,

$i = 0, 1, 2, \dots, q-1$. We then examine conditions that $A(s)^{-1}$ has to satisfy so that $\beta(t)$ has no impulsive behaviour at $t = 0$ for every set of initial values $\beta^{(i)}(0-)$, $i = 0, 1, 2, \dots, q-1$. A necessary and sufficient condition of the continuity of $\beta(t)$ at $t = 0$ and for every set of initial values $\beta^{(i)}(0-)$, $i = 0, 1, 2, \dots, q-1$ in terms of the coefficients in the Laurent expansion of $A(s)^{-1}$ at $s = \infty$ is given in Proposition 1. This result is then generalized by giving necessary and sufficient conditions for the continuity of $\beta(t)$ and of all its derivatives $\beta^{(i)}(t)$ up to order $j \leq q-1$ and for every set of initial values at $t = 0- : \beta^{(i)}(0-)$, $i = 0, 1, 2, \dots, q-1$. The results about the continuity of $\beta(t)$ and its derivatives at $t = 0$ presented here, are comparable to those in Geerts (1993, 1996) where the notions of consistency and weak constistency have been introduced.

We thus generalize to the general polynomial matrix case some results obtained in Verghese (1978) and Verghese *et al.* (1981) regarding the response and the impulsive behaviour of the generalized state vector $x(t) : (0-, \infty) \rightarrow \mathbb{R}^n$ of input free *generalized state space* systems, i.e. linear systems whose state vector $x(t)$ is governed by the generalized state space equation

$$E\dot{x}(t) = Ax(t) \quad t \geq 0 \quad (1.3)$$

where $E \in \mathbb{R}^{n \times n}$, $A \in \mathbb{R}^{n \times n}$ with $\text{rank}_{\mathbb{R}} E \leq n$ and which are associated with finite and infinite zero structure of the *matrix pencil* $sE - A$.

2. Background

In this section we review a number of results required in the sequel. This background comes mainly from Vardulakis (1991). In the following \mathbb{R} denotes the field of reals, $\mathbb{R}[s]$ the ring of polynomials, $\mathbb{R}(s)$ the field of rational functions and $\mathbb{R}_p(s)$ the ring of *proper* rational functions all in the indeterminate s and with coefficients in \mathbb{R} . If k is a set then $k^{p \times m}$ denotes the set of $p \times m$ matrices with elements in k . If $T(s) \in \mathbb{R}(s)^{p \times m}$, $\delta_M(T(s))$ denotes the *McMillan degree* of the $T(s)$, i.e.

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its total number of poles (finite and at $s = \infty$ and multiplicities accounted for).

Consider a polynomial matrix

$$A(s) = A_q s^q + A_{q-1} s^{q-1} + \cdots + A_0 \in \mathbb{R}[s]^{r \times r} \quad (2.1)$$

where $A_i \in \mathbb{R}^{r \times r}$, $i = 0, 1, \dots, q$, $A_q \neq 0$ with $\text{rank}_{\mathbb{R}(s)} A(s) = r$, $q \geq 1$ and let

$$S_{A(s)}^\infty = \text{diag} \left[\begin{array}{c} \xleftrightarrow[v]{v} \xleftrightarrow[r-v]{r-v} \\ s^{q_1}, s^{q_2}, \dots, s^{q_k}, I_{v-k}, \frac{1}{s^{q_{v+1}}}, \dots, \frac{1}{s^{q_r}} \end{array} \right] \quad (2.2)$$

be the Smith–McMillan form of $A(s)$ at $s = \infty$ (Vardulakis 1991) where $0 \leq k \leq v \leq r$, and

$$q_1 \geq q_2 \geq \cdots \geq q_k > 0 = q_{k+1} = \cdots = q_v \\ \hat{q}_r \geq \hat{q}_{r-1} \geq \cdots \geq \hat{q}_{v+1} > 0$$

are respectively the orders of the *poles* and the *zeros* at $s = \infty$ of $A(s)$. Then the following facts hold true:

Fact 1 (Vardulakis 1991):

$$q_1 = q \quad (2.3)$$

Fact 2: The Laurent series expansion at $s = \infty$ of the rational matrix $A(s)^{-1} \in \mathbb{R}(s)^{r \times r}$ has the form (Vardulakis 1991)

$$A(s)^{-1} = H_{q_r} s^{q_r} + H_{q_r-1} s^{q_r-1} + \cdots + H_1 s + H_0 \\ + H_{-1} s^{-1} + H_{-2} s^{-2} + \cdots \\ = H_{pol}(s) + H_{sp}(s) \quad (2.4)$$

where

$H_{pol}(s) = H_{q_r} s^{q_r} + H_{q_r-1} s^{q_r-1} + \cdots + H_1 s + H_0 \in \mathbb{R}[s]^{r \times r}$, $H_i \in \mathbb{R}^{r \times r}$, $i = 0, 1, \dots, q_r$, $H_{q_r} \neq 0$ and $H_{sp}(s) = H_{-1} s^{-1} + H_{-2} s^{-2} + \cdots \in \mathbb{R}_{pr}(s)^{r \times r}$ is strictly proper. From the fact that $A(s)^{-1} A(s) = I_r$, it is obvious that the terms H_i in (2.4) satisfy the following identities

$$H_{i-q_1} A_{q_1} + H_{i-q_1+1} A_{q_1-1} + \cdots + H_i A_0 \quad (2.5)$$

$$= A_{q_1} H_{i-q_1} + A_{q_1-1} H_{i-q_1+1} + \cdots + A_0 H_i = \delta_i I_r, \quad \forall i \quad (2.6)$$

where $\delta_i = 0$ for $i \neq 0$ and $\delta_0 = 1$ (terms H_i , with $i > q_r$ are zero).

If we consider the matrix pair $[I_r, A(s)]$ which is trivially right coprime then from the polynomial matrix (right) division of I_r by $A(s)$ (Kalaith 1980) there exist $Q(s)$, $R(s) \in \mathbb{R}[s]^{r \times r}$ such that

$$I_r = Q(s)A(s) + R(s) \quad (2.7)$$

or

$$A(s)^{-1} = Q(s) + R(s)A(s)^{-1} = H_{pol}(s) + H_{sp}(s) \quad (2.8)$$

where $H_{pol}(s) := Q(s)$ and $H_{sp}(s) := R(s)A(s)^{-1}$. Equation (2.7) can be written as

$$\begin{bmatrix} I_r \\ A(s) \end{bmatrix} = \begin{bmatrix} I_r & Q(s) \\ 0_{r,r} & I_r \end{bmatrix} \begin{bmatrix} R(s) \\ A(s) \end{bmatrix} \quad (2.9)$$

which implies that the pair $[R(s), A(s)]$ is also right coprime and thus we have

Fact 3: $\delta_M(H_{sp}(s)) = \deg |A(s)| =: n$.

Fact 4: The Smith–McMillan form of $H_{pol}(s)$ at $s = \infty$ has the form (Vardulakis 1991)

$$S_{H_{pol}(s)}^\infty = \text{diag} \left[\begin{array}{c} \xleftrightarrow[d]{d} \xleftrightarrow[r-v]{r-v} \\ s^{\hat{q}_r}, s^{\hat{q}_{r-1}}, \dots, s^{\hat{q}_{v+1}}, I_{d-(r-v)}, \frac{1}{s^{\hat{q}_{d+1}}}, \dots, \frac{1}{s^{\hat{q}_\sigma}}, 0_{r-\sigma, r-\sigma} \end{array} \right] \quad (2.10)$$

where $1 \leq (r-v) \leq d \leq \sigma = \text{rank}_{\mathbb{R}(s)} H_{pol}(s)$ and $\hat{q}_\sigma \geq \hat{q}_{\sigma-1} \geq \cdots \geq \hat{q}_{d+1} > 0$ are the orders of the zeros at $s = \infty$ of $H_{pol}(s)$, i.e. the *pole structure* at $s = \infty$ of $A(s)^{-1}$ (which is the *zero structure* at $s = \infty$ of $A(s)$) coincides with the *pole structure* at $s = \infty$ of its polynomial part $H_{pol}(s)$.

