On the discretization of singular systems

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Abstract

This note proposes two new discretization methods. The proposed sampled systems are described in terms of the Markov parameters of the system and therefore the proposed methods are easily implemented. The methodology we use is a zero-order hold discretization for the input and first-order approximation of its derivatives..

1 Introduction

Consider the linear time-invariant singular system

$$E\dot{x}(t) = Ax(t) + Bu(t) \tag{1}$$

where $E, A \in R^{n\times n}, B \in R^{m\times n}, C \in R^{p\times n}$ and $D \in R^{p\times m}$. The system is assumed to be regular $\det[sE - A] \neq 0$. Systems of the above form are usually called singular systems, descriptor systems, generalized state space systems, semistate systems etc. It is easily seen that when E is non-singular then equation (1) may be rewritten in a state space form as :

$$\dot{x}(t) = E^{-1}Ax(t) + E^{-1}Bu(t) \tag{2}$$

which is the well-known state space representation. Therefore, descriptor systems constitute a more general class of linear systems than state space systems. Descriptor systems appear in the modelling of many physical phenomena, such as engineering systems (power systems, electrical networks, aerospace engineering, chemical processes), social economic systems, network analysis, biological systems, etc. An extended reference on descriptor systems may be found in [2],[3] and [4].

It is known that the zero-order hold discretized model of (2) is given by [1] :

$$x((k+1)T) = \tilde{A}x(kT) + \tilde{B}u(kT) \tag{3}$$

where

$$\begin{aligned}\tilde{A} &= e^{E^{-1}AT} \\ \tilde{B} &= \left[\int_0^T e^{E^{-1}Aw}dw\right]E^{-1}B\end{aligned}$$

In case where E is singular, we may use the forward or backward Euler method, or even the Gears method proposed in [6] in order to get a discretized singular model of (1). In this work, based a) in a recent work of [7] concerning the solution of singular systems in terms of the Laurent expansion terms of $(sE-A)^{-1}$, and b) in the same techinques applied for the derivation of a zero order hold state space system [1] and first order approximation of singular system [9], we propose two discretization methods of the singular system (1). More specifically, after some preliminary results concerning the solution and the Markov parameters of the system, we present in second and third section a state space discretization method for homogeneous and nonhomogeneous singular systems respectively. The methodology that we are using is a zero-order hold discretization for the input $u(t)$ of the system and first-order approximations for the derivatives of $u(t)$. The proposed models are described in terms of the Markov parameters of the system. Iin the fourth section, and using the same methodology we proposed in previous sections, we propose a singular system discretization instead of a state space. The whole theory has been illustrated by examples and a Mathematica programming code for the implementation of the proposed procedures.

2 Preliminary results

Consider the singular system described by (1). Its resolvent matrix can be expressed in a power series expansion of s as follows

$$\Phi(s)=\Phi_{-\mu}s^{\mu-1}+\cdots+\Phi_{-1}s^{0}+\Phi_{0}s^{-1}+\cdots+\Phi_{k}s^{-k-1}+\cdots=\sum_{k=-\mu}^{\infty}\Phi_{k}(E,A)s^{-k-1} \tag{4}$$

where μ is the index of nilpotency of the pencil $sE-A$. The matrices $\Phi_i\,(E,A)$ are uniquely defined by the relations [5] :

$$E\Phi_k = A\Phi_{k-1}, k=-\mu,...,-2,-1 \text{ with } \Phi_{-\mu-1}=0 \tag{5}$$

$$E\Phi_0 - A\Phi_{-1} = I_n \tag{6}$$

$$\Phi_k = (\Phi_0 A)^k\,\Phi_0 = \Phi_{k-1}A\Phi_0 \ (k=1,2,...) \tag{7}$$

$$\Phi_{-k} = -\Phi_{-k+1}E\Phi_{-1} = (-\Phi_{-1}E)^{k-1}\,\Phi_{-1} \ (k=2,3...,\mu) \tag{8}$$

or equivalently by the relations

$$\begin{aligned}\Phi_k E &= \Phi_{k-1} A, k=-\mu,...,-2,-1 \text{ with } \Phi_{-\mu-1}=0 &(9)\\ \Phi_0 E-\Phi_{-1} A &= I_n &(10)\\ \Phi_k &= \Phi_0 \left(A \Phi_0\right)^k=\Phi_0 A \Phi_{k-1} \ (k=1,2,...) &(11)\\ \Phi_{-k} &= -\Phi_{-1} E \Phi_{-k+1}=\Phi_{-1}\left(-E \Phi_{-1}\right)^{k-1} \ (k=2,3...,\mu) &(12)\end{aligned}$$

Based on the above relations, the following properties can be derived

$$\Phi_i E \Phi_j = \Phi_j E \Phi_i \ (\forall i, j) \tag{13}$$

$$\Phi_i E \Phi_j = \left\{\begin{array}{cc}-\Phi_{i+j} & i<0, j<0 \\ \Phi_{i+j} & i \geq 0, j \geq 0 \\ 0 & ij \leq 0 \text{ and } |i|+|j| \neq 0\end{array}\right\} \tag{14}$$

$$\Phi_i A \Phi_j = \left\{\begin{array}{cc}-\Phi_{i+j+1} & i<0, j<0 \\ \Phi_{i+j+1} & i \geq 0, j \geq 0 \\ 0 & ij \leq 0 \text{ and } |i|+|j| \neq 0\end{array}\right\} \tag{15}$$

where $|\cdot|$ is the absolute value of the argument matrix.

The solution of (1) in terms of the resolvent matrix of the singular systems is given by [7] :

$$\begin{aligned} x(t) = {} & e^{\Phi_0 A t} \Phi_0 E x(0-)+\int_0^t e^{\Phi_0 A(t-\tau)} \Phi_0 B u(\tau) d\tau+\sum_{i=0}^{\mu-1} \Phi_{-i-1} B u^{(i)}(t) \\ & +\sum_{i=0}^{\mu-1} \Phi_{-i-1}\left(\delta^{(i)}(t) E x(0-)+\sum_{j=0}^{i-1} \delta^{(i-j-1)} B u^{(j)}(0-)\right) \end{aligned} \tag{16}$$

3 State space discretization of homogeneous singular systems

It is easily seen from (16) that the smooth solution of the homogeneous system

$$E \dot{x}(t)=A x(t) \tag{17}$$

is given by

$$x(t)=e^{\Phi_0 A t} \Phi_0 E x(0-) \tag{18}$$

Based on equation (18) we can prove the following Theorem.

