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On the solution space of discrete time AR-representations over a finite time horizon

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Abstract

The main purpose of this work is to determine the forward and backward solution space of a nonregular discrete time AR-representation i.e. $A(\sigma)\xi(k) = 0$, in a finite time horizon where $A(\sigma)$ is a polynomial matrix and σ is the forward shift operator. The construction of the behavior is based on the structural invariants of the polynomial matrix that describes the AR-representation i.e. the finite and infinite elementary divisors and the right and left minimal indices of $A(\sigma)$.

Keywords: Behaviour; Elementary divisors; Minimal indices; Autoregressive representations; Discrete-time systems

Let R, C denote the fields of real and complex numbers respectively, $R[\sigma]$ the ring of polynomials with real coefficients and $R(\sigma)$ the field of rational functions. By $R(\sigma)^{p\times m}$, $R_{pr}(\sigma)^{p\times m}$ and $R[\sigma]^{p\times m}$ we denote the sets of $p \times m$ rational, proper rational and polynomial matrices respectively with real coefficients and indeterminate σ. Consider a system of linear homogeneous difference and algebraic equations described in matrix form by

$$A(\sigma)\xi(k) = 0, \tag{1}$$

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where σ denotes the forward shift operator i.e. $\sigma\xi(k) = \xi(k+1)$, $A(\sigma) = A_0 + A_1\sigma + \cdots + A_q\sigma^q \in R[\sigma]^{p\times m}$ with $\text{rank}_{R(\sigma)} A(\sigma) = r \leqslant \min(p, m)$ and $\xi(k) : [0, N] \to R^m$. We call [10,11] the set of equations (1) an AR-representation of B_D (behavior) where B_D is defined as

$$B_D := \left\{\xi(k) : [0, N] \to R^m \mid (1) \text{ is satisfied } \forall k \in [0, N-q]\right\}$$

and N is supposed to be large enough.

If $A(\sigma)$ is regular ($r = p = m$ and $\det[A(\sigma)] \neq 0$), then the solution vector space of (1) over a finite time interval has been studied by [1,6]. More specifically, it has been shown that the behavior of (1) constitutes of forward and backward solutions and its dimension is equal to rq or equivalent to the sum of the degrees of the finite and infinite elementary divisors of $A(\sigma)$. However, certain questions concerning the behavior over a finite time interval of nonregular discrete time AR-representations still remain.

In this paper we study the behavior B_D of (1) in case where $A(\sigma)$ is nonregular. More specifically, we show that B_D contains an infinite number of linear independent forward and backward solutions due to the right null space of $A(\sigma)$. In the sequel we correspond all the forward and backward solutions which are due to a specific boundary value (initial–final condition) to an element $[\xi(k)]$. According to this way the behavior space B_D is divided into equivalence classes and a new space is created, named $\hat{B}_D$. In Section 1, we present some preliminary results which concern the structural invariants of a polynomial matrix i.e. finite and infinite elementary divisors, left and right minimal indices. If now we denote by n the sum of the degrees of the finite elementary divisors of $A(\sigma)$, by μ the sum of the degrees of the infinite elementary divisors of $A(\sigma)$, by ε the sum of the right minimal indices of $A(\sigma)$, then we prove in Section 2 that the dimension of $\hat{B}_D$ is equal to $n + \mu + 2\varepsilon$. In the following three Sections (3.1–3.3) we produce linearly independent vectors $\xi(k) : [0, N] \to R^m$ which are due to the structural invariants of the system. Finally, in Section 3.4 we show that the equivalence classes produced by these specific vectors i.e. $[\xi(k)]$, constitute a linearly independent basis for $\hat{B}_D$. Furthermore, the construction of the basis that produces the space $\hat{B}_D$ helps us to construct also the space B_D. Necessary and sufficient conditions are proposed in Section 4 for the existence of solution of (1) under given boundary conditions. The whole theory is illustrated via an example which is, nevertheless, implicit in all the sessions of the paper.

1. Structural invariants of a polynomial matrix

In this section, we present four basic invariants for the study of polynomial matrices: the finite and infinite elementary divisors and the right and left minimal indices, that will play a crucial role in the construction of the forward and backward solution space of (1).

Definition 1 [9]. Let $A(\sigma) \in R[\sigma]^{p\times m}$, with $\operatorname{rank}_{R(\sigma)} A(\sigma) = r \leqslant \min\{p, m\}$. Then there exist unimodular matrices $U_L(\sigma) \in R[\sigma]^{p\times p}$, $U_R(\sigma) \in R[\sigma]^{m\times m}$ (i.e. $\det[U_L(\sigma)], \det[U_R(\sigma)] \in R\backslash\{0\}$), such that

$$\begin{aligned} S^C_{A(\sigma)}(\sigma) &:= U_L(\sigma)A(\sigma)U_R(\sigma)\\ &= \operatorname{blockdiag}\left[1, 1, \ldots, 1, f_z(\sigma), f_{z+1}(\sigma), \ldots, f_r(\sigma), 0_{p-r,m-r}\right] \end{aligned}$$

with $1 \leqslant z \leqslant r$ and $f_i(\sigma)/f_{i+1}(\sigma)$ $i = z, z+1, \ldots, r-1$. $S^C_{A(\sigma)}(\sigma)$ is called the *Smith form* of $A(\sigma)$ (in C), where $f_i(\sigma) \in R[\sigma]$ are the *invariant polynomials* of $A(\sigma)$. Assume that the partial multiplicities of the zeros $\lambda_i \in C, i \in k$ are $0 \leqslant \sigma_{i,z} \leqslant \sigma_{i,z+1} \leqslant \cdots \leqslant \sigma_{i,r}$ i.e.

$$f_j(\sigma) = (\sigma - \lambda_i)^{\sigma_{i,j}} \hat{f}_j(\sigma), \quad j = z, z+1, \ldots, r \text{ with } \hat{f}_j(\lambda_\iota) \neq 0.$$

The terms $(\sigma - \lambda_i)^{\sigma_{i,j}}$ are called *finite elementary divisors* of $A(\sigma)$ at $\sigma = \lambda_i$. We also denote by n the sum of the degrees of the finite elementary divisors of $A(\sigma)$ i.e.

$$n := \deg\left[\prod_{j=z}^{r} f_j(\sigma)\right] = \sum_{i=1}^{k}\sum_{j=z}^{r} \sigma_{i,j}.$$

Similarly, we can find $U_L(\sigma) \in R(\sigma)^{p\times p}$, $U_R(\sigma) \in R(\sigma)^{m\times m}$ having no poles and zeros at the point $\sigma = \lambda_0$ such that

$$\begin{aligned} S^{\lambda_0}_{A(\sigma)}(\sigma) := U_L(\sigma)A(\sigma)U_R(\sigma) = \operatorname{blockdiag}\big[&1, 1, \ldots, 1, (\sigma - \lambda_0)^{\sigma_z},\\ &(\sigma - \lambda_0)^{\sigma_{z+1}}, \ldots, (\sigma - \lambda_0)^{\sigma_r}, 0_{p-r,m-r}\big]. \end{aligned}$$

In that case, $S^{\lambda_0}_{A(\sigma)}(\sigma)$ is called the Smith form of $A(\sigma)$ at the local point $\sigma = \lambda_0$.