Finally let $C_\infty \in \mathbb{R}^{r \times \mu}$, $J_\infty \in \mathbb{R}^{\mu \times \mu}$, $B_\infty \in \mathbb{R}^{\mu \times r}$, be an *irreducible* at $s = \infty$ (Verghese 1978, Verghese et al. 1981) *generalized state space realization* of $H_{pol}(s)$, i.e. let $(1/w)H_{pol}(1/w) = C_\infty(wI_\mu - J_\infty)^{-1}B_\infty$ so that with $(1/w) = s$:

$$H_{pol}(s) = C_\infty(I_\mu - sJ_\infty)^{-1}B_\infty \quad (2.11)$$

where (Vardulakis 1991)

$$\mu = \delta_M \left[\frac{1}{w} H_{pol} \left(\frac{1}{w} \right) \right] = \sum_{i=v+1}^r (\hat{q}_i + 1) + [d - (r-v)] \quad (2.12)$$

$$J_\infty = \text{block diag} [0_{d-(r-v), d-(r-v)}, \tilde{J}_{\infty v+1}, \tilde{J}_{\infty v+2}, \dots, \tilde{J}_{\infty r}] \\ \in \mathbb{R}^{\mu \times \mu} \quad (2.13)$$

$$\tilde{J}_{\infty i} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix} \in \mathbb{R}^{(\hat{q}_i+1) \times (\hat{q}_i+1)} \\ i = v+1, \dots, r \quad (2.14)$$

and

$$\text{rank}_{\mathbb{R}} \begin{bmatrix} C_{\infty} \\ C_{\infty} J_{\infty} \\ \vdots \\ C_{\infty} J_{\infty}^{\mu-1} \end{bmatrix} = \text{rank}_{\mathbb{R}} [B_{\infty} \quad J_{\infty} B_{\infty} \quad \dots \quad J_{\infty}^{\mu-1} B_{\infty}] = \mu \quad (2.15)$$

Let also $C \in \mathbb{R}^{r \times n}$, $J \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times r}$ be a *minimal* state space realization of $H_{sp}(s)$ with J in Jordan normal form, i.e. let

$$H_{sp}(s) = C(sI_n - J)^{-1} B \quad (2.16)$$

Then from (2.4), (2.11) and (2.16) we have

Fact 5:

$$H_i = C_{\infty} J_{\infty}^i B_{\infty} \quad i = 0, 1, 2, \dots, \hat{q}_r \quad (2.17)$$

$$H_{-i} = C J^{i-1} B \quad i = 1, 2, \dots \quad (2.18)$$

3. Solution of linear homogeneous matrix differential equations and impulsive behaviour of their solution at $t = 0$

Consider the homogeneous matrix differential equation in (1.1) and (1.2). In this section we examine the solution of (1.1) for every possible value of $\beta(t)$ and its derivatives at $t = 0^-$ using the Laplace transform method. Let $\beta(0^-)$, $\beta^{(1)}(0^-)$, \dots , $\beta^{(q-1)}(0^-)$ be the initial values of the pseudo-state $\beta(t)$ and its derivatives up to order $q-1$ at $t = 0^-$. As it will be seen in the sequel by allowing $\beta(t)$ and its derivatives $\beta^{(i)}(t)$, $i = 1, 2, \dots$ to have arbitrary values at $t = 0^-$ we do not guarantee that $\beta(t)$ is continuous at $t = 0$, i.e. we might have that $\beta^{(i)}(0^-) \neq \beta^{(i)}(0^+)$, $i = 0, 1, 2, \dots$

Considering the L_- Laplace transform $\hat{\beta}(s)$ of $\beta(t) : \hat{\beta}(s) := L_- \beta(t) = \int_{0^-}^{\infty} \beta(t) e^{-st} dt$ and taking the L_- Laplace transform of (1.1) we obtain

$$L_- \{A(\rho)\beta(t)\} = A(s)\hat{\beta}(s) - \hat{\alpha}(s) = 0 \quad (3.1)$$

where $A(s) = A_{q_1} s^{q_1} + A_{q_1-1} s^{q_1-1} + \dots + A_0 \in \mathbb{R}[s]^{r \times r}$ and

$$\hat{\alpha}(s) = [s^{q_1-1} I_r, s^{q_1-2} I_r, \dots, s I_r, I_r] \times \begin{bmatrix} A_{q_1} & 0 & \dots & 0 & 0 \\ A_{q_1-1} & A_{q_1} & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ A_2 & A_3 & \dots & A_{q_1} & 0 \\ A_1 & A_2 & \dots & A_{q_1-1} & A_{q_1} \end{bmatrix}$$

$$\times \begin{bmatrix} \beta(0^-) \\ \beta^{(1)}(0^-) \\ \vdots \\ \beta^{(q_1-2)}(0^-) \\ \beta^{(q_1-1)}(0^-) \end{bmatrix} \mathbb{R}[s]^{r \times 1} \quad (3.2)$$

is the *initial condition* vector associated with the initial values $\beta(0^-)$, $\beta^{(1)}(0^-)$, \dots , $\beta^{(q-1)}(0^-)$. From (3.1) we obtain

$$\hat{\beta}(s) = A(s)^{-1} \hat{\alpha}(s) \in \mathbb{R}(s)^{r \times 1} \quad (3.3)$$

i.e. the L_- Laplace transform $\hat{\beta}(s)$ of $\beta(t)$ will be in general a possibly non-proper rational vector. Going back to (3.3) and using (2.4) and (3.2) for $\hat{q}_r \geq q_1$ (and with appropriate changes for $\hat{q}_r < q_1$) we obtain

$$\begin{aligned} \hat{\beta}(s) &= \left[H_{\hat{q}_r} s^{\hat{q}_r} + H_{\hat{q}_r-1} s^{\hat{q}_r-1} + \dots + H_1 s + H_0 + H_{-1} \frac{1}{s} \right. \\ &\quad \left. + \dots \right] \hat{\alpha}(s) \\ &= \begin{bmatrix} s^{\hat{q}_r} I_r, s^{\hat{q}_r-1} I_r, \dots, s I_r, I_r, \frac{1}{s} I_r, \dots \end{bmatrix} \begin{bmatrix} H_{\hat{q}_r} \\ H_{\hat{q}_r-1} \\ \vdots \\ H_0 \\ H_{-1} \\ \vdots \end{bmatrix} \hat{\alpha}(s) \\ &= \begin{bmatrix} s^{\hat{q}_r+q_1-1} I_r, s^{\hat{q}_r+q_1-2} I_r, \dots, s I_r, I_r, \frac{1}{s} I_r, \frac{1}{s^2} I_r, \dots \end{bmatrix} \\ &\quad \times \begin{bmatrix} H_{\hat{q}_r} & 0 & \dots & 0 & 0 \\ H_{\hat{q}_r-1} & H_{\hat{q}_r} & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ H_{\hat{q}_r-(q_1-1)} & H_{\hat{q}_r-(q_1-2)} & \dots & H_{\hat{q}_r-1} & H_{\hat{q}_r} \\ H_{\hat{q}_r-q_1} & H_{\hat{q}_r-(q_1-1)} & \dots & H_{\hat{q}_r-2} & H_{\hat{q}_r-1} \\ H_{\hat{q}_r-(q_1+1)} & H_{\hat{q}_r-q_1} & \dots & H_{\hat{q}_r-3} & H_{\hat{q}_r-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ H_{-(q_1-2)} & H_{-(q_1-3)} & \dots & H_0 & H_1 \\ H_{-(q_1-1)} & H_{-(q_1-2)} & \dots & H_{-1} & H_0 \\ - & - & - & - & - \\ H_{-q_1} & H_{-(q_1-1)} & \dots & H_{-2} & H_{-1} \\ H_{-(q_1+1)} & H_{-q_1} & \dots & H_{-3} & H_{-2} \\ \vdots & \vdots & & \vdots & \vdots \end{bmatrix} \end{aligned}$$