Theorem 1 *The repsonse of the singular system (17) at the sample times $kT, k=0,1,...$ is given by the response of the discrete time system*

$$x((k+1) T)=\tilde{A} x(kT), \quad x(0)=\Phi_0 E x(0-) \tag{19}$$

where

$$\tilde{A}=e^{\Phi_0 A T} \tag{20}$$

Proof. The solution (18) at the sample time $kT, k = 0, 1, ...$ is given by

$$x(kT) = e^{\Phi_0 AkT} \Phi_0 Ex(0-) \tag{21}$$

The same solution at the sample time $(k+1)T$ is :

$$\begin{aligned} x((k+1)T) &= e^{\Phi_0 A(k+1)T} \Phi_0 Ex(0-) = \\ &= e^{\Phi_0 AT} \left[e^{\Phi_0 AkT} \Phi_0 Ex(0-) \right] = \\ &\overset{(21)}{=} \underbrace{e^{\Phi_0 AT}}_{\tilde{A}} x(kT) \end{aligned} \tag{22}$$

We have also that for $k = 0$

$$x(0) = x(0T) = e^{\Phi_0 A0T} \Phi_0 Ex(0-) = \Phi_0 Ex(0-)$$

Note that according to [7], $\Phi_0 Ex(0-)$ is the part of the initial conditions $Ex(0-)$ that produces the smooth part of the homogeneous solution of (1). ■

Example 2 *Consider the singular system*

$$\begin{bmatrix} -\rho+38 & 12\rho+54 & 37\rho+47 \\ 2\rho-3 & 6\rho+11 & 13\rho+32 \\ -\rho+3 & 2\rho+9 & 8\rho+13 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = 0, \rho x(t) = dx(t)/dt$$

Then

$$E = \begin{bmatrix} -1 & 12 & 37 \\ 2 & 6 & 13 \\ -1 & 2 & 8 \end{bmatrix} \quad ; \quad A = \begin{bmatrix} -38 & -54 & -47 \\ 3 & -11 & -32 \\ -3 & -9 & -13 \end{bmatrix}$$

We compute the Laurent expansion of the inverse of $[sE - A]$

$$\begin{bmatrix} -s+38 & 12s+54 & 37s+47 \\ 2s-3 & 6s+11 & 13s+32 \\ -s+3 & 2s+9 & 8s+13 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{15s-22s^2+145}{520s+1040} & \frac{161s+22s^2+279}{520s+1040} & \frac{-397s+66s^2-1211}{520s+1040} \\ \frac{-5s+29s^2-135}{520s+1040} & \frac{-227s-29s^2-353}{520s+1040} & \frac{479s-87s^2+1357}{520s+1040} \\ \frac{-s-2s^2+12}{104s+208} & \frac{17s+2s^2+36}{104s+208} & \frac{-29s+6s^2-116}{104s+208} \end{bmatrix} =$$

$$= \underbrace{\begin{bmatrix} -\frac{11}{260} & \frac{11}{260} & \frac{33}{260} \\ \frac{29}{520} & -\frac{29}{520} & \frac{87}{520} \\ -\frac{1}{52} & \frac{1}{52} & \frac{3}{52} \end{bmatrix}}_{\Phi_{-2}} s + \underbrace{\begin{bmatrix} \frac{59}{520} & \frac{9}{40} & -\frac{529}{520} \\ -\frac{63}{520} & -\frac{13}{40} & \frac{653}{520} \\ \frac{3}{104} & \frac{1}{8} & -\frac{41}{104} \end{bmatrix}}_{\Phi_{-1}} +$$

$$+ \underbrace{\begin{bmatrix} \frac{27}{520} & \frac{9}{104} & -\frac{153}{520} \\ -\frac{9}{520} & -\frac{3}{104} & \frac{51}{520} \\ \frac{3}{52} & \frac{5}{52} & -\frac{17}{52} \end{bmatrix}}_{\Phi_0} \frac{1}{s} + \underbrace{\begin{bmatrix} -\frac{27}{260} & -\frac{9}{52} & \frac{153}{260} \\ \frac{9}{260} & \frac{3}{52} & -\frac{51}{260} \\ -\frac{3}{26} & -\frac{5}{26} & \frac{17}{26} \end{bmatrix}}_{\Phi_1} \frac{1}{s^2} + \cdots$$

and thus

$$\tilde{A} = e^{\Phi_0 AT} = \begin{bmatrix} \frac{27}{65}e^{-2T} + \frac{38}{65} & \frac{36}{65}e^{-2T} - \frac{36}{65} & \frac{9}{13}e^{-2T} - \frac{9}{13} \\ -\frac{9}{65}e^{-2T} + \frac{9}{65} & -\frac{12}{65}e^{-2T} + \frac{77}{65} & -\frac{3}{13}e^{-2T} + \frac{3}{13} \\ \frac{6}{13}e^{-2T} - \frac{6}{13} & \frac{8}{13}e^{-2T} - \frac{8}{13} & \frac{10}{13}e^{-2T} + \frac{3}{13} \end{bmatrix}$$

whereas the discretize model will be

$$\begin{bmatrix} x_1((k+1)T) \\ x_2((k+1)T) \\ x_3((k+1)T) \end{bmatrix} = \begin{bmatrix} \frac{27}{65}e^{-2T}+\frac{38}{65} & \frac{36}{65}e^{-2T}-\frac{36}{65} & \frac{9}{13}e^{-2T}-\frac{9}{13} \\ -\frac{9}{65}e^{-2T}+\frac{9}{65} & -\frac{12}{65}e^{-2T}+\frac{77}{65} & -\frac{3}{13}e^{-2T}+\frac{3}{13} \\ \frac{6}{13}e^{-2T}-\frac{6}{13} & \frac{8}{13}e^{-2T}-\frac{8}{13} & \frac{10}{13}e^{-2T}+\frac{3}{13} \end{bmatrix} \begin{bmatrix} x_1(kT) \\ x_2(kT) \\ x_3(kT) \end{bmatrix}$$

with solution

$$x(kT) = \tilde{A}^k x(0) =$$

$$= \begin{bmatrix} \frac{27}{65}e^{-2Tk}+\frac{38}{65} & \frac{36}{65}e^{-2Tk}-\frac{36}{65} & \frac{9}{13}e^{-2Tk}-\frac{9}{13} \\ -\frac{9}{65}e^{-2Tk}+\frac{9}{65} & -\frac{12}{65}e^{-2Tk}+\frac{77}{65} & -\frac{3}{13}e^{-2Tk}+\frac{3}{13} \\ \frac{6}{13}e^{-2Tk}-\frac{6}{13} & \frac{8}{13}e^{-2Tk}-\frac{8}{13} & \frac{10}{13}e^{-2Tk}+\frac{3}{13} \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{bmatrix}$$

We have also that

$$x(0) = \Phi_0 E x(0-) = \begin{bmatrix} \frac{27}{65} & \frac{36}{65} & \frac{9}{13} \\ -\frac{9}{65} & -\frac{12}{65} & -\frac{3}{13} \\ \frac{6}{13} & \frac{8}{13} & \frac{10}{13} \end{bmatrix} \begin{bmatrix} x_1(0-) \\ x_2(0-) \\ x_3(0-) \end{bmatrix}$$

Under the above condition the discretized solution may be rewritten as

$$x(kT) = \begin{bmatrix} \frac{27}{65}e^{-2Tk} & \frac{36}{65}e^{-2Tk} & \frac{9}{13}e^{-2Tk} \\ -\frac{9}{65}e^{-2Tk} & -\frac{12}{65}e^{-2Tk} & -\frac{3}{13}e^{-2Tk} \\ \frac{6}{13}e^{-2Tk} & \frac{8}{13}e^{-2Tk} & \frac{10}{13}e^{-2Tk} \end{bmatrix} \begin{bmatrix} x_1(0-) \\ x_2(0-) \\ x_3(0-) \end{bmatrix}$$

The smooth solution of the continuous time system is

$$x(t) = e^{\Phi_0 A t}\Phi_0 E x(0-) = \begin{bmatrix} \frac{27}{65}e^{-2t} & \frac{36}{65}e^{-2t} & \frac{9}{13}e^{-2t} \\ -\frac{9}{65}e^{-2t} & -\frac{12}{65}e^{-2t} & -\frac{3}{13}e^{-2t} \\ \frac{6}{13}e^{-2t} & \frac{8}{13}e^{-2t} & \frac{10}{13}e^{-2t} \end{bmatrix} \begin{bmatrix} x_1(0-) \\ x_2(0-) \\ x_3(0-) \end{bmatrix}$$

and thus the discretized solution coincides with the solution $x(t)$ at the sample instants $t = kT, k = 0, 1, ...$