We shall show in Section 3.1 that given the matrices $U_R(\sigma)$ and $S^C_{A(\sigma)}(\sigma)$, we can construct n forward solutions of the AR-representation (1). Now based on the definition of the Smith form of a polynomial matrix at a local point, we can define the infinite elementary divisors of a polynomial matrix.

Definition 2 [8]. Define the "dual" polynomial matrix $\tilde{A}(\sigma)$ of $A(\sigma)$ as

$$\tilde{A}(\sigma) := A_0\sigma^q + A_1\sigma^{q-1} + \cdots + A_q = \sigma^q A\left(\frac{1}{\sigma}\right) \in R[\sigma]^{p\times m}.$$

Let $\tilde{U}_L(\sigma) \in R(\sigma)^{p\times p}$, $\tilde{U}_R(\sigma) \in R(\sigma)^{m\times m}$ be rational matrices having no poles and zeros at $\sigma = 0$ and such that

$$\tilde{U}_L(\sigma)\tilde{A}(\sigma)\tilde{U}_R(\sigma) = S^0_{\tilde{A}(\sigma)}(\sigma) = \operatorname{diag}\left[\sigma^{\mu_1}, \sigma^{\mu_2}, \ldots, \sigma^{\mu_r}, 0_{p-r,m-r}\right],$$

where $S^0_{\tilde{A}(\sigma)}(\sigma)$ is the Smith form of $\tilde{A}(\sigma)$ at $\sigma = 0$. The terms σ^{μ_j} are called the *infinite elementary divisors* of $A(\sigma)$ i.e. are actually the finite elementary divisors of the dual matrix of $A(\sigma)$ at $\sigma = 0$. We also denote by μ the sum of the degrees of the infinite elementary divisors of $A(\sigma)$ i.e.

$$\mu := \sum_{j=1}^{r} \mu_j.$$

We shall show in Section 3.2 that given the matrices $\tilde{U}_R(\sigma)$ and $S^0_{\tilde{A}(\sigma)}(\sigma)$, we can construct μ backward solutions of the AR-representation (1). In case now where the polynomial matrix is regular i.e. $p = m = r$ and $\det[A(\sigma)] \neq 0$, then a connection exists between n, μ defined above, the rank r of the matrix $A(\sigma)$ and the greatest degree q among all the polynomial entries of $A(\sigma)$, as can be seen by the following lemma.

Lemma 3 [6]. *Let $A(\sigma) = A_q\sigma^q + \cdots + A_1\sigma + A_0 \in R[\sigma]^{p\times p}$, with $\det[A(\sigma)] \neq 0$. Let also, n, μ be the sum of the degrees of the finite and infinite elementary divisors of $A(\sigma)$, as has been defined above. Then*

$$n + \mu = r \times q,$$

where q is the highest degree among all the polynomial entries of $A(\sigma)$.

Actually, it has been proved in [1], that in case where $A(\sigma)$ is regular, the dimension of the solution space of (1) is equal to $n + \mu = r \times q$. In case now where the polynomial matrix is nonregular, i.e. $p \neq m$ or $p = m$ and $r < \min\{p, m\}$, then except for the finite and infinite elementary divisors, extra invariants are involved in the algebraic structure of $A(\sigma)$ due to its right and left null space.

Definition 4 [4]. Let $A(\sigma) \in R[\sigma]^{p\times m}$, with $\operatorname{rank}_{R(\sigma)} A(\sigma) = r < \min\{p, m\}$. Define by

$$V_R := \left\{ v(\sigma) \in R(\sigma)^{m\times 1} : A(\sigma)v(\sigma) = 0_{p,1} \right\}$$

the $m - r$ dimensional vector space of the right kernel of $A(\sigma)$ over $R(\sigma)$. Let $\{\varepsilon_{r+1}(\sigma), \varepsilon_{r+2}(\sigma), \ldots, \varepsilon_m(\sigma)\}$ be a minimal polynomial basis of V_R i.e. the matrix $[\varepsilon_{r+1}(\sigma) \quad \varepsilon_{r+2}(\sigma) \quad \cdots \quad \varepsilon_m(\sigma)]$ that has least order among all polynomial matrix bases for V_R. Then, we define by ε_i the degrees of the m-tuples $\varepsilon_i(\sigma)$ i.e. $\varepsilon_i := \deg[\varepsilon_i(\sigma)]$, where degree of an m-tuple is the greatest degree among its components. The indices $\{\varepsilon_{r+1}, \varepsilon_{r+2}, \ldots, \varepsilon_m\}$ are called right minimal indices of $A(\sigma)$ and constitute invariant elements of the polynomial matrix $A(\sigma)$. We define also by ε the sum of the right minimal indices i.e.

$$\varepsilon := \sum_{i=r+1}^{m-r} \varepsilon_i.$$

Similarly we can define the left minimal indices of a polynomial matrix.

Definition 5 [4]. Let $A(\sigma) \in R[\sigma]^{p\times m}$, with $\operatorname{rank}_{R(\sigma)} A(\sigma) = r < \min\{p, m\}$. Define by

$$V_L := \left\{ v(\sigma) \in R(\sigma)^{1\times p} : v(\sigma)A(\sigma) = 0_{1,m} \right\}$$

the $p - r$ dimensional vector space of the left kernel of $A(\sigma)$ over $R(\sigma)$. Let $\{\eta_{r+1}(\sigma), \eta_{r+2}(\sigma), \dots, \eta_p(\sigma)\}$ be a minimal polynomial basis of V_L, i.e. the matrix $[\eta_{r+1}^{\mathrm{T}}(\sigma) \quad \eta_{r+2}^{\mathrm{T}}(\sigma) \quad \cdots \quad \eta_p^{\mathrm{T}}(\sigma)]^{\mathrm{T}}$ that has least order among all polynomial bases for V_L. Then, we define by η_i the degrees of the p-tuples $\eta_i(\sigma)$ i.e. $\eta_i := \deg[\eta_i(\sigma)]$, where degree of an m-tuple is the greatest degree among its components. The indices $\{\eta_{r+1}, \eta_{r+2}, \dots, \eta_p\}$ are called left minimal indices of $A(\sigma)$ and constitute invariant elements of the polynomial matrix $A(\sigma)$. We define also by η the sum of the left minimal indices i.e.

$$\eta := \sum_{i=r+1}^{p} \eta_i.$$

As in the regular case, a connection exists between the structural invariants of a nonregular polynomial matrix, which we can easily see by the following lemma.

Lemma 6 [8]. *Let* $A(\sigma) = A_0 + A_1\sigma + \cdots + A_q\sigma^q \in R[\sigma]^{p\times m}$, *with* $\operatorname{rank}_{R(\sigma)} A(\sigma) = r < \min\{p, m\}$. *Let* $n, \mu, \varepsilon, \eta$ *as defined above, be respectively the sum of the degrees of the finite and infinite elementary divisors and the sum of the right and left minimal indices. Then*

$$n + \mu + \varepsilon + \eta = q \times r,$$

where q is the highest degree among all the polynomial entries of $A(\sigma)$.

It is easily seen that Lemma 6 is a simple extension of Lemma 3 to nonregular polynomial matrices. The structural invariants of a polynomial matrix that has been presented above, are illustrated in the following example.