$\xrightarrow{\quad r q_1 \quad}$

$$\begin{aligned}
& \times \begin{bmatrix} A_{q_1} & 0 & \dots & 0 \\ A_{q_1-1} & A_{q_1} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_1 & A_2 & \dots & A_{q_1} \end{bmatrix} \\
& \times \begin{bmatrix} \beta(0-) \\ \beta^{(1)}(0-) \\ \vdots \\ \beta^{(q_1-1)}(0-) \end{bmatrix} \\
& = [s^{\hat{q}_r+q_1-1}I_r, s^{\hat{q}_r+q_1-2}I_r, \dots, sI_r, I_r] \\
& \times \begin{bmatrix} H_{\hat{q}_r} & 0 & \dots & 0 \\ H_{\hat{q}_r-1} & H_{\hat{q}_r} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ H_{\hat{q}_r-(q_1-1)} & H_{\hat{q}_r-(q_1-2)} & \dots & H_{\hat{q}_r} \\ - & - & - & - \\ H_{\hat{q}_r-q_1} & H_{\hat{q}_r-(q_1-1)} & \dots & H_{\hat{q}_r-1} \\ H_{\hat{q}_r-(q_1+1)} & H_{\hat{q}_r-q_1} & \dots & H_{\hat{q}_r-2} \\ \vdots & \vdots & & \vdots \\ \vdots & \vdots & & \vdots \\ H_{-(q_1-2)} & H_{-(q_1-3)} & \dots & H_1 \\ H_{-(q_1-1)} & H_{-(q_1-2)} & \dots & H_0 \end{bmatrix} \\
& \times \begin{bmatrix} A_{q_1} & 0 & \dots & 0 \\ A_{q_1-1} & A_{q_1} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_1 & A_2 & \dots & A_{q_1} \end{bmatrix} \\
& \times \begin{bmatrix} H_{\hat{q}_r} & 0 & \dots & 0 \\ H_{\hat{q}_r-1} & H_{\hat{q}_r} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ H_{\hat{q}_r-(q_1-1)} & H_{\hat{q}_r-(q_1-2)} & \dots & H_{\hat{q}_r} \\ \vdots & \vdots & & \vdots \\ H_{-(q_1-1)} & H_{-(q_1-2)} & \dots & H_0 \end{bmatrix} \\
& \times \begin{bmatrix} A_{q_1} & 0 & \dots & 0 \\ A_{q_1-1} & A_{q_1} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_1 & A_2 & \dots & A_{q_1} \end{bmatrix} \begin{bmatrix} \beta(0-) \\ \beta^{(1)}(0-) \\ \vdots \\ \beta^{(q_1-1)}(0-) \end{bmatrix} \\
& + \left[\frac{1}{s}I_r, \frac{1}{s^2}I_r, \dots \right] \begin{bmatrix} H_{-q_1} & H_{-(q_1-1)} & \dots & H_{-1} \\ H_{-(q_1+1)} & H_{-q_1} & \dots & H_{-2} \\ \vdots & \vdots & & \vdots \end{bmatrix} \\
& \times \begin{bmatrix} A_{q_1} & 0 & \dots & 0 \\ A_{q_1-1} & A_{q_1} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_1 & A_2 & \dots & A_{q_1} \end{bmatrix} \begin{bmatrix} \beta(0-) \\ \beta^{(1)}(0-) \\ \vdots \\ \beta^{(q_1-1)}(0-) \end{bmatrix} \\
& = \hat{\beta}_{pol}(s) + \hat{\beta}_{sp}(s) \tag{3.4}
\end{aligned}$$

where $\hat{\beta}_{pol}(s) \in \mathbb{R}[s]^{r \times 1}$ is the polynomial part and $\hat{\beta}_{sp}(s) \in \mathbb{R}_{sp}(s)^{r \times 1}$ is the strictly proper part of $\hat{\beta}(s)$.

Now from (2.5) we obtain the relation

$$\begin{aligned}
& \begin{bmatrix} H_{\hat{q}_r} & 0 & \dots & 0 \\ H_{\hat{q}_r-1} & H_{\hat{q}_r} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ H_{\hat{q}_r-(q_1-1)} & H_{\hat{q}_r-(q_1-2)} & \dots & H_{\hat{q}_r} \\ - & - & - & - \\ H_{\hat{q}_r-q_1} & H_{\hat{q}_r-(q_1-1)} & \dots & H_{\hat{q}_r-1} \\ H_{\hat{q}_r-(q_1+1)} & H_{\hat{q}_r-q_1} & \dots & H_{\hat{q}_r-2} \\ \vdots & \vdots & & \vdots \\ \vdots & \vdots & & \vdots \\ H_{-(q_1-2)} & H_{-(q_1-3)} & \dots & H_1 \\ H_{-(q_1-1)} & H_{-(q_1-2)} & \dots & H_0 \end{bmatrix} \\
& \times \begin{bmatrix} A_{q_1} & 0 & \dots & 0 \\ A_{q_1-1} & A_{q_1} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_1 & A_2 & \dots & A_{q_1} \end{bmatrix} \\
& = (-1) \begin{bmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 0 \\ - & - & - & - \\ H_{\hat{q}_r} & 0 & \dots & 0 \\ H_{\hat{q}_r-1} & H_{\hat{q}_r} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ H_{\hat{q}_r-(q_1-1)} & H_{\hat{q}_r-(q_1-2)} & \dots & H_{\hat{q}_r} \\ \vdots & \vdots & & \vdots \\ H_2 & H_3 & \dots & H_{q_1+1} \\ H_1 & H_2 & \dots & H_{q_1} \end{bmatrix} \\
& \times \begin{bmatrix} A_0 & A_1 & \dots & A_{q_1-1} \\ 0 & A_0 & \dots & A_{q_1-2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & A_0 \end{bmatrix} \tag{3.5}
\end{aligned}$$

so that from (3.4) we have

$$S_{A(s)}^\infty = \text{diag} \left[I_{r-k}, \frac{1}{s^{q_k}}, \frac{1}{s^{q_{k-1}}}, \dots, \frac{1}{s^{q_1}} \right] \in \mathbb{R}_{pr}(s)^{r \times r}$$

$$\Rightarrow A(s)^{-1} \in \mathbb{R}_{pr}(s)^{r \times r} \quad (3.11)$$

so that the Laurent expansion at $s = \infty$ of $A(s)^{-1}$ will have the form

$$A(s)^{-1} = H_0 + H_{-1} \frac{1}{s} + H_{-2} \frac{1}{s^2} + \dots \quad (3.12)$$

and thus (2.5) gives

$$\left[\begin{array}{ccccc} \xleftarrow{(q_1+1)r} & & & & \\ H_0 & 0 & \dots & 0 & 0 \\ H_{-1} & H_0 & \dots & 0 & 0 \\ H_{-2} & H_{-1} & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ H_{-q_1} & H_{-(q_1-1)} & \dots & H_{-1} & H_0 \\ - & - & - & - & - \\ H_{-(q_1+1)} & H_{-q_1} & \dots & H_{-2} & H_{-1} \\ H_{-(q_1+2)} & H_{-(q_1+1)} & \dots & H_{-3} & H_{-2} \\ \vdots & \vdots & & \vdots & \vdots \end{array} \right] \begin{bmatrix} A_{q_1} \\ A_{q_1-1} \\ \vdots \\ A_1 \\ A_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ I_r \\ - \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad (3.13)$$

Then (3.3) gives

$$\hat{\beta}(s) = \left[H_0 + H_{-1} \frac{1}{s} + H_{-2} \frac{1}{s^2} + \dots \right] [s^{q_1-1} I_r, \dots, s I_r, I_r]$$

$$\times \begin{bmatrix} A_{q_1} & 0 & \dots & 0 \\ A_{q_1-1} & A_{q_1} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_1 & A_2 & \dots & A_{q_1} \end{bmatrix} \begin{bmatrix} \beta(0-) \\ \beta^{(1)}(0-) \\ \vdots \\ \beta^{(q_1-1)}(0-) \end{bmatrix}$$

$$= \left[\xleftarrow{q_1 r} s^{q_1-1} I_r, s^{q_1-2} I_r, \dots, s I_r, I_r, \left| \frac{1}{s} I_r, \frac{1}{s^2} I_r, \dots \right. \right]$$

$$\times \begin{bmatrix} H_0 & 0 & \dots & 0 \\ H_{-1} & H_0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ H_{-(q_1-1)} & H_{-(q_1-2)} & \dots & H_0 \\ - & - & - & - \\ H_{-q_1} & H_{-(q_1-1)} & \dots & H_{-1} \\ H_{-(q_1+1)} & H_{-q_1} & \dots & H_{-2} \\ \vdots & \vdots & & \vdots \end{bmatrix}$$

$$\times \begin{bmatrix} A_{q_1} & 0 & \dots & 0 \\ A_{q_1-1} & A_{q_1} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_1 & A_2 & \dots & A_{q_1} \end{bmatrix} \begin{bmatrix} \beta(0-) \\ \beta^{(1)}(0-) \\ \vdots \\ \beta^{(q_1-1)}(0-) \end{bmatrix} \quad (3.14)$$

but from (3.13)

$$\begin{bmatrix} H_0 & 0 & \dots & 0 \\ H_{-1} & H_0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ H_{-(q_1-1)} & H_{-(q_1-2)} & \dots & H_0 \end{bmatrix} \begin{bmatrix} A_{q_1} & 0 & \dots & 0 \\ A_{q_1-1} & A_{q_1} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_1 & A_2 & \dots & A_{q_1} \end{bmatrix} = 0_{q_1 r \times q_1 r}$$

and so from (3.14)