4 State space discretization of nonhomogeneous singular systems

Consider the nonhomogeneous singular system defined in (1) with solution given by (16). The (16) may be rewritten under the relation (8) as follows :

$$\begin{aligned} x(t) \quad &= \quad e^{\Phi_0 A t}\Phi_0 E x(0-) + \int_0^t e^{\Phi_0 A(t-\tau)}\Phi_0 B u(\tau) d\tau + \sum_{i=0}^{\mu-1} \left(-\Phi_{-1}E\right)^i \Phi_{-1} B u^{(i)}(t) \\ &+ \sum_{i=0}^{\mu-1} \left(-\Phi_{-1}E\right)^i \Phi_{-1} \left(\delta^{(i)}(t) E x(0-) + \sum_{j=0}^{i-1} \delta^{(i-j-1)} B u^{(j)}(0-) \right) \end{aligned} \quad (23)$$

Theorem 3 *Using a zero order hold approximation of the input $u(t)$ and first order hold approximation of the derivatives of the input $u(t)$, the continuous time nonhomogeneous singular system (17) is discretized to yield the state space system*

$$\begin{aligned} x\left((k+1)T\right) &= \tilde{A}x\left(kT\right) + \hat{B}\left(\sigma\right)u(kT) \qquad (24)\\ x(0) &= \Phi_0 Ex(0-) + \sum_{i=0}^{\mu-1}\left(-\Phi_{-1}E\right)^i \Phi_{-1}Bu^{(i)}(0-) \end{aligned}$$

where

$$\tilde{A} = e^{\Phi_0 AT} \qquad (25)$$

$$\hat{B}_0 = \left[\int_0^T e^{\Phi_0 Aw}dw\right]\Phi_0 B + \sum_{i=1}^{\mu}(-1)^i \Phi_{-i}BT^{1-i} \qquad (26)$$

$$\hat{B}_l = \sum_{j=l}^{\mu}(-1)^{j-l}\Phi_{-j}BT^{1-j}\begin{pmatrix} j \\ j-l \end{pmatrix} \quad l = 1,2,..,\mu \qquad (27)$$

$$\hat{B}\left(\sigma\right) = \sum_{i=0}^{\mu}\hat{B}_i\sigma^i \quad \textit{with} \quad \sigma u\left(kT\right) = u\left(\left(k+1\right)T\right) \qquad (28)$$

Proof. At the sample times $kT, k = 0, 1, ...$ (where $0 \equiv 0+$) the state of (1) is given by

$$x(kT) = e^{\Phi_0 AkT}\Phi_0 Ex(0-) + \int_0^{kT} e^{\Phi_0 A(kT-\tau)}\Phi_0 Bu(\tau)d\tau + \qquad (29)$$

$$+\sum_{i=0}^{\mu-1}\left(-\Phi_{-1}E\right)^i \Phi_{-1}Bu^{(i)}(kT) \quad \text{for } k = 1,2,...$$

and

$$x(0) = \Phi_0 Ex(0-) + \sum_{i=0}^{\mu-1}\left(-\Phi_{-1}E\right)^i \Phi_{-1}Bu^{(i)}(0-) \text{ for } k = 0$$

The state at the $(k+1)$th step can be expressed in terms of the state at the kth step as follows

$$x((k+1)T) = e^{\Phi_0 A(kT+T)}\Phi_0 Ex(0-) + \int_0^{kT+T} e^{\Phi_0 A(kT+T-\tau)}\Phi_0 Bu(\tau)d\tau +$$

$$+\sum_{i=0}^{\mu-1}\Phi_{-i-1}Bu^{(i)}((k+1)T)$$

$$= e^{\Phi_0 AT}e^{\Phi_0 AkT}\Phi_0 Ex(0-) + e^{\Phi_0 AT}\int_0^{kT} e^{\Phi_0 A(kT-\tau)}\Phi_0 Bu(\tau)d\tau +$$

$$+e^{\Phi_0 AT}\int_{kT}^{(k+1)T} e^{\Phi_0 A(kT-\tau)}\Phi_0 Bu(\tau)d\tau + \sum_{i=0}^{\mu-1}\Phi_{-i-1}Bu^{(i)}((k+1)T) =$$

$$= e^{\Phi_0 AT}\left(e^{\Phi_0 AkT}\Phi_0 Ex(0-) + \int_0^{kT} e^{\Phi_0 A(kT-\tau)}\Phi_0 Bu(\tau)d\tau\right) +$$

$$+ \int_{kT}^{(k+1)T} e^{\Phi_0 A(kT+T-\tau)}\Phi_0 Bu(\tau)d\tau + \sum_{i=0}^{\mu-1}\Phi_{-i-1}Bu^{(i)}((k+1)T)$$

In order to obtain a zero-order hold discretization we assume that the input $u(t)$ is constant in the interval $[kT, kT+T)$ i.e. $u(\tau) = u(kT)\ \forall\tau \in [kT, kT+T)$ and thus :

$$\int_{kT}^{(k+1)T} e^{\Phi_0 A(kT+T-\tau)}\Phi_0 Bu(\tau)d\tau = \int_{kT}^{(k+1)T} e^{\Phi_0 A(kT+T-\tau)}\Phi_0 Bu(kT)d\tau$$

Let

$$\tilde{B}_0 = \left[\int_{kT}^{(k+1)T} e^{\Phi_0 A(kT+T-\tau)}d\tau\right]\Phi_0 B$$

and

$$w = kT + T - \tau \text{ and } dw = -d\tau$$

we have

$$\tilde{B}_0 = \left[\int_0^T e^{\Phi_0 Aw}dw\right]\Phi_0 B$$

Therefore

$$x((k+1)T) = e^{\Phi_0 AT}\left(e^{\Phi_0 AkT}\Phi_0 Ex(0-) + \int_0^{kT} e^{\Phi_0 A(kT-\tau)}\Phi_0 Bu(\tau)d\tau\right) + \tilde{B}_0 u(kT) +$$

$$+ \sum_{i=0}^{\mu-1}\Phi_{-i-1}Bu^{(i)}((k+1)T) =$$

$$= e^{\Phi_0 AT}\left(e^{\Phi_0 AkT}\Phi_0 Ex(0-) + \int_0^{kT} e^{\Phi_0 A(kT-\tau)}\Phi_0 Bu(\tau)d\tau + \sum_{i=0}^{\mu-1}\Phi_{-i-1}Bu^{(i)}(kT)\right)$$

$$+\tilde{B}_0 u(kT) - e^{\Phi_0 AT}\sum_{i=0}^{\mu-1}\Phi_{-i-1}Bu^{(i)}(kT) + \sum_{i=0}^{\mu-1}\Phi_{-i-1}Bu^{(i)}((k+1)T) =$$

$$= e^{\Phi_0 AT}x(kT) + \tilde{B}_0 u(kT)$$

$$-e^{\Phi_0 AT}\sum_{i=0}^{\mu-1}\Phi_{-i-1}Bu^{(i)}(kT) + \sum_{i=0}^{\mu-1}\Phi_{-i-1}Bu^{(i)}((k+1)T) \tag{30}$$