Example 7. Let

$$A(\sigma) = \begin{bmatrix} \sigma & \sigma^4 & \sigma^2+\sigma \\ 1 & \sigma^3 & \sigma+1 \\ 0 & \sigma+1 & 0 \end{bmatrix}.$$

Then there exist unimodular matrices $U_L(\sigma), U_R(\sigma) \in R[\sigma]^{3\times 3}$, such that

$$U_L(\sigma)A(\sigma)U_R(\sigma) = S^C_{A(\sigma)}(\sigma) \Longleftrightarrow \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -\sigma & 0 \end{bmatrix} \begin{bmatrix} \sigma & \sigma^4 & \sigma^2+\sigma \\ 1 & \sigma^3 & \sigma+1 \\ 0 & \sigma+1 & 0 \end{bmatrix} \times \begin{bmatrix} 1 & -\sigma^3 & -1-\sigma \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sigma+1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad (2)$$

and therefore $r = 2 = z$, $\lambda_1 = -1$, $\sigma_{1,2} = 1$ and $n = \sigma_{1,2} = 1$. Consider the dual polynomial matrix $\tilde{A}(\sigma)$ of $A(\sigma)$ defined by

$$\tilde{A}(\sigma) = \sigma^4 A\left(\frac{1}{\sigma}\right) = \begin{bmatrix} \sigma^3 & 1 & \sigma^2+\sigma^3 \\ \sigma^4 & \sigma & \sigma^3+\sigma^4 \\ 0 & \sigma^3+\sigma^4 & 0 \end{bmatrix}.$$

There exist matrices $\tilde{U}_L(\sigma)$ and $\tilde{U}_R(\sigma)$ having no poles and zeros at $\sigma = 0$ such that

$$\tilde{U}_L(\sigma)\tilde{A}(\sigma)\tilde{U}_R(\sigma) = S^0_{\tilde{A}(\sigma)}(\sigma) \Longleftrightarrow \begin{bmatrix} 1 & 0 & 0 \\ \sigma^3 & 0 & -\frac{1}{\sigma+1} \\ \sigma & -1 & 0 \end{bmatrix} \begin{bmatrix} \sigma^3 & 1 & \sigma^2+\sigma^3 \\ \sigma^4 & \sigma & \sigma^3+\sigma^4 \\ 0 & \sigma^3+\sigma^4 & 0 \end{bmatrix}$$
$$\times \begin{bmatrix} 0 & -1 & 1+\sigma \\ 1 & -\sigma^2 & 0 \\ 0 & 1 & -\sigma \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sigma^5 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and therefore $\mu = \mu_2 = 5$. We can easily see from (2) that

$$\underbrace{\begin{bmatrix} \sigma & \sigma^4 & \sigma^2+\sigma \\ 1 & \sigma^3 & \sigma+1 \\ 0 & \sigma+1 & 0 \end{bmatrix}}_{A(\sigma)} \underbrace{\begin{bmatrix} -1-\sigma \\ 0 \\ 1 \end{bmatrix}}_{\varepsilon_3(\sigma)} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

and

$$\underbrace{\begin{bmatrix} 1 & -\sigma & 0 \end{bmatrix}}_{\eta_3(\sigma)} \underbrace{\begin{bmatrix} \sigma & \sigma^4 & \sigma^2+\sigma \\ 1 & \sigma^3 & \sigma+1 \\ 0 & \sigma+1 & 0 \end{bmatrix}}_{A(\sigma)} = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}.$$

Therefore, $[\varepsilon_3(\sigma)]$ (resp. $[\eta_3(\sigma)]$) constitutes a minimal bases of the right (resp. left) kernel of $A(\sigma)$ and $\varepsilon = \varepsilon_3 = 1$ (resp. $\eta = \eta_3 = 1$). Note also that

$$1 + 5 + 1 + 1 = n + \mu + \varepsilon + \eta \equiv r \times q = 2 \times 4.$$

2. Determination of the dimension of the behavior space of a discrete time AR-representations

In this section we are trying to determine the dimension of the behavior space B_D of (1).

Eq. (1) may be rewritten as

$$\underbrace{\begin{bmatrix} A_0 & A_1 & \cdots & A_q & 0 & \cdots & 0 \\ 0 & A_0 & A_1 & \cdots & A_q & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & A_0 & A_1 & \cdots & A_q \end{bmatrix}}_{S_N} \underbrace{\begin{bmatrix} \xi(0) \\ \xi(1) \\ \vdots \\ \xi(N) \end{bmatrix}}_{\xi_N} = 0 \iff \tag{3}$$

$$S_N \xi_N = 0, \quad S_N \in R^{(N-q+1)p \times (N+1)m}, \xi_N \in R^{(N+1)m},$$

where N is assumed to be "large enough". Then by applying Z transform on $\xi(k)$ we get [5],

$$\xi(z) \overset{\text{def}}{=} Z[\xi(k)] = \sum_{k=0}^{N} \xi(k) z^{-k}$$

and by applying Z transforms to (1) we also get:

$$Z[A(\sigma)\xi(k)] = Z[0] \Leftrightarrow A(z)\xi(z) = \underbrace{\begin{bmatrix} z^q I_p & \cdots & z I_p & I_p \end{bmatrix}}_{\mathscr{Z}_q}$$

$$\times \underbrace{\begin{bmatrix} A_q & 0 & \cdots & 0 \\ A_{q-1} & A_q & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_1 & A_2 & \cdots & A_q \end{bmatrix}}_{\Re_q} \underbrace{\begin{bmatrix} \xi(0) \\ \xi(1) \\ \vdots \\ \xi(q-1) \end{bmatrix}}_{\tilde{\xi}(0)}$$

$$+ \underbrace{\begin{bmatrix} z^{-N} I_p & \cdots & z^{-N+q-2} I_p & z^{-N+q-1} I_p \end{bmatrix}}_{\mathscr{Z}_N}$$

$$\times \underbrace{\begin{bmatrix} A_0 & 0 & \cdots & 0 \\ A_1 & A_0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_{q-1} & A_{q-2} & \cdots & A_0 \end{bmatrix}}_{\Re_0} \underbrace{\begin{bmatrix} \xi(N) \\ \xi(N-1) \\ \vdots \\ \xi(N-q+1) \end{bmatrix}}_{\tilde{\xi}(N)}$$

$$=: \mathscr{Z}_q \Re_q \tilde{\xi}(0) + \mathscr{Z}_N \Re_0 \tilde{\xi}(N). \tag{4}$$

Let $\{\varepsilon_{r+1}(\sigma), \varepsilon_{r+2}(\sigma), \ldots, \varepsilon_m(\sigma)\}$ be a minimal polynomial basis of the right kernel of $A(\sigma)$[1] where

$$\varepsilon_i(\sigma) := \varepsilon_{i,0} + \varepsilon_{i,1}\sigma + \cdots + \varepsilon_{i,\varepsilon_i}\sigma^{\varepsilon_i} \quad i = r+1, r+2, \ldots, m.$$

Then the rational vectors of the form

$$\hat{\varepsilon}_i(\sigma) := \varepsilon_{i,0}\frac{1}{\sigma^{\varepsilon_i}} + \varepsilon_{i,1}\frac{1}{\sigma^{\varepsilon_i-1}} + \cdots + \varepsilon_{i,\varepsilon_i} \quad i = r+1, r+2, \ldots, m \tag{5}$$

still constitute a minimal proper rational basis of the right kernel of $A(\sigma)$. Then one can prove the following.