$$\hat{\beta}(s) = \left[\frac{1}{s} I_r, \frac{1}{s^2} I_r, \dots \right] \begin{bmatrix} H_{-q_1} & H_{-(q_1-1)} & \dots & H_{-1} \\ H_{-(q_1+1)} & H_{-q_1} & \dots & H_{-2} \\ \vdots & \vdots & & \vdots \end{bmatrix}$$

$$\times \begin{bmatrix} A_{q_1} & 0 & \dots & 0 \\ A_{q_1-1} & A_{q_1} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_1 & A_2 & \dots & A_{q_1} \end{bmatrix} \begin{bmatrix} \beta(0-) \\ \beta^{(1)}(0-) \\ \vdots \\ \beta^{(q_1-1)}(0-) \end{bmatrix}$$

$$=: \hat{\beta}_{sp}(s) \in \mathbb{R}_{pr}(s)^{r \times 1} \quad (3.15)$$

is strictly proper for every set of initial values $\beta(0-), \beta^{(1)}(0-), \dots, \beta^{(q_1-1)}(0-)$. Conversely if $\hat{\beta}_{pol}(s) = 0$ for every set of initial values $\beta(0-), \beta^{(1)}(0-), \dots, \beta^{(q_1-1)}(0-)$ then from (3.6) it follows that we must have that

$$\begin{bmatrix} H_{\hat{q}_r} & 0 & \dots & 0 \\ H_{\hat{q}_r-1} & H_{\hat{q}_r} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ H_{\hat{q}_r-(q_1-1)} & H_{\hat{q}_r-(q_1-2)} & \dots & H_{\hat{q}_r} \\ \vdots & \vdots & \ddots & \vdots \\ H_2 & H_3 & \dots & H_{q_1+1} \\ H_1 & H_2 & \dots & H_{q_1} \end{bmatrix}$$

$$\times \begin{bmatrix} A_0 & A_1 & \dots & A_{q_1-1} \\ 0 & A_0 & \dots & A_{q_1-2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & A_0 \end{bmatrix} = 0_{\hat{q}_r, q_1 r} \quad (3.16)$$

Now (3.16) implies

$$H_{\hat{q}_r} [A_0 \ A_1 \ \dots \ A_{q_1-1}] = 0_{r, q_1 r} \quad (3.17)$$

but from (2.5) for $i = \hat{q}_r + q_1$

$$H_{\hat{q}_r} A_{q_1} = 0 \quad (3.18)$$

Combining (3.17) and (3.18) gives

$$H_{\hat{q}_r} [A_0 \ A_1 \ \dots \ A_{q_1-1} \ A_{q_1}] = 0_{r, q_1(r+1)}$$

which, since

$$\begin{aligned} \text{rank}_{\mathbb{R}(s)} A(s) = r &\Rightarrow \text{rank}_{\mathbb{R}} [A_0 \ A_1 \ \dots \ A_{q_1-1} \ A_{q_1}] \\ &= r, \end{aligned}$$

(see Exercise 4.10 in Vardoulakis 1991) implies that $H_{\hat{q}_r} = 0$. Putting $H_{\hat{q}_r} = 0$ into (3.16) and using similar arguments it can be shown successively that $H_{\hat{q}_r-1} = \dots = H_1 = 0$, i.e. that $A(s)^{-1} \in \mathbb{R}_{pr}[s]^{r \times r}$. In view of the above, the absence of impulsive behaviour from $\beta(t)$ at $t = 0$ for every set of initial values is characterized by the absence from $A(s)$ of zeros at $s = \infty$. These facts can be stated as

Theorem 1: Consider the linear homogeneous matrix differential equation

$$A(\rho)\beta(t) = 0 \quad t \geq 0$$

where $A(\rho) \in \mathbb{R}[\rho]^{r \times r}$, $\text{rank}_{\mathbb{R}(\rho)} A(\rho) = r$. Then $\beta(t) : (0-, \infty) \rightarrow \mathbb{R}^r$ does not contain impulses at $t = 0$ for every set of initial values $\beta(0-)$, $\beta^{(1)}(0-), \dots, \beta^{(q_1-1)}(0-)$, if and only if the following equivalent conditions hold true:

- (i) $\hat{\beta}(s) = A(s)^{-1} \hat{\alpha}(s) \in \mathbb{R}_{pr}^{r \times 1}(s)$ is strictly proper.
- (ii) $\hat{\beta}_{pol}(s) = 0$.
- (iii) $A(s)$ has no zeros at $s = \infty$.
- (iv) $A(s)^{-1}$ has no poles at $s = \infty \Leftrightarrow A(s)^{-1} \in \mathbb{R}_{pr}[s]^{r \times r}$.

Remark 1: Notice that absence of impulsive behaviour from $\beta(t)$ at $t = 0$ for every set of initial values does not necessarily imply continuity of $\beta(t)$ at $t = 0$, i.e. we might have $\beta(0-) \neq \beta(0+)$ (see also Liu *et al.* (1995)). A necessary and sufficient condition for the continuity of $\beta(t)$ and its derivatives up to order $q_1 - 1$ at $t = 0$ for every set of initial values $\beta(0-)$, $\beta^{(1)}(0-), \dots, \beta^{(q_1-1)}(0-)$ is given in Proposition 1 in the following section.

4. A closed formula for the solution of the homogeneous matrix differential equation $A(\rho)\beta(t) = 0$.

Conditions for the continuity of the solution

From (3.6) and (2.17) and after some algebra (see Vardoulakis 1991) we obtain

$$\hat{\beta}_{pol}(s) = C_\infty (sJ_\infty - I_\mu)^{-1} J_\infty x_f(0-) \quad (4.1)$$

where

$$\begin{aligned} x_f(0-) &:= [B_\infty \ J_\infty B_\infty \ \dots \ J_\infty^{q_1-1}] \\ &\times \begin{bmatrix} A_0 & A_1 & \dots & A_{q_1-1} \\ 0 & A_0 & \dots & A_{q_1-2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & A_0 \end{bmatrix} \\ &\times \begin{bmatrix} \beta(0-) \\ \beta^{(1)}(0-) \\ \vdots \\ \beta^{(q_1-1)}(0-) \end{bmatrix} \in \mathbb{R}^{\mu \times 1} \end{aligned} \quad (4.2)$$

also from the second part of (3.4) and (2.18) after some algebra we obtain that

$$\hat{\beta}_{sp}(s) = C(sI_n - J)^{-1} x_s(0-) \quad (4.3)$$

where

$$\begin{aligned} x_s(0-) &:= [J^{q_1-1} B \ J^{q_1-2} B \ \dots \ B] \\ &\times \begin{bmatrix} A_{q_1} & 0 & \dots & 0 \\ A_{q_1-1} & A_{q_1} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_1 & A_2 & \dots & A_{q_1} \end{bmatrix} \\ &\times \begin{bmatrix} \beta(0-) \\ \beta^{(1)}(0-) \\ \vdots \\ \beta^{(q_1-1)}(0-) \end{bmatrix} \in \mathbb{R}^{n \times 1} \end{aligned} \quad (4.4)$$

Combining (4.1), (4.3) with (3.4) we finally obtain

$$\begin{aligned} \hat{\beta}(s) &= \hat{\beta}_{sp}(s) + \hat{\beta}_{pol}(s) \\ &= [C \ C_\infty] \begin{bmatrix} sI_n - J & 0_{n, \mu} \\ 0_{\mu, n} & sJ_\infty - I_\mu \end{bmatrix}^{-1} \begin{bmatrix} x_s(0-) \\ J_\infty x_f(0-) \end{bmatrix} \end{aligned} \quad (4.5)$$

Definition 1 (Vardoulakis 1991): The vector

$$\begin{aligned} & \begin{bmatrix} x_s(0-) \\ J_\infty x_f(0-) \end{bmatrix} \\ &:= \begin{bmatrix} I_n & 0_{n,\mu} \\ 0_{\mu,n} & J_\infty \end{bmatrix} \\ & \times \begin{bmatrix} J^{q_1-1}B, J^{q_1-2}B, \dots, B & 0_{n,q_1\mu} \\ 0_{\mu,q_1n} & B_\infty, J_\infty B_\infty, \dots, J_\infty^{q_1-1} \end{bmatrix} \\ & \times \begin{bmatrix} A_{q_1} & 0 & \dots & 0 \\ A_{q_1-1} & A_{q_1} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_1 & A_2 & \dots & A_{q_1} \\ A_0 & A_1 & \dots & A_{q_1-1} \\ 0 & A_0 & \dots & A_{q_1-2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & A_0 \end{bmatrix} \begin{bmatrix} \beta(0-) \\ \beta^{(1)}(0-) \\ \vdots \\ \beta^{(q_1-1)}(0-) \end{bmatrix} \\ & \in \mathbb{R}^{(n+\mu) \times 1} \end{aligned} \quad (4.6)$$

is defined as the state at $t=0-$ of the homogeneous matrix differential equation $A(\rho)\beta(t)=0$, $t \geq 0$. $x_s(0-)$ is the slow state at $t=0-$ and $x_f(0-)$ is the fast state at $t=0-$.