Now since

$$e^{\Phi_0 AT}\sum_{i=0}^{\mu-1}\Phi_{-i-1}Bu^{(i)}(kT)=$$

$$=\sum_{i=0}^{\mu-1}\left(I+\frac{\Phi_0 AT}{1!}+\frac{(\Phi_0 AT)(\Phi_0 AT)}{2!}+....\right)\Phi_{-i-1}Bu^{(i)}(kT)\overset{(15)}{=}$$

$$=\sum_{i=0}^{\mu-1}\Phi_{-i-1}Bu^{(i)}(kT)$$

we rewrite (30) as

$$x((k+1)T)=e^{\Phi_0 AT}x(kT)+\tilde{B}_0 u(kT)+$$

$$+\sum_{i=0}^{\mu-1}\Phi_{-i-1}B\left(u^{(i)}((k+1)T)-u^{(i)}(kT)\right)=$$

$$=e^{\Phi_0 AT}x(kT)+\tilde{B}_0 u(kT)+$$

$$+\sum_{i=0}^{\mu-1}\Phi_{-i-1}BT\frac{\left(u^{(i)}((k+1)T)-u^{(i)}(kT)\right)}{T}$$

Now by using first order approximations of the derivatives of u i.e.

$$u^{(i+1)}(kT)\simeq\frac{\left(u^{(i)}((k+1)T)-u^{(i)}(kT)\right)}{T}$$

we obtain that

$$x((k+1)T)=e^{\Phi_0 AT}x(kT)+\tilde{B}_0 u(kT)+\sum_{i=0}^{\mu-1}\Phi_{-i-1}BTu^{(i+1)}(kT)=$$

$$=e^{\Phi_0 AT}x(kT)+\tilde{B}_0 u(kT)+\sum_{i=1}^{\mu}\Phi_{-i}BTu^{(i)}(kT)\qquad(31)$$

The use of the approximation

$$\begin{aligned}
u^{(1)}(kT) &\simeq \frac{1}{T}\left(u((k+1)\,T)-u(kT)\right)\\
u^{(2)}(kT) &\simeq \frac{1}{T^2}\left(u\left((k+2)\,T\right)-2u\left((k+1)\,T\right)+u\left(kT\right)\right)\\
&\quad .....\\
u^{(i)}\left(kT\right) &= \frac{1}{T^i}\sum_{\ell=0}^{i}(-1)^{\ell}\begin{pmatrix} i\\ \ell\end{pmatrix}u\left((k+i-\ell)\,T\right)
\end{aligned}\qquad(32)$$

in equation (31) yields

$$x((k+1)T) = e^{\Phi_0 AT}x(kT) + \tilde{B}_0 u(kT) +$$
$$+\sum_{i=1}^{\mu}\Phi_{-i}BT\left(\frac{1}{T^i}\sum_{\ell=0}^{i}(-1)^{\ell}\left(\begin{array}{c} i \\ \ell \end{array}\right)u\left((k+i-\ell)\,T\right)\right) =$$
$$= e^{\Phi_0 AT}x(kT) + \tilde{B}_0 u(kT) +$$
$$+\left(\sum_{i=1}^{\mu}(-1)^{i}\,\Phi_{-i}BT^{1-i}\right)u\left(kT\right) + \left(\sum_{i=2}^{\mu}(-1)^{i-1}\,\Phi_{-i}BT^{1-i}\left(\begin{array}{c} i \\ i-1 \end{array}\right)\right)u\left((k+1)\,T\right)$$
$$+\cdots+\left(\Phi_{-\mu}BT^{1-\mu}\right)u\left((k+\mu)\,T\right)$$

or in more compact form

$$\begin{aligned} x((k+1)T) \quad &= \quad e^{\Phi_0 AT}x(kT) + \left(\tilde{B}_0 + \sum_{i=1}^{\mu}(-1)^{i}\,\Phi_{-i}BT^{1-i}\right)u(kT) + \\ &\quad + \sum_{\ell=2}^{\mu}\left(\sum_{j=\ell}^{\mu}(-1)^{j-\ell}\,\Phi_{-j}BT^{1-j}\left(\begin{array}{c} j \\ j-\ell \end{array}\right)\right)u\left((k+l)\,T\right) \end{aligned}$$

Thus the approximated discrete time model will be the ones given by relation (24-25-28). ■

Example 4 *Consider the singular system*

$$\left[\begin{array}{ccc} -\rho+38 & 12\rho+54 & 37\rho+47 \\ 2\rho-3 & 6\rho+11 & 13\rho+32 \\ -\rho+3 & 2\rho+9 & 8\rho+13 \end{array}\right]\left[\begin{array}{c} x_1(t) \\ x_2(t) \\ x_3(t) \end{array}\right] = \left[\begin{array}{c} 0 \\ 0 \\ 1 \end{array}\right]u(t), \rho x(t) = dx(t)/dt$$

Then

$$E = \left[\begin{array}{ccc} -1 & 12 & 37 \\ 2 & 6 & 13 \\ -1 & 2 & 8 \end{array}\right]; A = \left[\begin{array}{ccc} -38 & -54 & -47 \\ 3 & -11 & -32 \\ -3 & -9 & -13 \end{array}\right], B = \left[\begin{array}{c} 0 \\ 0 \\ 1 \end{array}\right]$$

From Example 2 we have that

$$\tilde{A} = e^{\Phi_0 AT} = \left[\begin{array}{ccc} \frac{27}{65}e^{-2T}+\frac{38}{65} & \frac{36}{65}e^{-2T}-\frac{36}{65} & \frac{9}{13}e^{-2T}-\frac{9}{13} \\ -\frac{9}{65}e^{-2T}+\frac{9}{65} & -\frac{12}{65}e^{-2T}+\frac{77}{65} & -\frac{3}{13}e^{-2T}+\frac{3}{13} \\ \frac{6}{13}e^{-2T}-\frac{6}{13} & \frac{8}{13}e^{-2T}-\frac{8}{13} & \frac{10}{13}e^{-2T}+\frac{3}{13} \end{array}\right]$$

We have also that

$$\tilde{B}_0 = \left[\int_0^T e^{\Phi_0 Aw}dw\right]\Phi_0 B = \left[\begin{array}{c} \frac{153}{1040}e^{-2T}-\frac{153}{1040} \\ -\frac{51}{1040}e^{-2T}+\frac{51}{1040} \\ \frac{17}{104}e^{-2T}-\frac{17}{104} \end{array}\right]$$

$$\hat{B}_0 = \tilde{B}_0 + \sum_{i=1}^{\mu} (-1)^i \, \Phi_{-i} B T^{1-i} = \tilde{B}_0 - \Phi_{-1} B + \Phi_{-2} B T^{-1} =$$

$$= \begin{bmatrix} \frac{33}{260T} + \frac{153}{1040e^{2T}} + \frac{181}{208} \\ -\frac{87}{520T} - \frac{51}{1040e^{2T}} - \frac{251}{208} \\ \frac{3}{52T} + \frac{17}{104e^{2T}} + \frac{3}{13} \end{bmatrix}$$

$$\tilde{B}_1 = \sum_{j=1}^{\mu} (-1)^{j-1} \, \Phi_{-j} B T^{1-j} \begin{pmatrix} j \\ j-1 \end{pmatrix} = \Phi_{-1} B - 2\Phi_{-2} B T^{-1} =$$