Proposition 8. *The AR-representation* (1) *with boundary conditions*

$$\tilde{\xi}(0) \in \mathrm{Ker}[\Re_q] \quad \textit{and} \quad \tilde{\xi}(N) \in \mathrm{Ker}[\Re_0] \tag{6}$$

has the solution

$$\xi(k) := \sum_{i=0}^{k} \hat{\varepsilon}_{r+1}(i) z_1(k-i) + \sum_{i=0}^{k} \hat{\varepsilon}_{r+2}(i) z_2(k-i) + \cdots + \sum_{i=0}^{k} \hat{\varepsilon}_m(i) z_{m-r}(k-i), \tag{7}$$

where $z_i(k)$, $i = 1, 2, \ldots, m-r$ are arbitrary discrete time functions and $\hat{\varepsilon}_i(k) = Z^{-1}[\hat{\varepsilon}_i(z)]$, $i = r+1, \ldots, m$.

Proof. Taking into account relations (6), the relation (4) may be rewritten as

$$A(z)\xi(z) = 0$$

and so the solution $\xi(z)$ belongs to the right kernel of $A(z)$ and, therefore, can be written as a linear combination of the minimal proper rational basis of $A(z)$ defined in (5) i.e.

$$\xi(z) = \sum_{i=r+1}^{m} \hat{\varepsilon}_i(z) z_{i-r}(z), \tag{8}$$

where $z_i(k) = \sum_{k=0}^{N} z_{i,k} \times z^{-k}$ with $z_{i,k}$ arbitrary for $i = 1, 2, \ldots, m-r$. Taking inverse Z-transforms in (8) we obtain the solution (7) which verifies the proposition. □

[1] For the simplicity of the proofs of the main theorems we give a specific construction of the minimal polynomial basis in (19), although in practice other constructions may also be used i.e. [2].

A necessary and sufficient condition for the uniqueness of solution is given in the following theorem.

Theorem 9. *In case where* (1) *has a solution then this solution is unique iff the following conditions are satisfied*:

$$p \geqslant m \quad \textit{and} \quad \operatorname{rank}_{R(\sigma)} A(\sigma) = m.$$

Proof. Suppose that (1) has not a unique solution but has two solutions $\xi_1(k), \xi_2(k)$ $(\xi_1(k) \neq \xi_2(k))$ under the same initial–final conditions or equivalently that

$$A(z)\xi_1(z) = \mathscr{Z}_q \Re_q \tilde{\xi}(0) + \mathscr{Z}_N \Re_0 \tilde{\xi}(N) \tag{9}$$

and

$$A(z)\xi_2(z) = \mathscr{Z}_q \Re_q \tilde{\xi}(0) + \mathscr{Z}_N \Re_0 \tilde{\xi}(N). \tag{10}$$

Equating the left hand sides of (9) and (10), since the right hand sides coincide, we obtain that

$$A(z)\xi_1(z) = A(z)\xi_2(z) \Leftrightarrow A(z)(\xi_1(z) - \xi_2(z)) = 0$$

and so the difference between $\xi_1(z)$ and $\xi_2(z)$ belongs to the right kernel of $A(z)$. Thus, these two solutions will coincide iff the right kernel of $A(z)$ is the null space or equivalently iff the required conditions are satisfied. □

Corollary 10. *It follows from the above theorem that in case where the right kernel of* $A(z)$ *is not the null space and* $\xi_0(k)$ *be a solution of* (1) *then*

$$\xi(k) = \xi_0(k) + \sum_{i=0}^{k} \hat{\varepsilon}_{r+1}(i) z_1(k-i) + \sum_{i=0}^{k} \hat{\varepsilon}_{r+2}(i) z_2(k-i) + \cdots + \sum_{i=0}^{k} \hat{\varepsilon}_m(i) z_{m-r}(k-i),$$

where $z_i(k)$, $i = 1, 2, \ldots, m-r$ *are arbitrary discrete time functions*, *are also solutions of* (1).

Let now

$$X = \left\{ \begin{array}{c} \xi(k) : \tilde{\xi}(0) \in \operatorname{Ker}[\Re_q] \text{ and } \tilde{\xi}(N) \in \operatorname{Ker}[\Re_0] \\ \xi(k) = \sum_{i=0}^{k} \hat{\varepsilon}_{r+1}(i) z_1(k-i) + \sum_{i=0}^{k} \hat{\varepsilon}_{r+2}(i) z_2(k-i) \\ + \cdots + \sum_{i=0}^{k} \hat{\varepsilon}_m(i) z_{m-r}(k-i) \end{array} \right\}$$

be the solution space of (1) which comes under initial–final conditions of the form (6). An interesting question arisen from above concerns the dimension of the spaces B_D and X. In order to find out these dimensions we first propose the following lemma.

Lemma 11 [3]. *Consider the matrix S_k defined by*

$$S_k = \begin{bmatrix} A_0 & A_1 & \cdots & A_q & 0 & \cdots & 0 \\ 0 & A_0 & A_1 & \cdots & A_q & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & A_0 & A_1 & \cdots & A_q \end{bmatrix} \in R^{p(k-q+1)\times(k+1)m}. \quad (11)$$

Then

$$\operatorname{rank}_R S_k = p(k-q+1) - \sum_{\{i:k>\eta_i\}} (k-q+1-\eta_i), \quad (12)$$

where $\eta_i, i = 1, 2, \ldots, l$ are the left minimal indices of $A(\sigma)$.

Using the above lemma we can now easily prove the following:

Theorem 12. *Let $N_{\min} = \max_{i=r+1,\ldots,m\ j=r+1,\ldots,p} \{\varepsilon_i + 1, \eta_j\}$, where $\{\varepsilon_i, i = r+1, \ldots, m\}$ and $\{\eta_i, j = r+1, \ldots, p\}$ are the left and right minimal indices of $A(\sigma)$. Then for $N > N_{\min}$ we have that*

$$\dim B_D = (N+1)(m-r) + qr - \eta, \quad (13)$$

$$\dim X = (N+1)(m-r) - \varepsilon, \quad (14)$$

where ε (resp. η) is the sum of the right (resp. left) minimal indices.