Taking the inverse Laplace transform of (4.5) we have

$$\begin{aligned} \beta(t) &= L^{-1}\{\hat{\beta}(s)\} \\ &= C e^{Jt} x_s(0-) - C_\infty [\delta(t) J_\infty + \delta(t)^{(1)} J_\infty^2 + \dots \\ & \quad + \delta(t)^{(q_1-1)} J_\infty^{q_1}] x_f(0-) \end{aligned}$$

So that

$$\begin{aligned} \beta^{(i)}(t) &= C J^i e^{Jt} x_s(0-) \\ & \quad - C_\infty [\delta(t)^{(i)} J_\infty + \delta(t)^{(i+1)} J_\infty^2 + \dots \\ & \quad + \delta(t)^{(i+q_1-1)} J_\infty^{q_1}] x_f(0-) \quad i=0,1,2,\dots \end{aligned} \quad (4.7)$$

Since $\delta^{(i)}(t)=0 \forall t \neq 0$ equations (4.7) for $t=0+$, give:

$$\beta^{(i)}(0+) = C J^i x_s(0-) \quad i=0,1,2,\dots \quad (4.8)$$

Writing (4.8) for $i=0,1,\dots,q_1-1$ in matrix form we get

$$\begin{bmatrix} \beta(0+) \\ \beta^{(1)}(0+) \\ \vdots \\ \beta^{(q_1-1)}(0+) \end{bmatrix} = \begin{bmatrix} C \\ CJ \\ \vdots \\ CJ^{q_1-1} \end{bmatrix} x_s(0-) \quad (4.9)$$

Substituting $x_s(0-)$ from (4.4) into (4.9) we obtain that in general the relation between $\beta^{(i)}(0+)$ and $\beta^{(i)}(0-)$ for $i=0,1,2,\dots,q_1-1$ is given by

$$\begin{aligned} & \begin{bmatrix} \beta(0+) \\ \beta^{(1)}(0+) \\ \vdots \\ \beta^{(q_1-1)}(0+) \end{bmatrix} \\ &= \begin{bmatrix} C \\ CJ \\ \vdots \\ CJ^{q_1-1} \end{bmatrix} [J^{q_1-1}B \quad J^{q_1-2}B \quad \dots \quad B] \\ & \times \begin{bmatrix} A_{q_1} & 0 & \dots & 0 \\ A_{q_1-1} & A_{q_1} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_1 & A_2 & \dots & A_{q_1} \end{bmatrix} \begin{bmatrix} \beta(0-) \\ \beta^{(1)}(0-) \\ \vdots \\ \beta^{(q_1-1)}(0-) \end{bmatrix} \\ &= \begin{bmatrix} CJ^{q_1-1}B & CJ^{q_1-2}B & \dots & CB \\ CJ^{q_1}B & CJ^{q_1-1}B & \dots & CJB \\ \vdots & \vdots & \ddots & \vdots \\ CJ^{2q_1-2}B & CJ^{2q_1-3}B & \dots & CJ^{q_1-1}B \end{bmatrix} \\ & \times \begin{bmatrix} A_{q_1} & 0 & \dots & 0 \\ A_{q_1-1} & A_{q_1} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_1 & A_2 & \dots & A_{q_1} \end{bmatrix} \begin{bmatrix} \beta(0-) \\ \beta^{(1)}(0-) \\ \vdots \\ \beta^{(q_1-1)}(0-) \end{bmatrix} \\ &= \begin{bmatrix} H_{-q_1} & H_{-(q_1-1)} & \dots & H_{-1} \\ H_{-(q_1+1)} & H_{-q_1} & \dots & H_{-2} \\ \vdots & \vdots & \ddots & \vdots \\ H_{-(2q_1-1)} & H_{-(2q_1-2)} & \dots & H_{-q_1} \end{bmatrix} \begin{bmatrix} \beta(0-) \\ \beta^{(1)}(0-) \\ \vdots \\ \beta^{(q_1-1)}(0-) \end{bmatrix} \end{aligned}$$

$$\times \begin{bmatrix} A_{q_1} & 0 & \dots & 0 \\ A_{q_1-1} & A_{q_1} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_1 & A_2 & \dots & A_{q_1} \end{bmatrix} \begin{bmatrix} \beta(0-) \\ \beta^{(1)}(0-) \\ \vdots \\ \beta^{(q_1-1)}(0-) \end{bmatrix} \quad (4.10)$$

where we made use of equation (2.18).

Equation (4.10) indicates that if $A(s)$ has zeros at $s = \infty$ and the initial values $\beta^{(i)}(0-)$, $i = 0, 1, 2, \dots, q_1 - 1$ are chosen arbitrarily then in general $\beta^{(i)}(0+) \neq \beta^{(i)}(0-)$ for $i = 0, 1, 2, \dots, q_1 - 1$, i.e. there will be a discontinuity in $\beta(t)$ and its derivatives $\beta^{(i)}(t)$, $i = 1, 2, \dots, q_1 - 1$ at $t = 0$. These discontinuities will be described by the given $\beta^{(i)}(0-)$ and the $\beta^{(i)}(0+)$, $i = 0, 1, 2, \dots, q_1 - 1$ which are obtained from (4.10). If we demand that $\beta(t)$ and its derivatives $\beta^{(i)}(t)$, $i = 1, 2, \dots, q_1 - 1$ are all continuous at $t = 0$ so that $\beta^{(i)}(0-) = \beta^{(i)}(0+)$, $i = 0, 1, 2, \dots, q_1 - 1$ then from (4.10) we see that the given initial values $\beta^{(i)}(0-)$, $i = 0, 1, \dots, q_1 - 1$ cannot be completely arbitrary but must satisfy the relation

$$\begin{bmatrix} \beta(0-) \\ \beta^{(1)}(0-) \\ \vdots \\ \beta^{(q_1-1)}(0-) \end{bmatrix} = \begin{bmatrix} H_{-q_1} & H_{-(q_1-1)} & \dots & H_{-1} \\ H_{-(q_1+1)} & H_{-q_1} & \dots & H_{-2} \\ \vdots & \vdots & \ddots & \vdots \\ H_{-(2q_1-1)} & H_{-(q_1-2)} & \dots & H_{-q_1} \end{bmatrix} \times \begin{bmatrix} A_{q_1} & 0 & \dots & 0 \\ A_{q_1-1} & A_{q_1} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_1 & A_2 & \dots & A_{q_1} \end{bmatrix} \times \begin{bmatrix} \beta(0-) \\ \beta^{(1)}(0-) \\ \vdots \\ \beta^{(q_1-1)}(0-) \end{bmatrix} \quad (4.11)$$

or equivalently for $\beta(t)$ and its derivatives $\beta^{(i)}(t)$, $i = 1, 2, \dots, q_1 - 1$ to be continuous at $t = 0$ the given initial values at $t = -0$: $\beta^{(i)}(0-)$, $i = 0, 1, 2, \dots, q_1 - 1$ must satisfy

$$\begin{bmatrix} \beta(0-) \\ \beta^{(1)}(0-) \\ \vdots \\ \beta^{(q_1-1)}(0-) \end{bmatrix} \in \ker \left[I_{rq_1} - \begin{bmatrix} H_{-q_1} & H_{-(q_1-1)} & \dots & H_{-1} \\ H_{-(q_1+1)} & H_{-q_1} & \dots & H_{-2} \\ \vdots & \vdots & \ddots & \vdots \\ H_{-(2q_1-1)} & H_{-(q_1-2)} & \dots & H_{-q_1} \end{bmatrix} \times \begin{bmatrix} A_{q_1} & 0 & \dots & 0 \\ A_{q_1-1} & A_{q_1} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_1 & A_2 & \dots & A_{q_1} \end{bmatrix} \right] \quad (4.12)$$

or equivalently using (2.5)

$$\begin{bmatrix} \beta(0-) \\ \beta^{(1)}(0-) \\ \vdots \\ \beta^{(q_1-1)}(0-) \end{bmatrix} \in \ker \begin{bmatrix} H_0 & H_1 & \dots & H_{q_1-1} \\ H_{-1} & H_0 & & \vdots \\ \vdots & & \ddots & H_1 \\ H_{-q_1+1} & \dots & H_{-1} & H_0 \end{bmatrix} \times \begin{bmatrix} A_0 & A_1 & \dots & A_{q_1-1} \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & A_0 & A_1 \\ 0 & \dots & 0 & A_0 \end{bmatrix} \quad (4.13)$$