$$= \begin{bmatrix} -\frac{33}{130T} - \frac{529}{520} \\ \frac{87}{260T} + \frac{653}{520} \\ -\frac{3}{26T} - \frac{41}{104} \end{bmatrix}$$

$$\tilde{B}_2 = \sum_{j=2}^{\mu} (-1)^{j-2} \, \Phi_{-j} B T^{1-j} \begin{pmatrix} j \\ j-2 \end{pmatrix} = \Phi_{-2} B T^{-1} = \begin{bmatrix} \frac{33}{260T} \\ -\frac{87}{520T} \\ \frac{3}{52T} \end{bmatrix}$$

and the discretized model will be

$$\begin{bmatrix} x_1((k+1)\,T) \\ x_2((k+1)\,T) \\ x_3((k+1)\,T) \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{27}{65}e^{-2T} + \frac{38}{65} & \frac{36}{65}e^{-2T} - \frac{36}{65} & \frac{9}{13}e^{-2T} - \frac{9}{13} \\ -\frac{9}{65}e^{-2T} + \frac{9}{65} & -\frac{12}{65}e^{-2T} + \frac{77}{65} & -\frac{3}{13}e^{-2T} + \frac{3}{13} \\ \frac{6}{13}e^{-2T} - \frac{6}{13} & \frac{8}{13}e^{-2T} - \frac{8}{13} & \frac{10}{13}e^{-2T} + \frac{3}{13} \end{bmatrix}}_{\tilde{A}} \begin{bmatrix} x_1(kT) \\ x_2(kT) \\ x_3(kT) \end{bmatrix} +$$

$$+ \underbrace{\begin{bmatrix} \frac{33}{260T} + \frac{153}{1040e^{2T}} + \frac{181}{208} \\ -\frac{87}{520T} - \frac{51}{1040e^{2T}} - \frac{251}{208} \\ \frac{3}{52T} + \frac{17}{104e^{2T}} + \frac{3}{13} \end{bmatrix}}_{\hat{B}_0} u\,(kT) + \underbrace{\begin{bmatrix} -\frac{33}{130T} - \frac{529}{520} \\ \frac{87}{260T} + \frac{653}{520} \\ -\frac{3}{26T} - \frac{41}{104} \end{bmatrix}}_{\hat{B}_1} u\,((k+1)\,T) + \underbrace{\begin{bmatrix} \frac{33}{260T} \\ -\frac{87}{520T} \\ \frac{3}{52T} \end{bmatrix}}_{\hat{B}_2} u((k+2)\,T)$$

Note that

$$\left(zI_3 - \tilde{A}\right)^{-1} \tilde{B}(z) = \begin{bmatrix} \frac{153e^{-2T} - 153}{1040z - 1040e^{-2T}} \\ \frac{-51e^{-2T} + 51}{1040z - 1040e^{-2T}} \\ \frac{17e^{-2T} - 17}{104z - 104e^{-2T}} \end{bmatrix} + \begin{bmatrix} \frac{1}{520}T^{-1}\,(66z - 529T - 66) \\ (-\frac{1}{520})\,T^{-1}\,(87z - 653T - 87) \\ \frac{1}{104}T^{-1}\,(6z - 41T - 6) \end{bmatrix} =$$

$$= \left(1 - z^{-1}\right) Z \left\{ \frac{H_{spr}\,(s)}{s} \right\} + H_{pol}\left(\frac{z-1}{T} \right)$$

where

$$H(s) = \begin{bmatrix} -s+38 & 12s+54 & 37s+47 \\ 2s-3 & 6s+11 & 13s+32 \\ -s+3 & 2s+9 & 8s+13 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \underbrace{\begin{bmatrix} -\frac{153}{520(s+2)} \\ \frac{51}{520(s+2)} \\ -\frac{17}{52(s+2)} \end{bmatrix}}_{H_{spr}(s)} + \underbrace{\begin{bmatrix} \frac{33}{260}s - \frac{529}{520} \\ -\frac{87}{520}s + \frac{653}{520} \\ \frac{3}{52}s - \frac{41}{104} \end{bmatrix}}_{H_{pol}(s)}$$

5 Singular system discretization of nonhomogeneous singular systems

Consider the continuous time singular system

$$\underbrace{\begin{bmatrix} \rho I_n - \Phi_0 A & 0 \\ 0 & I_n + \rho\Phi_{-1}E \end{bmatrix}}_{\rho\tilde{E}-\tilde{A}} \underbrace{\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}}_{\tilde{x}(t)} = \underbrace{\begin{bmatrix} \Phi_0 B \\ \Phi_{-1}B \end{bmatrix}}_{\tilde{B}} u(t) \tag{33}$$

Theorem 5 *There is a bijective map between the solution spaces-initial conditions of the systems (1) and (33).*

Proof. Since the matrix $\begin{bmatrix} \Phi_0^T & \Phi_{-1}^T \end{bmatrix}^T$ has full column rank [7], we define the following bijective map between the initial conditions of the two systems ($Ex(0-)$ and $\tilde{E}\tilde{x}(0-)$ respectively)

$$\tilde{E}\tilde{x}(0-) = \begin{bmatrix} I_n & 0 \\ 0 & \Phi_{-1}E \end{bmatrix} \begin{bmatrix} x_1(0-) \\ x_2(0-) \end{bmatrix} := \begin{bmatrix} \Phi_0 \\ \Phi_{-1} \end{bmatrix} Ex(0-) \tag{34}$$

Then by taking Laplace transforms in (33) we get

$$\begin{aligned}
\begin{bmatrix} \tilde{x}_1(s) \\ \tilde{x}_2(s) \end{bmatrix} &= \begin{bmatrix} sI_n - \Phi_0 A & 0 \\ 0 & I_n + s\Phi_{-1}E \end{bmatrix}^{-1} \begin{bmatrix} I_n & 0 \\ 0 & \Phi_{-1}E \end{bmatrix} \begin{bmatrix} x_1(0-) \\ x_2(0-) \end{bmatrix} + \\
&+ \begin{bmatrix} sI_n - \Phi_0 A & 0 \\ 0 & I_n + s\Phi_{-1}E \end{bmatrix}^{-1} \begin{bmatrix} \Phi_0 B \\ \Phi_{-1}B \end{bmatrix} \tilde{u}(s) \overset{(34)}{=} \\
&= \begin{bmatrix} (sI_n - \Phi_0 A)^{-1} [\Phi_0 Ex(0-) + \Phi_0 B\tilde{u}(s)] \\ (I_\mu + s\Phi_{-1}E)^{-1} [\Phi_{-1}Ex(0-) + \Phi_{-1}B\tilde{u}(s)] \end{bmatrix} = \\
= &\begin{bmatrix} \left(I_n s^{-1} + (\Phi_0 A) s^{-2} + (\Phi_0 A)^2 s^{-2} + \cdots\right) [\Phi_0 Ex(0-) + \Phi_0 B\tilde{u}(s)] \\ \left(I_n + (-\Phi_{-1}E) s + (-\Phi_{-1}E)^2 s^2 + \cdots + (-\Phi_{-1}E)^\mu s^{\mu-1}\right) [\Phi_{-1}Ex(0-) + \Phi_{-1}B\tilde{u}(s)] \end{bmatrix} \overset{(7)}{\underset{(8)}{=}} \\
&= \begin{bmatrix} \left(\Phi_0 s^{-1} + \Phi_1 s^{-2} + \Phi_2 s^{-2} + \cdots\right) [Ex(0-) + B\tilde{u}(s)] \\ \left(\Phi_{-1} + \Phi_{-2}s + \cdots + \Phi_{-\mu}s^{\mu-1}\right) [Ex(0-) + B\tilde{u}(s)] \end{bmatrix}
\end{aligned} \tag{35}$$