Proof. B_D is isomorphic to $\ker S_N$. According to previous lemma we have that

$$\operatorname{rank}_R S_N = p(N-q+1) - \sum_{\{j:N>\eta_j\}} (N-q+1-\eta_j).$$

For $N > N_{\min}$ we have that

$$\begin{aligned} \operatorname{rank}_R S_N &= p(N-q+1) - \sum_{j=r+1,\ldots,p} (N-q+1-\eta_j) \\ &= p(N-q+1) - (N-q+1)(p-r) + \eta. \end{aligned}$$

Thus

$$\dim \operatorname{Ker} S_N = (N+1)m - \operatorname{rank}_R S_N = (N+1)(m-r) + qr - \eta.$$

This proves that $\dim B_D = (N+1)(m-r) + qr - \eta$. In the same way we can prove that X is isomorphic to $\ker C_N$ (by adding Eq. (6) in Eq. (3)) where

$$C_N^{\mathrm{T}} = \begin{bmatrix} A_0^{\mathrm{T}} & A_1^{\mathrm{T}} & \cdots & A_q^{\mathrm{T}} & 0 & \cdots & 0 \\ 0 & A_0^{\mathrm{T}} & A_1^{\mathrm{T}} & \cdots & A_q^{\mathrm{T}} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & A_0^{\mathrm{T}} & A_1^{\mathrm{T}} & \cdots & A_q^{\mathrm{T}} \end{bmatrix} \in R^{(N+1)m \times p(N+q+1)}.$$

Applying the previous lemma to C_N we have that

$$\mathrm{rank}_R C_N = \mathrm{rank}_R C_N^{\mathrm{T}} = (N+1)m - \sum_{\{i:N>\varepsilon_i\}} (N+1-\varepsilon_i),$$

where the left indices of $A(\sigma)^{\mathrm{T}}$ are the right indices of $A(\sigma)$. Thus for $N > N_{\min}$ we have that

$$\mathrm{rank}_R C_N = (N+1)m - (N+1)(m-r) + \varepsilon$$

and

$$\dim \mathrm{Ker} C_N = (N+1)m - \mathrm{rank}_R C_N = (N+1)(m-r) - \varepsilon.$$

This proves that $\dim X = (N+1)(m-r) - \varepsilon$. □

It is easily checked that

$$\underbrace{\begin{bmatrix} A_0 & A_1 & \cdots & A_q & 0 & \cdots & 0 \\ 0 & A_0 & A_1 & \cdots & A_q & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & A_0 & A_1 & \cdots & A_q \end{bmatrix}}_{S_N} \begin{bmatrix} \varepsilon_{i,0} & 0 & \cdots & 0 \\ \varepsilon_{i,1} & \varepsilon_{i,0} & \cdots & 0 \\ \vdots & \vdots & \ddots & \varepsilon_{i,0} \\ \varepsilon_{i,\varepsilon_i} & \varepsilon_{i,\varepsilon_i-1} & \cdots & \vdots \\ 0 & \varepsilon_{i,\varepsilon_i} & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \varepsilon_{i,\varepsilon_i} \end{bmatrix} = 0$$

and thus the columns of the matrix

$$U_i := \begin{bmatrix} \varepsilon_{i,0} & 0 & \cdots & 0 \\ \varepsilon_{i,1} & \varepsilon_{i,0} & \cdots & 0 \\ \vdots & \vdots & \ddots & \varepsilon_{i,0} \\ \varepsilon_{i,\varepsilon_i} & \varepsilon_{i,\varepsilon_i-1} & \cdots & \vdots \\ 0 & \varepsilon_{i,\varepsilon_i} & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \varepsilon_{i,\varepsilon_i} \end{bmatrix} \in R^{(N+1)m \times (N-\varepsilon_i+1)}, \quad i = r+1, \ldots, m \tag{15}$$

are linearly independent solutions of (1) with its initial and final conditions satisfying (6), i.e. the linear independence can be proved using the properties of the column vectors of the minimal polynomial bases [4]. The number of those linearly independent solutions which are due to the right minimal indices is equal to

$$\sum_{i=r+1}^{m} (N - \varepsilon_i + 1) = (N + 1)(m - r) - \varepsilon \tag{16}$$

and thus produce the vector space X. Let $\hat{B}_D := B_D/X$. It is important to note here that each element of $\hat{B}_D$ corresponds to the whole set of solutions of (1) under some specific initial–final conditions i.e.

$$[\xi(k)] := \left\{ \xi(k) + \sum_{i=0}^{k} \hat{\varepsilon}_{r+1}(i) z_1(k-i) + \sum_{i=0}^{k} \hat{\varepsilon}_{r+2}(i) z_2(k-i) + \cdots + \sum_{i=0}^{k} \hat{\varepsilon}_{m}(i) z_{m-r}(k-i) \right\},$$

where $z_i(k), i = 1, 2, \ldots, m - r$ are arbitrary discrete time functions. It is now easy to show the following theorem.

Theorem 13. *The space* $\hat{B}_D := B_D/X$ *is a vector space and in case where* $N > N_{\min} = \max_{i=r+1,\ldots,m \; j=r+1,\ldots,p} \{\varepsilon_i + 1, \eta_j\}$, *it has dimension*

$$f := \dim \hat{B}_D = n + \mu + 2\varepsilon$$

f *is called the generalized order of* (1).

Proof. In order to prove the theorem it is enough to show that $\dim B_D - \dim X = n + \mu + 2\varepsilon$. It is easily seen from the Theorem 12 and Lemma 6, that the number of linearly independent solutions which belong to B_D but not to X is equal to

$$\begin{aligned} \dim B_D - \dim X &= ((N + 1)(m - r) + qr - \eta) - ((N + 1)(m - r) - \varepsilon) \\ &= n + \mu + 2\varepsilon \end{aligned}$$

The rest of the proof that $\hat{B}_D$ is a vector space and consists of $n + \mu + 2\varepsilon$ linearly independent vectors is left to the reader. □

Remark 14. Note that if $A(\sigma)$ is regular then $\varepsilon = 0$ and f coincides with $n + \mu$, as has been proved in [1].

In the following section we shall try to determine a basis for the vector space $\hat{B}_D$ in terms of the structural invariants of the polynomial matrix $A(\sigma)$ and thus, to find out the solution space B_D.

3. Construction of the behavior of a discrete time AR-representation

In this section we shall try to define n linearly independent forward solutions of (1) due to the finite elementary divisors of $A(\sigma)$, μ linearly independent backward solutions of (1) due to the infinite elementary divisors of $A(\sigma)$, ε linearly independent forward solutions of (1) due to the right minimal indices of $A(\sigma)$ and ε linearly independent backward solutions of (1) due to right minimal indices of $A(\sigma)$.

3.1. Finite elementary divisors and solutions of discrete time AR-representations

Let us assume that $A(\sigma)$ has k distinct zeros $\lambda_1, \lambda_2, \ldots, \lambda_k$ where, for simplicity of notation, we assume that $\lambda_i \in C$, $i \in k$ and let

$$U_L(\sigma)A(\sigma)U_R(\sigma) = \text{blockdiag}\big[1, 1, \ldots, 1, f_z(\sigma), f_{z+1}(\sigma), \ldots, f_r(\sigma), 0_{p-r,m-r}\big] \tag{17}$$

be the Smith form of $A(\sigma)$ (in C). Assume that the partial multiplicities of the zeros $\lambda_i \in C, i \in k$ are $0 \leqslant \sigma_{i,z} \leqslant \sigma_{i,z+1} \leqslant \cdots \leqslant \sigma_{i,r}$ i.e. $f_j(\sigma) = (\sigma - \lambda_i)^{\sigma_{i,j}} \hat{f}_j(\sigma)$, $j = z, z+1, \ldots r$ with $\hat{f}_j(\lambda_i) \neq 0$. Let $u_j(\sigma) \in R[\sigma]^{m\times 1}$, $j \in R$ be the columns of $U_R(\sigma)$ and $u_j^{(q)}(\sigma) := (\mathrm{d}^q/\mathrm{d}\sigma^q)u_j(\sigma)$, $q = 0, 1, \ldots, \sigma_{i,j} - 1$. Let also

$$x^i_{j,q} := \frac{1}{q!}u_j^{(q)}(\lambda_i) \quad i \in k \text{ and } j = z, z+1, \ldots, r.$$