Remark 2: Notice that if $A(s)$ has at least one zero at $s = \infty$, i.e. if $\hat{q}_r \geq 1$ then this implies that $\text{rank}_{\mathbb{R}} A_{q_1} < r$ (Vardoulakis 1991). In such a case if the initial values at $t = 0-$: $\beta^{(i)}(0-)$, $i = 0, 1, 2, \dots, q_1 - 1$ are chosen so that (4.13) is not satisfied, i.e. if $\beta^{(i)}(0-) \neq \beta^{(i)}(0+)$, $i = 0, 1, 2, \dots, q_1 - 1$ then steps

that result from the components of $\beta(t)$ falling from their initial values in $\beta(0-)$ to the values described by $\beta(0+)$ in (4.10) for $t \geq 0$ are differentiated in accordance with the differential equation $A(\rho)\beta(t) = 0$ giving rise to impulsive behaviour in $\beta(t)$ at $t = 0$ according to equation (3.9). If on the other hand the initial values $\beta^{(i)}(0-)$ are chosen so that (4.13) is satisfied (equivalently condition (4.9) is satisfied with $\beta^{(i)}(0+) = \beta^{(i)}(0-)$, $i = 0, 1, 2, \dots, q_1 - 1$), i.e. if we demand that $\beta(t)$ and its derivatives $\beta^{(i)}(t)$, $i = 1, 2, \dots, q_1 - 1$ are continuous at $t = 0$ then the 'fast state' at $t = 0-: x_f(0-) = 0$ (see Proposition 3 below) and from (4.1) $\hat{\beta}_{pol}(s) = 0$, so that there will be no impulsive behaviour in $\beta(t)$ at $t = 0$. In this case condition (4.13) imposes certain restrictions on the choice of $\beta^{(i)}(0-)$, $i = 0, 1, 2, \dots, q_1 - 1$.

Remark 3: Notice that if $A(s)$ is monic, i.e. $\text{rank}_{\mathbb{R}} A_{q_1} = r$, then $A(s)$ will be both row and column reduced at $s = \infty$ (Kalaith 1980) and all its row degrees will be equal to $q_1 \geq 1$. Consequently $A(s)$ will have no zeros at $s = \infty$ (i.e. $q_i = q_1 > 0$, $i = 1, 2, \dots, r$) and its row degrees q_1 will be the orders of its poles at $s = \infty$, so that the Smith-McMillan form at $s = \infty$: $S_{A(s)}^\infty$ of $A(s)$ will be given by $S_{A(s)}^\infty = \text{diag}[s^{q_1}, s^{q_1}, \dots, s^{q_1}] = s^{q_1} I_r$. This in turn implies that $A(s)^{-1} \in \mathbb{R}_{pr}(s)^{r \times r}$ will be strictly proper with Smith-McMillan form at $s = \infty$: $S_{A(s)^{-1}}^\infty = [S_{A(s)}^\infty]^{-1} = (1/s^{q_1}) I_r$. So in this case the Laurent expansion of $A(s)^{-1}$ at $s = \infty$ will 'start' from the term $(1/s^{q_1})H_{-q_1}$, i.e. $H_0 = H_{-1} = H_{-2} = \dots = H_{-(q_1-1)} = 0$ and $H_{-q_1} \neq 0$, so that $A(s)^{-1} = (1/s^{q_1})H_{-q_1} + (1/s^{q_1+1})H_{-(q_1+1)} + \dots$. Due to this fact if $\text{rank}_{\mathbb{R}} A_{q_1} = r$ condition (4.11) becomes

$$\begin{bmatrix} \beta(0-) \\ \beta^{(1)}(0-) \\ \vdots \\ \beta^{(q_1-1)}(0-) \end{bmatrix} = \begin{bmatrix} H_{-q_1} & 0 & \dots & 0 \\ H_{-(q_1+1)} & H_{-q_1} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ H_{-(2q_1-1)} & H_{-(2q_1-2)} & \dots & H_{-q_1} \end{bmatrix} \times \begin{bmatrix} A_{q_1} & 0 & \dots & 0 \\ A_{q_1-1} & A_{q_1} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_1 & A_2 & \dots & A_{q_1} \end{bmatrix} \begin{bmatrix} \beta(0-) \\ \beta^{(1)}(0-) \\ \vdots \\ \beta^{(q_1-1)}(0-) \end{bmatrix} \quad (4.14)$$

which is an identity since from (2.5) we have that

$$\begin{bmatrix} H_{-q_1} & 0 & \dots & 0 \\ H_{-(q_1+1)} & H_{-q_1} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ H_{-(2q_1-1)} & H_{-(2q_1-2)} & \dots & H_{-q_1} \end{bmatrix} \times \begin{bmatrix} A_{q_1} & 0 & \dots & 0 \\ A_{q_1-1} & A_{q_1} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_1 & A_2 & \dots & A_{q_1} \end{bmatrix} = I_{q_1 r} \quad (4.15)$$

Conversely if for every $\beta^{(i)}(0-)$, $i = 0, 1, 2, \dots, q_1 - 1$, $\beta^{(i)}(0-) = \beta^{(i)}(0+)$ from (4.10) it follows that we must have

$$\begin{bmatrix} H_{-q_1} & H_{-(q_1-1)} & \dots & H_{-1} \\ H_{-(q_1+1)} & H_{-q_1} & \dots & H_{-2} \\ \vdots & \vdots & \ddots & \vdots \\ H_{-(2q_1-1)} & H_{-(2q_1-2)} & \dots & H_{-q_1} \end{bmatrix} \times \begin{bmatrix} A_{q_1} & 0 & \dots & 0 \\ A_{q_1-1} & A_{q_1} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_1 & A_2 & \dots & A_{q_1} \end{bmatrix} = I_{r q_1} \quad (4.16)$$

which implies that both block matrices in (4.16) must have full rank. From the second block lower triangular matrix in (4.16) this in turn implies that $\text{rank}_{\mathbb{R}} A_{q_1} = r$, i.e. that $A(s)$ is monic. These considerations give rise to

Proposition 1: *There is no discontinuity in $\beta(t)$ and its derivatives $\beta^{(i)}(t)$, $i = 1, 2, \dots, q_1 - 1$ at $t = 0$, i.e. $\beta^{(i)}(0-) = \beta^{(i)}(0+)$, $i = 0, 1, 2, \dots, q_1 - 1$ for every set of initial values $\beta^{(i)}(0-)$, $i = 1, 2, \dots, q_1 - 1$ iff $A(s)$ is monic, i.e. iff $\text{rank}_{\mathbb{R}} A_{q_1} = r$.*

If $A(s)^{-1} \in \mathbb{R}_{pr}(s)^{r \times r}$ and we consider only the first equation in (4.11) we obtain

$$\begin{aligned}
\beta(0+) &= C X_s(0-) = C [J^{q_1-1} B \quad J^{q_1-2} B \quad \dots \quad B] \\
&\times \begin{bmatrix} A_{q_1} & 0 & \dots & 0 \\ A_{q_1-1} & A_{q_1} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_1 & A_2 & \dots & A_{q_1} \end{bmatrix} \\
&\times \begin{bmatrix} \beta(0-) \\ \beta^{(1)}(0-) \\ \vdots \\ \beta^{(q_1-1)}(0-) \end{bmatrix} \\
&= [H_{-q_1} \quad H_{-(q_1-1)} \quad \dots \quad H_{-1}] \\
&\times \begin{bmatrix} A_{q_1} & 0 & \dots & 0 \\ A_{q_1-1} & A_{q_1} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_1 & A_2 & \dots & A_{q_1} \end{bmatrix} \begin{bmatrix} \beta(0-) \\ \beta^{(1)}(0-) \\ \vdots \\ \beta^{(q_1-1)}(0-) \end{bmatrix} \\
&\quad (4.17)
\end{aligned}$$