where $\tilde{x}_1(s), \tilde{x}_2(s)$ and $\tilde{u}(s)$ are the Laplace transforms of $x_1(t), x_2(t)$ and $u(t)$ respectively. Now by taking Laplace transforms in (1) we get

$$\begin{aligned}
x(s) = (sE - A)^{-1} Ex(0-) + (sE - A)^{-1} B\tilde{u}(s) = \\
\overset{(4)}{=} \left(\Phi_{-\mu}s^{\mu-1} + \cdots + \Phi_{-1}s^0 + \Phi_0 s^{-1} + \cdots + \Phi_k s^{-k-1} + \cdots\right) [Ex(0-) + B\tilde{u}(s)]
\end{aligned} \tag{36}$$

Now by comparing relations (35) and (36) we conclude that any solution $x(t)$ is related with the solution $\tilde{x}(t)$ under the surjective map

$$x(t) = \underbrace{\begin{bmatrix} I_n & I_n \end{bmatrix}}_{\tilde{C}} \tilde{x}(t) \tag{37}$$

The above map is also an injective map since the compound matrix

$$\begin{bmatrix} s\tilde{E}-\tilde{A} \\ -\tilde{C} \end{bmatrix} = \begin{bmatrix} sI_n - \Phi_0 A & 0 \\ 0 & I_\mu + s\Phi_{-1}E \\ I_n & I_n \end{bmatrix}$$

has no finite decouling zeros [8] i.e. the determinant defined by the second and third row blocks is constant since the matrix $I_n + s\Phi_{-1}E$ is unimodular. ■

It seems quite natural from the above Theorem, that in order to discretize the singular model (1) we may discretize the model (33), as we describe in the following Theorem.

Theorem 6 *Using a zero order hold approximation of the input $u(t)$ and first order hold approximation of the derivatives of the input $u(t)$, the continuous time nonhomogeneous singular system (17) is discretized to yield the singular state space system*

$$\left\{ \begin{array}{c} x_1\left((k+1)T\right) = \tilde{A}x_1\left(kT\right) + \tilde{B}_1 u\left(kT\right) \\ \tilde{E}_1 x_2\left((k+1)T\right) = x_2\left(kT\right) + \tilde{B}_2 u\left(kT\right) \end{array} \right\} \tag{38}$$

$$x\left(kT\right) = \begin{bmatrix} I_n & I_n \end{bmatrix} \begin{bmatrix} x_1\left(kT\right) \\ x_2\left(kT\right) \end{bmatrix}$$

where

$$\tilde{A} = e^{\Phi_0 AT} \quad ; \quad \tilde{B}_1 = \int_0^T e^{\Phi_0 A\tau} d\tau \left(\Phi_0 B\right)$$

$$\tilde{E}_1 = \left(\Phi_{-1}E - T \times I_n\right)^{-1} \Phi_{-1}E \quad ; \quad \tilde{B}_2 = T\left(\Phi_{-1}E - T\times I_n\right)^{-1}\Phi_{-1}B$$

Proof. By applying a zero-order hold discretization on the first subsystem of (33)

$$\dot{x}_1\left(t\right) = \left[\Phi_0 A\right] x_1\left(t\right) + \left[\Phi_0 B\right] u\left(t\right)$$

we get

$$x_1\left((k+1)T\right) = \tilde{A}x_1\left(kT\right) + \tilde{B}_1 u\left(kT\right)$$

$$\tilde{A} = e^{\Phi_0 AT} \quad ; \quad \tilde{B}_1 = \int_0^T e^{\Phi_0 A\tau} d\tau \left(\Phi_0 B\right)$$

Consider now the second subsystem of (33)

$$\left[\Phi_{-1}E\right]\dot{x}_2\left(t\right) = -x_2\left(t\right) + \left[\Phi_{-1}B\right] u\left(t\right)$$

The smooth solution of the above system is

$$x_2\left(t\right) = \Phi_{-1}u\left(t\right) + \Phi_{-2}u^{(1)}\left(t\right) + \cdots + \Phi_{-\mu}u^{(\mu-1)}\left(t\right)$$

In case where we aplly a zero-order hold discretization on the above system the input derivatives will be lost. In order to avoid such kind of problems we are applying a first-order approximations of the derivatives of $u(t)$ i.e. (see (32)). However, according to [9], this kind of discretization lead to the same discrete time system with the ones that we get if we apply the known Euler approximation

$$\dot{x}_2(t) = \frac{x_2((k+1)T) - x_2(kT)}{T}$$

that gives rise to the discrete time system

$$\tilde{E}_1 x_2((k+1)T) = x_2(kT) + \tilde{B}_2 u(kT)$$
$$\tilde{E}_1 = (\Phi_{-1}E - T \times I_n)^{-1} \Phi_{-1}E \quad ; \quad \tilde{B}_2 = T(\Phi_{-1}E - T \times I_n)^{-1} \Phi_{-1}B$$

Finally the solution of (1) is given by the discretized map between the solutions of the two systems

$$x(kT) = \begin{bmatrix} I_n & I_n \end{bmatrix} \begin{bmatrix} x_1(kT) \\ x_2(kT) \end{bmatrix}$$

and the bijective map between the initial conditions (34). ■

In order to preserve stability under the above discretization we have to consider the relation between the zeros of the matrix $sI_n - \Phi_0 A$ and the zeros of the pencil $sE - A$. The following Theorem establish the connection between these two sets of zeros.

Theorem 7 *Let λ_i, μ_i and ν_i be the zeros of the matrix pencils $sI_n - \Phi_0 A$, $sE - A$ and $\begin{bmatrix} sI_n & \Phi_0 \end{bmatrix}$ i.e. $rank_R[\lambda_i I_n - \Phi_0 A] \leq n$, $rank_R[\mu_i E - A] \leq n$ and $rank_R \begin{bmatrix} \nu_i I_n & \Phi_0 \end{bmatrix} \leq n$. Then*

$$\{\lambda_i\} = \{\mu_i\} + \{\nu_i\}$$

where $\{\cdot\}$ denotes the set of the specific zeros.