Define the vector valued functions

$$\xi^i_{j,q}(k) := \lambda_i^k x^i_{j,q} + k\lambda_i^{k-1}x^i_{j,q-1} + \cdots + \binom{k}{q}\lambda_i^{k-q}x^i_{j,0} \quad \text{if } \lambda_i \neq 0$$

$$\xi^i_{j,q}(k) := \delta(k)x^i_{j,q} + \delta(k-1)x^i_{j,q-1} + \cdots + \delta(k-q)x^i_{j,0} \quad \text{if } \lambda_i = 0$$

$$i \in k;\ j = z, z+1, \ldots, r;\ q = 0, 1, \ldots, \sigma_{i,j} - 1$$

where by $\delta(k)$ we denote the known Kronecker delta function i.e.

$$\delta(k) = \begin{cases} 1 & k = 0, \\ 0 & k \neq 0. \end{cases}$$

Let

$$\Psi_{i,j}(k) := \begin{bmatrix} \xi^i_{j,0}(k) & \xi^i_{j,1}(k) & \cdots & \xi^i_{j,\sigma_{i,j}-2}(k) & \xi^i_{j,\sigma_{i,j}-1}(k) \end{bmatrix},$$

$$C_{i,j} := \begin{bmatrix} x^i_{j,0} & x^i_{j,1} & \cdots & x^i_{j,\sigma_{i,j}-2} & x^i_{j,\sigma_{i,j}-1} \end{bmatrix},$$

$$J_{i,j} := \begin{bmatrix} \lambda_i & 1 & \cdots & 0 & 0 \\ 0 & \lambda_i & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \lambda_i & 1 \\ 0 & 0 & \cdots & 0 & \lambda_i \end{bmatrix} \in R^{\sigma_{i,j}\times\sigma_{i,j}},$$

where $i \in k$ and $j = z, z+1, \dots, r$ and

$$\begin{aligned}
\Psi_i^F(k) &:= \left[\Psi_{i,z}(k) \quad \Psi_{i,z+1}(k) \quad \cdots \quad \Psi_{i,r-1}(k) \quad \Psi_{i,r}(k)\right],\\
C_i^F &:= \left[C_{i,z}(k) \quad C_{i,z+1}(k) \quad \cdots \quad C_{i,r-1}(k) \quad C_{i,r}(k)\right],\\
J_i^F &:= \text{blockdiag}\left[J_{i,z}(k) \quad J_{i,z+1}(k) \quad \cdots \quad J_{i,r-1}(k) \quad J_{i,r}(k)\right],
\end{aligned}$$

where $m_i := \sigma_{i,z} + \sigma_{i,z+1} + \cdots \sigma_{i,r}$ and $i \in k$. Finally let

$$\begin{aligned}
\Psi_F^D(k) &:= \left[\Psi_1^F(k) \quad \Psi_2^F(k) \quad \cdots \quad \Psi_{k-1}^F(k) \quad \Psi_k^F(k)\right],\\
C_F^D &:= \left[C_1^F(k) \quad C_2^F(k) \quad \cdots \quad C_{k-1}^F(k) \quad C_k^F(k)\right],\\
J_F^D &:= \text{blockdiag}\left[J_1^F(k) \quad J_2^F(k) \quad \cdots \quad J_{k-1}^F(k) \quad J_k^F(k)\right],
\end{aligned}$$

where

$$n := \deg\left[\prod_{j=z}^{r} f_j(\sigma)\right].$$

Denote the space B_F^D that is spanned by the columns of the matrix $\Psi_F^D(k)$ i.e.

$$B_F^D = \left\langle \Psi_F^D(k)\right\rangle = \left\langle C_F^D (J_F^D)^k\right\rangle.$$

Then we have the following.

Theorem 15. *The generators of B_F^D lie in $B_D - X$ and*

$$\dim B_F^D = n := \text{total sum of the degrees of the finite elementary divisors of } A(\sigma).$$

Proof. It is easily seen (see also [6]) that the pair (C_F^D, J_F^D) constitutes a finite spectral pair of $A(\sigma)$ which satisfies the following conditions:

$$\sum_{k=0}^{q} A_k C_F^D (J_F^D)^k = 0 \quad \text{and} \quad \text{rank col}\left(C_F^D (J_F^D)^k\right)_{k=0}^{n-1} = n$$

and, therefore, the columns of the matrix $\Psi_F^D(k)$ satisfy the Eq. (1). The proof that the initial conditions $(C_F^D (J_F^D)^k)_{k=0}^{n-1}$ do not belong to the kernel of $\Re_q$ and, therefore, the solutions defined by the columns of the matrix $\Psi_F^D(k)$ do not belong to X is not presented here. The reason is that it includes a lot of technicalities. However, it is based (a) on the specific selection of the right minimal basis selected in the sequel and (b) on the linear independence of the columns of the transforming matrix $U_R(\sigma)$ defined above. The proof follows similar lines to the ones for the continuous time case presented in [7]. □

Any other finite spectral pair will also define an isomorphic space to B_F^D. However, our intention is twofold: (a) to give the reader a method for the construction of the finite spectral pair and (b) to simplify some of the proofs with the specific form of this spectral pair.

Example 16. Consider the AR-representation

$$\underbrace{\begin{bmatrix} \sigma & \sigma^4 & \sigma^2+\sigma \\ 1 & \sigma^3 & \sigma+1 \\ 0 & \sigma+1 & 0 \end{bmatrix}}_{A(\sigma)} \underbrace{\begin{bmatrix} \xi_1(k) \\ \xi_2(k) \\ \xi_3(k) \end{bmatrix}}_{\xi(k)} = 0_{3,1}.$$

Then, from Example 7, there exist unimodular matrices $U_L(\sigma)$ and $U_R(\sigma)$ such that

$$\underbrace{\begin{bmatrix} \sigma & \sigma^4 & \sigma^2+\sigma \\ 1 & \sigma^3 & \sigma+1 \\ 0 & \sigma+1 & 0 \end{bmatrix}}_{A(\sigma)} \underbrace{\begin{bmatrix} 1 & -\sigma^3 & -1-\sigma \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{U_R(s)} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & \sigma+1 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{U_L(\sigma)^{-1}} \underbrace{\begin{bmatrix} \sigma & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}}_{S^C_{A(\sigma)}(\sigma)}.$$

Define

$$x^1_{2,0} = \frac{1}{0!} u_2(-1) = \begin{bmatrix} -\sigma^3 \\ 1 \\ 0 \end{bmatrix}_{\sigma=-1} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$$

Thus

$$\Psi^D_F(k) := [x^1_{2,0}(k)] = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} (-1)^k =: C^D_F (J^D_F)^k$$

and therefore

$$B^D_F := \left\langle \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} (-1)^k \right\rangle$$

with $\dim B^D_F = n = 1$.