But from (2.5) we have

$$\begin{aligned}
&[H_{-q_1} \quad H_{-(q_1-1)} \quad \dots \quad H_{-1} \mid H_0] \\
&\times \begin{bmatrix} A_{q_1} & 0 & \dots & 0 \\ A_{q_1-1} & A_{q_1} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_1 & A_2 & \dots & A_{q_1} \\ - & - & - & - \\ A_0 & A_1 & \dots & A_{q_1-1} \end{bmatrix} = [I_r \quad 0 \quad 0 \quad \dots \quad 0] \\
&\quad (4.18)
\end{aligned}$$

which can be written as

$$\begin{aligned}
&[H_{-q_1} \quad H_{-(q_1-1)} \quad \dots \quad H_{-1}] \\
&\times \begin{bmatrix} A_{q_1} & 0 & \dots & 0 \\ A_{q_1-1} & A_{q_1} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_1 & A_2 & \dots & A_{q_1} \end{bmatrix} \\
&= [I_r \quad 0 \quad 0 \quad \dots \quad 0] - H_0 [A_0 \quad A_1 \quad \dots \quad A_{q_1-1}] \\
&\quad (4.19)
\end{aligned}$$

so that (4.17) gives

$$\begin{aligned}
\beta(0+) &= \beta(0-) - H_0 [A_0 \beta(0-) + A_1 \beta^{(1)}(0-) + \dots \\
&\quad + A_{q_1-1} \beta^{(q_1-1)}(0-)] \\
&\quad (4.20)
\end{aligned}$$

which implies that if $H_0 = 0$, i.e. if $A(s)^{-1}$ is *strictly proper* then $\beta(t)$ (but not necessarily its derivatives) is continuous at $t = 0$, i.e. $\beta(0+) = \beta(0-)$. Conversely if we require that $\beta(0+) = \beta(0-)$ for every set of initial values $\beta^{(i)}(0-)$, $i = 0, 1, 2, \dots, q_1 - 1$ then from (4.20) it follows that we must have

$$H_0 [A_0 \quad A_1 \quad \dots \quad A_{q_1-1}] = 0 \quad (4.21)$$

but from (2.5)

$$H_0 A_{q_1} = 0 \quad (4.22)$$

Now (4.21) and (4.22) can be written as

$$H_0 [A_0 \quad A_1 \quad \dots \quad A_{q_1-1} \quad A_{q_1}] = 0 \quad (4.23)$$

which again, since

$$\begin{aligned}
\text{rank}_{\mathbb{R}(s)} A(s) &= r \Rightarrow \text{rank}_{\mathbb{R}} [A_0 \quad A_1 \quad \dots \quad A_{q_1-1} \quad A_{q_1}] \\
&= r,
\end{aligned}$$

implies that $H_0 = 0$, i.e. $A(s)^{-1}$ is *strictly proper*. The above argument gives rise to the following

Proposition 2: If $A(s)^{-1} \in \mathbb{R}_{pr}(s)^{r \times r}$ then $\beta(t)$ (but not necessarily its derivatives) is continuous at $t = 0$, i.e. $\beta(0+) = \beta(0-)$ for every set of initial values at $t = 0-: \beta^{(i)}(0-)$, $i = 0, 1, 2, \dots, q_1 - 1$ iff $H_0 = 0$, i.e. iff $A(s)^{-1}$ is strictly proper (compare this with Theorem 1).

Remark 4: Similarly it can be shown that if $A(s)^{-1} \in \mathbb{R}_{pr}(s)^{r \times r}$

$$\begin{aligned}
\beta^{(1)}(0+) &= \beta^{(1)}(0-) - H_{-1} [A_0 \beta(0-) + A_1 \beta^{(1)}(0-) + \dots \\
&\quad + A_{q_1-1} \beta^{(q_1-1)}(0-)]
\end{aligned}$$

so that $\beta^{(1)}(t)$ is continuous at $t = 0$, i.e. $\beta^{(1)}(0+) = \beta^{(1)}(0-)$ for every set of initial values at $t = 0-: \beta^{(i)}(0-)$, $i = 0, 1, 2, \dots, q_1 - 1$ iff $H_{-1} = 0$. This result can be generalized by showing that continuity at $t = 0$ of all derivatives $\beta^{(j)}(t)$ of $\beta(t)$ up to order $j \leq q_1 - 1$ and for every set of initial values at $t = 0-: \beta^{(i)}(0-)$, $i = 0, 1, 2, \dots, q_1 - 1$ is guaranteed iff $H_0 = H_{-1} = H_{-2} = \dots = H_{-j} = 0$. (Note that $j \leq q_1 - 1$ because otherwise the conditions $H_0 = H_{-1} = H_{-2} = \dots = H_{-(q_1-1)} = H_{-q_1} = 0$ would imply that $A(s)$ has a pole at $s = \infty$ of order greater than q_1 .)

Finally we state

Proposition 3: Assume that the given initial values $\beta^{(i)}(0-)$, $i = 0, 1, 2, \dots, q_1 - 1$ satisfy (4.13) so that $\beta(t)$ and its derivatives $\beta^{(i)}(t)$, $i = 1, 2, \dots, q_1 - 1$ are all continuous at $t = 0$, i.e. $\beta^{(i)}(0-) = \beta^{(i)}(0+) =: \beta^{(i)}(0)$,

$i = 0, 1, 2, \dots, q_1 - 1$. Then $x_f(0-) = 0$ and the solution of (1.1) is given by $\beta(t) = C e^{At} x_s(0-)$ where $x_s(0-)$ is given by (4.4).

Proof: Vardoulakis (1991). \square

Example 1: Consider the system of differential equations

$$\begin{aligned}\dot{\beta}_1(t) + \ddot{\beta}_2(t) &= -\beta_1(t) \quad t \geq 0 \\ \dot{\beta}_2(t) &= -\beta_2(t)\end{aligned}$$

which can be written in matrix form as

$$\begin{bmatrix} \rho + 1 & \rho^3 \\ 0 & \rho + 1 \end{bmatrix} \begin{bmatrix} \beta_1(t) \\ \beta_2(t) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

or $A(\rho)\beta(t) = 0$, $\beta(t) := [\beta_1(t) \ \beta_2(t)]^T$, $r = 2$, $q = 3$ where

$$A(\rho) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \rho + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \rho^2 + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \rho^3$$

Now the Smith-McMillan form of $A(s)$ at $s = \infty$ is $S_{A(s)}^\infty = \text{diag}[s^3, 1/s]$, i.e. $A(s)$ has a pole at $s = \infty$ of order $q = q_1 = 3$ and a zero at $s = \infty$ of order $\hat{q}_2 = 1$ and thus $A(s)^{-1}$ is a non-proper rational matrix:

$$\begin{aligned}A(s)^{-1} &= \begin{bmatrix} \frac{1}{s+1} & \frac{-s^3}{(s+1)^2} \\ 0 & \frac{1}{s+1} \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{s+1} & \frac{-(3s+2)}{(s+1)^2} \\ 0 & \frac{1}{s+1} \end{bmatrix} + \begin{bmatrix} 0 & 2-s \\ 0 & 0 \end{bmatrix} \\ &= H_{sp}(s) + H_{pol}(s)\end{aligned}$$

from which we obtain that

$$H_1 = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}, \quad H_0 = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$$

$H_j = 0_{2,2}$ for $j > 1$, and by long division

$$\begin{aligned}\frac{1}{(s+1)} &= 1s^{-1} - 1s^{-2} + 1s^{-3} - \dots, \\ \frac{-(3s+2)}{(s+1)} &= -3s^{-1} + 4s^{-2} - 5s^{-3} + \dots,\end{aligned}$$

i.e.

$$\begin{aligned}H_{-1} &= \begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix}, \quad H_{-2} = \begin{bmatrix} -1 & 4 \\ 0 & -1 \end{bmatrix} \\ H_{-3} &= \begin{bmatrix} 1 & -5 \\ 0 & 5 \end{bmatrix}, \dots\end{aligned}$$

From condition (4.13) for $\beta(t)$ and its derivatives $\beta^{(i)}(t)$, $i = 1, 2$ to be continuous at $t = 0$ so that $\beta^{(i)}(0-) = \beta^{(i)}(0+)$, $i = 0, 1, 2$ the initial values at $t = 0-$, $\beta_j^{(i)}(0-)$, $j = 1, 2$, $i = 0, 1, 2$ must satisfy

$$\begin{bmatrix} \beta(0-) \\ \beta^{(1)}(0-) \\ \beta^{(2)}(0-) \end{bmatrix} \in \ker \begin{bmatrix} H_0 & H_1 & H_2 \\ H_{-1} & H_0 & H_1 \\ H_{-2} & H_{-1} & H_0 \end{bmatrix} \begin{bmatrix} A_0 & A_1 & A_2 \\ 0 & A_0 & A_1 \\ 0 & 0 & A_0 \end{bmatrix}$$

$$= \ker \begin{bmatrix} 0 & 2 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -3 & 0 & 2 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 4 & 1 & -3 & 0 & 2 \\ 0 & -1 & 0 & 1 & 0 & 0 \end{bmatrix}$$

$$\times \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \ker \begin{bmatrix} 0 & 2 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -3 & 1 & -1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ -1 & 4 & 0 & 1 & 1 & -1 \\ 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