Proof. Note that the following realization of $(sE - A)^{-1}$

$$(sE - A)^{-1} = \underbrace{\begin{bmatrix} I_n & I_n \end{bmatrix}}_{\tilde{C}} \underbrace{\begin{bmatrix} sI_n - \Phi_0 A & 0 \\ 0 & I_n + s\Phi_{-1}E \end{bmatrix}^{-1}}_{s\tilde{E} - \tilde{A}} \underbrace{\begin{bmatrix} \Phi_0 \\ \Phi_{-1} \end{bmatrix}}_{\tilde{B}}$$

is observable in the finite sense, since the compound matrix $\begin{bmatrix} \tilde{C}^T & \left(s\tilde{E} - \tilde{A}\right)^T \end{bmatrix}^T$ has no finite zeros (see proof of the previous Theorem). According to [10] and since the above realization is observable we have that

$$\left\{\text{finite poles of } (sE-A)^{-1}\right\} = \left\{\text{finite zeros of } \left(s\tilde{E}-\tilde{A}\right)\right\} +$$
$$+\left\{\text{input decoupling zeros of } \left[\begin{array}{cc} s\tilde{E}-\tilde{A} & \tilde{B}\end{array}\right]\right\} \Leftrightarrow$$
$$\{\text{finite zeros of } (sE-A)\} = \{\text{finite zeros of } (sI_n-\Phi_0 A)\} +$$
$$+\left\{\text{input decoupling zeros of } \left[\begin{array}{cc} sI_n-\Phi_0 A & \Phi_0\end{array}\right]\right\} \Leftrightarrow$$
$$\{\text{finite zeros of } (sE-A)\} = \{\text{finite zeros of } (sI_n-\Phi_0 A)\} +$$
$$+\left\{\text{input decoupling zeros of } \left[\begin{array}{cc} sI_n & \Phi_0\end{array}\right]\right\} \Leftrightarrow$$
$$\{\lambda_i\} = \{\mu_i\} + \{\nu_i\}$$

In case where Φ_0 is nonsingular then $\{\nu_i\} = \varnothing$ i.e. E is nonsingular, and $\{\lambda_i\} = \{\mu_i\}$. ■

Corollary 8 *If the system (1) is marginally stable then the system (33) is marginally stable (or stable if Φ_0 is nonsingular).*

Proof. Since the matrix sI_n contains only finite zeros at $s=0$ of order 1, then the only input decoupling zeros of $\left[\begin{array}{cc} sI_n & \Phi_0\end{array}\right]$ that may added in the matrix $sE-A$ are finite zeros at $s=0$ of order 1 which are not capable to change the system stability. ■

Corollary 9 *If the system (1) is marginally stable then the sampled systems (24) and (38) are marginally stable (or stable if Φ_0 is nonsingular).*

Proof. Let $\{\mu_i, \mu_i \leq 0\}$ be the zeros of the marginally stable matrix pencil $sE-A$. Then the matrix pencil $sI_n - \Phi_0 A$ have as zeros the set $\{\mu_i\} + \{\nu_i = 0, \text{of order } 1\}$ and therefore the matrix $\tilde{A} = e^{\Phi_0 AT}$ will have as eigenvalues the set $\left\{e^{\mu_i T}, e^{\mu_i T} \leq 1\right\} + \left\{e^{0T} = 1 \equiv 1, \text{of order } 1\right\}$ that belongs into the unit circle. ■

A connection exists between the transfer function matrices of the systems (1) and (33) as can be easily seen in the following Corollary.

Corollary 10 *If the transfer function matrix of the system (1) is*

$$H(s) = H_{spr}(s) + H_{pol}(s)$$

where $H_{spr}(s)$ and $H_{pol}(s)$ denotes the strictly proper and polynomial part respectively, then the transfer function matrix of the discretized system in (38) is given by

$$\tilde{H}(z) = \left(1-z^{-1}\right) Z\left\{\frac{H_{spr}(s)}{s}\right\} + H_{pol}\left(\frac{z-1}{T}\right)$$

where $Z\{x(s)\}$ denotes the Z-transform of the function $x(s)$.

Proof. Fisrt note from the proof of Theorem 5, that the systems (1) and (33) have the same transfer function. Since we apply a zero-order hold discretization in the first subsystem of (33) we have that $\left(zI_3 - \tilde{A}\right)^{-1} \tilde{B}_1 = \left(1 - z^{-1}\right) Z\left\{\frac{H_{spr}(s)}{s}\right\}$. Similarly we apply a first-order approximation in the second subsystem of (33) and thus $\left(z\tilde{E}_1 + I_3\right)^{-1} \tilde{B}_2 = H_{pol}\left(\frac{z-1}{T}\right)$. Therefore,

$$\tilde{H}(z) = \left(zI_3 - \tilde{A}\right)^{-1} \tilde{B}_1 + \left(z\tilde{E}_1 + I_3\right)^{-1} \tilde{B}_2 = \left(1 - z^{-1}\right) Z\left\{\frac{H_{spr}(s)}{s}\right\} + H_{pol}\left(\frac{z-1}{T}\right)$$

which verifies the corollary. ■

We have presented until now two different discretized models of (1). The first one, has been presented in Section 3 and is in state space form while the second one has been presented in this section and is in singular form. We can easily notice that the state space system presented in Section 3 is coming from the singular system presented in this section under the replacement of the solution of the second subsystem in (38) $x_2(kT)$ to $x(kT)$ in the same sampled system. Therefore all the corollaries presented in this section are also hold for the sampled system presented in Section 3.

Example 11 *Consider the singular system of the Example 4*

$$\begin{bmatrix} -\rho + 38 & 12\rho + 54 & 37\rho + 47 \\ 2\rho - 3 & 6\rho + 11 & 13\rho + 32 \\ -\rho + 3 & 2\rho + 9 & 8\rho + 13 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t), \rho x(t) = dx(t)/dt$$

Then

$$\tilde{A} = e^{\Phi_0 A T} = \begin{bmatrix} \frac{27}{65}e^{-2T} + \frac{38}{65} & \frac{36}{65}e^{-2T} - \frac{36}{65} & \frac{9}{13}e^{-2T} - \frac{9}{13} \\ -\frac{9}{65}e^{-2T} + \frac{9}{65} & -\frac{12}{65}e^{-2T} + \frac{77}{65} & -\frac{3}{13}e^{-2T} + \frac{3}{13} \\ \frac{6}{13}e^{-2T} - \frac{6}{13} & \frac{8}{13}e^{-2T} - \frac{8}{13} & \frac{10}{13}e^{-2T} + \frac{3}{13} \end{bmatrix}$$

$$\tilde{B}_1 = \int_0^T e^{\Phi_0 A \tau} d\tau \left(\Phi_0 B\right) = \begin{bmatrix} \frac{153}{1040}e^{-2T} - \frac{153}{1040} \\ -\frac{51}{1040}e^{-2T} + \frac{51}{1040} \\ \frac{17}{104}e^{-2T} - \frac{17}{104} \end{bmatrix}$$

$$\tilde{E}_1 = \left(\Phi_{-1}E - T \times I_n\right)^{-1} \Phi_{-1}E = \begin{bmatrix} \left(-\frac{88}{65}\right)T^{-1} & \left(-\frac{44}{65}\right)T^{-1} & \frac{66}{65}T^{-1} \\ \frac{116}{65}T^{-1} & \frac{58}{65}T^{-1} & \left(-\frac{87}{65}\right)T^{-1} \\ \left(-\frac{8}{13}\right)T^{-1} & \left(-\frac{4}{13}\right)T^{-1} & \frac{6}{13}T^{-1} \end{bmatrix}$$

$$\tilde{B}_2 = T\left(\Phi_{-1}E - T \times I_n\right)^{-1} \Phi_{-1}B = \begin{bmatrix} \frac{1}{520}T^{-1}\left(529T + 66\right) \\ \left(-\frac{1}{520}\right)T^{-1}\left(653T + 87\right) \\ \frac{1}{104}T^{-1}\left(41T + 6\right) \end{bmatrix}$$