3.2. Infinite elementary divisors and solutions of discrete time AR-representations

Let $\tilde{U}_L(\sigma) \in R(\sigma)^{p\times p}$, $\tilde{U}_R(\sigma) \in R(\sigma)^{m\times m}$ be rational matrices having no poles and zeros at $\sigma = 0$ and such that

$$\tilde{U}_L(\sigma)\tilde{A}(\sigma)\tilde{U}_R(\sigma) = S^0_{\tilde{A}(\sigma)}(\sigma) = \operatorname{diag}\left[\sigma^{\mu_1}, \sigma^{\mu_2}, \ldots, \sigma^{\mu_r}, 0_{p-r,m-r}\right],$$

where $S^0_{\tilde{A}(\sigma)}(\sigma)$ is the Smith form of $\tilde{A}(\sigma)$ at $\sigma = 0$. Let also $\tilde{U}_R(\sigma) = \begin{bmatrix}\tilde{u}_1(\sigma) & \tilde{u}_2(\sigma) & \cdots & \tilde{u}_m(\sigma)\end{bmatrix}$ where $\tilde{u}_j(\sigma) \in R(\sigma)^{m\times 1}$ and $\tilde{u}_j^{(i)}(\sigma)$, $\tilde{A}^{(i)}(\sigma)$ be the qth derivatives of $\tilde{u}_j(\sigma)$ and $\tilde{A}(\sigma)$ with respect to σ for $i = 0, 1, \ldots, \mu_j - 1$ and $j \in r$ where μ_j are defined above as the multiplicities of the zeros of $\tilde{A}(\sigma)$ at $w = 0$ or equivalently the degrees of the infinite elementary divisors. Define

$$x_{j,i} := \frac{1}{i!}\tilde{u}_j^{(i)}(0)$$

for $i := 0, 1, \ldots, \mu_j - 1$ and $j \in r$. Then for final conditions

$$\begin{bmatrix} \xi(N) \\ \xi(N-1) \\ \vdots \\ \xi(N-q+1) \end{bmatrix} = \begin{bmatrix} x_{j,i} \\ \vdots \\ x_{j,0} \\ 0_{(q-i-1),1} \end{bmatrix},$$

we obtain respectively the linearly independent backward solutions

$$\xi^B_{j,i}(k) := x_{j,i}\delta(N-k) + x_{j,i-1}\delta(N-(k-1)) + \cdots + x_{j,0}\delta(N-(k-i)).$$

Define the vector valued functions

$$\begin{aligned} \xi^B_{j,i}(k) := {} & x_{j,i}\delta(N-k) + x_{j,i-1}\delta(N-(k-1)) \\ & + \cdots + x_{j,0}\delta(N-(k-i)) \quad i := 0, 1, \ldots, \mu_j - 1 \text{ and } j \in r. \end{aligned} \tag{18}$$

Let

$$\begin{aligned} \Psi^B_j(k) &:= \begin{bmatrix} \xi^B_{j,0}(k) & \xi^B_{j,1}(k) & \cdots & \xi^B_{j,\mu_j-1}(k) \end{bmatrix}, \\ C^B_j &:= \begin{bmatrix} x_{j,0} & x_{j,1} & \cdots & x_{j,\mu_j-1} \end{bmatrix}, \\ J^B_j &:= \begin{bmatrix} 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix} \in R^{\mu_j \times \mu_j}, \end{aligned}$$

where $j \in r$, and

$$\begin{aligned} \Psi^D_B(k) &:= \begin{bmatrix} \Psi^B_k(k) & \Psi^B_{k+1}(k) & \cdots & \Psi^B_r(k) \end{bmatrix}, \\ C^D_B &:= \begin{bmatrix} C^B_k(k) & C^B_{k+1}(k) & \cdots & C^B_r(k) \end{bmatrix}, \\ J^D_B &:= \text{blockdiag}\begin{bmatrix} J^B_k(k) & J^B_{k+1}(k) & \cdots & J^B_r(k) \end{bmatrix}, \end{aligned}$$

where

$$\mu := \sum_{j=1}^{r} \mu_j.$$

Denote the space B_B^D that is spanned by the columns of the matrix $\Psi_B^D(k)$ i.e.

$$B_B^D = \left\langle \Psi_B^D(k) \right\rangle = \left\langle C_B^D (J_B^D)^{N-k} \right\rangle.$$

Then, following similar lines to the the proof of Theorem 15, we have the following.

Theorem 17. *The generators of B_B^D lie in $B_D - X$ and*

$$\dim B_B^D = \mu := \textit{total sum of the degrees of the infinite elementary divisors}.$$

Any other infinite spectral pair which corresponds to infinite elementary divisors will also define an isomorphic space to B_B^D.

Proposition 18 [9]. *Let $A(\sigma) = A_0 + A_1\sigma + \cdots + A_q\sigma^q \in R[s]^{p\times m}$ and $U_L(\sigma) \in R_{pr}(\sigma)^{p\times p}$, $U_R(\sigma) \in R_{pr}(\sigma)^{m\times m}$ biproper matrices and such that*

$$U_L(\sigma)A(\sigma)U_R(\sigma) = S_{A(\sigma)}^{\infty}(\sigma) = \text{blockdiag}\left[\sigma^{q_1}, \ldots, \sigma^{q_k}, \frac{1}{\sigma^{\hat{q}_{k+1}}}, \ldots, \frac{1}{\sigma^{\hat{q}_r}}, 0_{p-r,m-r}\right].$$

Then the infinite elementary divisors of $A(\sigma)$ are given by

$$\sigma^{\mu_j}, \quad j = 1, 2, 3, \ldots, r,$$
$$\mu_j = q - q_j, \quad j = 1, 2, \ldots, k,$$
$$\mu_j = q + \hat{q}_j, \quad j = k+1, k+2, \ldots, r.$$

Therefore the elementary divisors are actually separated into two kinds. The first kind has order less than q and gives us complete information about the infinite pole structure of $A(\sigma)$ i.e. q_j, and we call them infinite pole infinite elementary divisors. The second kind has order greater than q and gives us complete information about the infinite zero structure of $A(\sigma)$ i.e. $\hat{q}_j$, and we call them infinite zero infinite elementary divisors.

Remark 19. It is easily seen due to the Proposition 18 and (18) that the infinite pole infinite elementary divisors produce solutions that belong to the space of final conditions $[N - q + 1, N]$, while the infinite zero infinite elementary divisors produce solutions that excess the discrete time space $[N - q + 1, N]$. Similar results also apply for the finite elementary divisors of $A(\sigma)$ at $\sigma = 0$ and concern the forward solutions defined in Theorem 15.