A basis for the right kernel of the above matrix is:

$$\left\{ \begin{bmatrix} 2 \\ 1 \\ -1 \\ -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

and thus we must have that

$$\begin{bmatrix} \beta(0-) \\ \beta^{(1)}(0-) \\ \beta^{(2)}(0-) \end{bmatrix} = \begin{bmatrix} \beta_1(0-) \\ \beta_2(0-) \\ - \\ \beta_1^{(1)}(0-) \\ \beta_2^{(1)}(0-) \\ - \\ \beta_1^{(2)}(0-) \\ \beta_2^{(2)}(0-) \end{bmatrix} = \alpha \begin{bmatrix} 2 \\ 1 \\ -1 \\ -1 \\ 0 \\ 1 \end{bmatrix} + \beta \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix},$$

$$\alpha, \beta \in \mathbb{R} \quad (4.24)$$

from which we obtain that $\alpha = \beta_2(0-)$, $\beta = \beta_1(0-) - 2\beta_2(0-)$ so that from 4.24 we obtain that $\beta_j^{(i)}(0-)$, $j = 1, 2$, $i = 0, 1, 2$ must satisfy the conditions

$$\beta_1^{(1)}(0-) = -\beta_1(0-) + \beta_2(0-) \quad (4.25)$$

$$\beta_2^{(1)}(0-) = -\beta_2(0-) \quad (4.26)$$

$$\beta_1^{(2)}(0-) = \beta_1(0-) - 2\beta_2(0-) \quad (4.27)$$

$$\beta_2^{(2)}(0-) = \beta_2(0-) \quad (4.28)$$

An irreducible at $s = \infty$ generalized state space realization of the polynomial part

$$H_{pol}(s) = \begin{bmatrix} 0 & 2-s \\ 0 & 0 \end{bmatrix}$$

of $A(s)^{-1}$ is given by the triple

$$C_\infty = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad J_\infty = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad B_\infty = \begin{bmatrix} 0 & -1 \\ 0 & 2 \end{bmatrix}$$

Formula (4.2) gives

$$\begin{aligned} x_f(0-) &= [B_\infty, J_\infty B_\infty, J_\infty^2 B_\infty] \begin{bmatrix} A_0 & A_1 & A_2 \\ 0 & A_0 & A_1 \\ 0 & 0 & A_0 \end{bmatrix} \begin{bmatrix} \beta(0-) \\ \beta^{(1)}(0-) \\ \beta^{(2)}(0-) \end{bmatrix} \\ &= \begin{bmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \end{bmatrix} \\ &\quad \times \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} &\times \begin{bmatrix} \beta_1(0-) \\ \beta_2(0-) \\ - \\ \beta_1^{(0)}(0-) \\ \beta_2^{(1)}(0-) \\ - \\ \beta_1^{(2)}(0-) \\ \beta_2^{(2)}(0-) \end{bmatrix} \\ &= \begin{bmatrix} -\beta_2(0-) - \beta_2^{(1)}(0-) \\ 2\beta_2(0-) + \beta_2^{(1)}(0-) - \beta_2^{(2)}(0-) \end{bmatrix} \\ &\stackrel{4.25}{=} \stackrel{4.28}{=} \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (4.29) \end{aligned}$$

i.e. we have that $x_f(0-) = 0$ and thus from (4.1)

$$\hat{\beta}_{pol}(s) = C_\infty(sJ_\infty - I_\mu)^{-1}J_\infty x_f(0-) = 0$$

as in Proposition 3 so that $\beta_\infty(t) := L^{-1}\{\hat{\beta}_{pol}(s)\} = 0$, and there is no impulsive behaviour in $\beta(t)$ at $t = 0$.

A minimal realization C, J, B of the strictly proper part of $A(s)^{-1}$:

$$H_{sp}(s) = \begin{bmatrix} \frac{1}{s+1} & -\frac{3s+2}{(s+1)^2} \\ 0 & \frac{1}{s+1} \end{bmatrix}$$

is given by

$$C = \begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix}, \quad J = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

($n = 2$) and formula (4.4) gives

$$\begin{aligned} x_s(0-) &:= [J^2 B \quad JB \quad B] \begin{bmatrix} A_3 & 0 & 0 \\ A_2 & A_3 & 0 \\ A_1 & A_2 & A_3 \end{bmatrix} \begin{bmatrix} \beta(0-) \\ \beta^{(1)}(0-) \\ \beta^{(2)}(0-) \end{bmatrix} \\ &= \begin{bmatrix} 1 & -2 & -1 & 1 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \end{bmatrix} \\ &\quad \times \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \beta_1(0-) \\ \beta_2(0-) \\ - \\ \beta_1^{(1)}(0-) \\ \beta_2^{(1)}(0-) \\ \beta_1^{(2)}(0-) \\ \beta_2^{(2)}(0-) \end{bmatrix} \end{aligned}$$

$$\begin{aligned}
&= \begin{bmatrix} \beta_1(0-) + \beta_2(0-) - \beta_2^{(1)}(0-) + \beta_2^{(2)}(0-) \\ \beta_2(0-) \end{bmatrix} \\
&= \begin{bmatrix} x_{s1}(0-) \\ x_{s2}(0-) \end{bmatrix}
\end{aligned}$$

which due to the constraints 4.25–4.12 gives that

$$x_s(0-) = \begin{bmatrix} x_{s1}(0-) \\ x_{s2}(0-) \end{bmatrix} = \begin{bmatrix} \beta_1(0-) + 3\beta_2(0-) \\ \beta_2(0-) \end{bmatrix} = x(0)$$

and thus the solution of the d.e. is

$$\begin{aligned}
\beta(t) &= \begin{bmatrix} \beta_1(t) \\ \beta_2(t) \end{bmatrix} = \mathcal{L}^{-1}\{\hat{\beta}_{sp}(s)\} \\
&= \mathcal{L}^{-1}\{C(sI_n - J)^{-1}x_s(0-)\} \\
&= C e^{Jt}x_s(0-) \\
&= \begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} e^{-t} & t e^{-t} \\ 0 & e^{-t} \end{bmatrix} \begin{bmatrix} \beta_1(0-) + 3\beta_2(0-) \\ \beta_2(0-) \end{bmatrix} \\
&= \begin{bmatrix} e^{-t} & t e^{-t} - 3 e^{-t} \\ 0 & e^{-t} \end{bmatrix} \begin{bmatrix} \beta_1(0-) + 3\beta_2(0-) \\ \beta_2(0-) \end{bmatrix}
\end{aligned}$$

i.e.

$$\begin{aligned}
\beta_1(t) &= \beta_1(0-) e^{-t} + \beta_2(0-) t e^{-t} \quad t \geq 0 \\
\beta_2(t) &= \beta_2(0-) e^{-t}
\end{aligned}$$

which for $t=0+$ and due to conditions 4.25–4.12 gives that $\beta_1^{(j)}(0+) = \beta_1^{(j)}(0-)$, $\beta_2^{(j)}(0+) = \beta_2^{(j)}(0-)$, $j=0,1,2$, i.e. that $\beta(t)$, $\beta^{(1)}(t)$, $\beta^{(2)}(t)$ are continuous at $t=0$.

References

- GEERTS, T., 1993, Solvability conditions, consistency and weak consistency, for linear differential algebraic equations and time-invariant singular systems: the general case. *Linear Algebra and its Applications*, **181**, 111–130.
- GEERTS, T., 1996, Higher-order continuous time implicit systems: consistency and weak consistency, impulse controllability, geometric concepts and invertibility properties. *Linear Algebra and its Applications*, **244**, 203–253.
- KAILATH, T., 1980, *Linear Systems* (Englewood Cliffs, NJ: Prentice Hall).
- LIU, W. Q., YAN, W. Y., and TEO, K. L., 1995, On initial instantaneous jumps of singular systems. *IEEE Transactions on Automatic Control*, **40**, 1650–1655.
- VARDOLAKIS, A. I. G., 1991, *Linear Multivariable Control—Algebraic Analysis and Synthesis Methods* (New York: Wiley).
- VERGHESE, G., 1978, Infinite frequency behavior in dynamical systems. PhD Dissertation, Department of Electrical Engineering, Stanford University, Stanford, CA, USA.
- VERGHESE, G., LEVY, B. C., and KAILATH, T., 1981, A generalized state space for singular systems. *IEEE Transactions of Automatic Control*, **AC-26**, 811–830.