The sampled system is given by

$$x_1((k+1)T) = \begin{bmatrix} \frac{27}{65}e^{-2T}+\frac{38}{65} & \frac{36}{65}e^{-2T}-\frac{36}{65} & \frac{9}{13}e^{-2T}-\frac{9}{13} \\ -\frac{9}{65}e^{-2T}+\frac{9}{65} & -\frac{12}{65}e^{-2T}+\frac{77}{65} & -\frac{3}{13}e^{-2T}+\frac{3}{13} \\ \frac{6}{13}e^{-2T}-\frac{6}{13} & \frac{8}{13}e^{-2T}-\frac{8}{13} & \frac{10}{13}e^{-2T}+\frac{3}{13} \end{bmatrix} x_1(kT) + \\ + \begin{bmatrix} \frac{153}{1040}e^{-2T}-\frac{153}{1040} \\ -\frac{51}{1040}e^{-2T}+\frac{51}{1040} \\ \frac{17}{104}e^{-2T}-\frac{17}{104} \end{bmatrix} u(kT)$$

$$\begin{bmatrix} \left(-\frac{88}{65}\right)T^{-1} & \left(-\frac{44}{65}\right)T^{-1} & \frac{66}{65}T^{-1} \\ \frac{116}{65}T^{-1} & \frac{58}{65}T^{-1} & \left(-\frac{87}{65}\right)T^{-1} \\ \left(-\frac{8}{13}\right)T^{-1} & \left(-\frac{4}{13}\right)T^{-1} & \frac{6}{13}T^{-1} \end{bmatrix} x_2((k+1)T) = x_2(kT) + \\ + \begin{bmatrix} \left(-\frac{1}{520}\right)T^{-1}(529T+66) \\ \frac{1}{520}T^{-1}(653T+87) \\ \left(-\frac{1}{104}\right)T^{-1}(41T+6) \end{bmatrix} u(kT)$$

$$x(kT) = \begin{bmatrix} I_n & I_n \end{bmatrix} \begin{bmatrix} x_1(kT) \\ x_2(kT) \end{bmatrix}$$

The zeros of the matrix $sE-A$ and $sI_n-\Phi_0 A$ are $\{-2\}$ and $\{1, e^{-2T}\}$ respectively. Note also that the transfer function of the system is

$$H(s) = \begin{bmatrix} -s+38 & 12s+54 & 37s+47 \\ 2s-3 & 6s+11 & 13s+32 \\ -s+3 & 2s+9 & 8s+13 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \underbrace{\begin{bmatrix} -\frac{153}{520(s+2)} \\ \frac{51}{520(s+2)} \\ -\frac{17}{52(s+2)} \end{bmatrix}}_{H_{spr}(s)} + \underbrace{\begin{bmatrix} \frac{33}{260}s-\frac{529}{520} \\ -\frac{87}{520}s+\frac{653}{520} \\ \frac{3}{52}s-\frac{41}{104} \end{bmatrix}}_{H_{pol}(s)}$$

and

$$\left(zI_3-\tilde{A}\right)^{-1}\tilde{B}_1 = \begin{bmatrix} \frac{153e^{-2T}-153}{1040z-1040e^{-2T}} \\ \frac{-51e^{-2T}+51}{1040z-1040e^{-2T}} \\ \frac{17e^{-2T}-17}{104z-104e^{-2T}} \end{bmatrix} = \left(1-z^{-1}\right) Z\left\{\frac{H_{spr}(s)}{s}\right\}$$

since the first subsystem is a zero-order hold discretization of the first subsystem in (38) and

$$\left(z\tilde{E}_1+I_3\right)^{-1}\tilde{B}_2 = \begin{bmatrix} \frac{1}{520}T^{-1}(66z-529T-66) \\ \left(-\frac{1}{520}\right)T^{-1}(87z-653T-87) \\ \frac{1}{104}T^{-1}(6z-41T-6) \end{bmatrix} = H_{pol}\left(\frac{z-1}{T}\right)$$

since the second subsystem is a first order approximation of the second subsystem in (38).

6 Numerical implementation

In this section we give two procedures in the Mathematica programming language, for the numerical implementation of the discretizations methods presented in Section 3 and 4.

6.1 State Space Discretization

Procedure : StateSpaceDiscretization

Parameters : pe, pa, pb, that corresponds to the matrices E, A and B respectively of the system (1).

Use : Returns the constant matrix $\tilde{A}$ and the polynomial matrix $\hat{B}(\sigma)$ that has been presented in (24).

Code :

```
StateSpaceDiscretization[pe_,pa_,pb_]:=
Module[{ps,psi,se,mi,fis,ah,bis,i,j},
ps=s*pe-pa;
psi=Inverse[ps];
se=Series[psi,{s,Infinity,2}];
mi=Max[Exponent[se,s]];
fis={Coefficient[se,s^(-1)]};
For[i=0,i\[LessEqual]mi,
fis=Append[fis,Coefficient[se*s,s^(i+1)]];
i++];
(* fis={f[0],f[-1],...}*)
ah=Simplify[MatrixExp[fis[[1]].pa*T]];
bis={Simplify[Integrate[MatrixExp[fis[[1]].pa*w],{w,0,T}].fis[[1]].pb+
Sum[(-1)^i*fis[[i+1]].pb*T^(1-i),{i,1,mi+1}]]//Expand};
For[j=1,j\[LessEqual]mi+1,
bis= Append[bis, Sum[(-1)^(i-j)*fis[[i+1]].pb*T^(1-i)*Binomial[i,i-j],{i,j, mi+1}]];
j++;
];
Return[{ah,Sum[bis[[i+1]]*s^i,{i,0,Dimensions[bis][[1]]-1}]}];
];
```

Example 12 *Consider the example 4*

6.2 Descriptor system discretization

Procedure : DescriptorStateSpaceDiscretization

Parameters : pe, pa, pb, that corresponds to the matrices E, A and B respectively of the system (1).

Use : Returns the constant matrix $\tilde{A}, \tilde{B}_1, \tilde{E}_1$ and $\tilde{B}_2$ that has been presented in (38).

Code :

```
DescriptorStateSpaceDiscretization[pe_,pa_,pb_]:=
Module[{ps,psi,se,mi,fis,ah,bis,i,j,eh},
ps=s*pe-pa;
psi=Inverse[ps];
se=Series[psi,{s,Infinity,2}];
mi=Max[Exponent[se,s]];
fis={Coefficient[se,s^(-1)]};
For[i=0,i\[LessEqual]mi,
fis=Append[fis,Coefficient[se*s,s^(i+1)]];
i++];
(* fis={f[0],f[-1],...}*)
ah=Simplify[MatrixExp[fis[[1]].pa*T]];
bis={Simplify[Integrate[MatrixExp[fis[[1]].pa*w],{w,0,T}].fis[[1]].pb]// Expand};
bis=Append[bis, Simplify[-T* Inverse[(fis[[2]].pe-
T*IdentityMatrix[Dimensions[pe][[1]]])].fis[[2]].pb]];
eh=Simplify[Inverse[fis[[2]].pe-T*IdentityMatrix[Dimensions[pe][[1]]]].fis[[2]].pe];
Return[{eh,ah,bis[[1]],bis[[2]]}];
];
```

Example 13 *Consider the example 11*

7 Conclusions

Two discretization methods of a singular system have been considered. The first one is a state space discretization while the second one is a singular system discretization. The method we used is a zero-order hold discretization for the input $u(t)$ and first-order approximation of the derivatives of the input. Both the proposed sampled systems are described in terms of the Markov parameters of the system and therefore are easily implemented. Certain questions regarding the stability of the sampled systems have been investigated. The whole theory has been illustrated by examples and procedures in the Mathematica programming language.

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