Example 20. Consider the AR-representation of Example 16. Then, according to Example 7 there exist matrices $\tilde{U}_L(\sigma)$ and $\tilde{U}_R(\sigma)$ having no poles and zeros at $\sigma = 0$ such that

$$\underbrace{\begin{bmatrix} \sigma^3 & 1 & \sigma^2+\sigma^3 \\ \sigma^4 & \sigma & \sigma^3+\sigma^4 \\ 0 & \sigma^3+\sigma^4 & 0 \end{bmatrix}}_{\tilde{A}(\sigma)} \underbrace{\begin{bmatrix} 0 & -1 & 1+\sigma \\ 1 & -\sigma^2 & 0 \\ 0 & 1 & -\sigma \end{bmatrix}}_{\tilde{U}_R(\sigma)} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ \sigma & 0 & -1 \\ \sigma^3+\sigma^4 & -\sigma-1 & 0 \end{bmatrix}}_{\tilde{U}_L^{-1}(\sigma)} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & \sigma^5 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{S^0_{\tilde{A}(\sigma)}(\sigma)},$$

where $\tilde{A}(\sigma)$ is the dual polynomial matrix of $A(\sigma)$, and therefore $\mu_2 = 5$. Define

$$x_{2,0} = \frac{1}{0!}\tilde{u}_2(0) = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}; \quad x_{2,1} = \frac{1}{1!}\tilde{u}_2^{(1)}(0) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix};$$

$$x_{2,2} = \frac{1}{2!}\tilde{u}_2^{(2)}(0) = \begin{bmatrix} 0 \\ -2 \\ 0 \end{bmatrix}; \quad x_{2,3} = \frac{1}{3!}\tilde{u}_2^{(3)}(0) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix};$$

$$x_{2,4} = \frac{1}{4!}\tilde{u}_2^{(4)}(0) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

and

$$C_B^D := \begin{bmatrix} x_{2,0} & x_{2,1} & \cdots & x_{2,4} \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$J_B^D := \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Thus

$$\begin{aligned} \Psi_B^D(k) &:= \begin{bmatrix} \xi_{2,0}^B(k) & \xi_{2,1}^B(k) & \cdots & \xi_{2,4}^B(k) \end{bmatrix} = C_B^D (J_B^D)^{N-k} \\ &= \begin{bmatrix} -\delta(N-k) & -\delta(N-(k-1)) & -\delta(N-(k-2)) & -\delta(N-(k-3) & -\delta(N-(k-4)) \\ 0 & 0 & -2\delta(N-k) & -2\delta(N-(k-1)) & -2\delta(N-(k-2)) \\ \delta(N-k) & \delta(N-(k-1)) & \delta(N-(k-2)) & \delta(N-(k-3)) & \delta(N-(k-4)) \end{bmatrix} \end{aligned}$$

and therefore

$$B_B^D := \left\langle \Psi_B^D(k) \right\rangle$$

with $\dim B_B^D = \mu = 5$.

3.3. Right minimal indices and solutions of discrete time AR-representations

$A(\sigma) \in R[\sigma]^{p\times m}$ according to our assumption has rank $r \leqslant \min\{p, m\}$ and therefore the dimension of the right null space of $A(\sigma)$ is equal to $m - r$. Consider a minimal polynomial base[2] of the right null space of $A(\sigma)$, let

$$\begin{bmatrix} \varepsilon_{r+1}(\sigma) & \varepsilon_{r+2}(\sigma) & \cdots & \varepsilon_m(\sigma) \end{bmatrix}. \tag{19}$$

Let also

$$x_{j,i} := \frac{1}{i!}\varepsilon_j^{(i)}(0) \quad i = 0, 1, \ldots, \varepsilon_i - 1 \text{ and } j = r+1, r+2, \ldots, m.$$

Define the vector valued functions

$$\begin{aligned} &\xi_{j,i}^F(k) := \delta(k)x_{j,i} + \delta(k-1)x_{j,i-1} + \cdots + \delta(k-i)x_{j,0} \\ &\quad i = 0, 1, \ldots, \varepsilon_j - 1 \text{ and } j = r+1, r+2, \ldots, m. \end{aligned} \tag{20}$$

Let

$$\Psi_j^F(k) := \begin{bmatrix} \xi_{j,0}^F(k) & \xi_{j,1}^F(k) & \cdots & \xi_{j,\varepsilon_j-1}^F(k) \end{bmatrix},$$

$$C_j^F := \begin{bmatrix} x_{j,0} & x_{j,1} & \cdots & x_{j,\varepsilon_j-1} \end{bmatrix},$$

$$J_j^F := \begin{bmatrix} 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix} \in R^{\varepsilon_j \times \varepsilon_j},$$

[2] The last $m - r$ columns of the transforming matrix $U_R(\sigma)$ defined in Section 3.1, constitute a basis of the right kernel of $A(\sigma)$. Under certain unimodular transformations i.e. column reduceness, we may do the above basis minimal (see [7]).

where $j = r+1, r+2, \ldots, m$, and

$$\Psi_F^{\varepsilon}(k) := \begin{bmatrix} \Psi_{r+1}^F(k) & \Psi_{r+2}^F(k) & \cdots & \Psi_m^F(k) \end{bmatrix},$$
$$C_F^{\varepsilon} := \begin{bmatrix} C_{r+1}^F(k) & C_{r+2}^F(k) & \cdots & C_m^F(k) \end{bmatrix},$$
$$J_F^{\varepsilon} := \text{blockdiag}\begin{bmatrix} J_{r+1}^F(k) & J_{r+2}^F(k) & \cdots & J_m^F(k) \end{bmatrix},$$

where

$$\varepsilon := \sum_{j=r+1}^{m} \varepsilon_i .$$

Denote the space B_F^{ε} that is spanned by the columns of the matrix $\Psi_F^{\varepsilon}(k)$ i.e.

$$B_F^{\varepsilon} = \left\langle \Psi_F^{\varepsilon}(k) \right\rangle = \left\langle C_F^{\varepsilon}(J_F^{\varepsilon})^k \right\rangle .$$

Then, following similar lines to the proof of Theorem 15, we have the following.

Theorem 21. *The generators of B_F^{ε} lie in $B_D - X$ and*

$$\dim B_F^{\varepsilon} = \varepsilon := \textit{total sum of the right minimal indices.}$$

Consider the dual minimal polynomial base of (19). It can be easily proved that it constitutes a minimal bases of the right null space of dual polynomial matrix $\tilde{A}(\sigma)$ of $A(\sigma)$, let

$$\begin{bmatrix} \tilde{\varepsilon}_{r+1}(\sigma) & \tilde{\varepsilon}_{r+2}(\sigma) & \cdots & \tilde{\varepsilon}_m(\sigma) \end{bmatrix}.$$

The greatest degrees of the columns $\tilde{u}_i(\sigma)$, $i = r+1, r+2, \ldots, m$ are the same as the right minimal indices of $A(\sigma)$ i.e. $\{\varepsilon_{r+1}, \varepsilon_{r+2}, \ldots, \varepsilon_m\}$.[3] Let also

$$x_{j,i} := \frac{1}{i!}\tilde{\varepsilon}_j^{(i)}(0) \quad i = 0, 1, \ldots, \varepsilon_i - 1 \text{ and } j = r+1, r+2, \ldots, m.$$

Define the vector valued functions

$$\xi_{j,i}^B(k) := \delta(N-k)x_{j,i} + \delta(N-(k-1))x_{j,i-1} + \cdots + \delta(N-(k-i))x_{j,0}$$
$$i = 0, 1, \ldots, \varepsilon_j - 1 \text{ and } j = r+1, r+2, \ldots, m.$$

[3] The vectors $u_i(\sigma)$ $i = r+1, \ldots, m$ are linearly independent and thus, its values at $s = 0$ i.e. $u_i(0)$ $i = r+1, \ldots, m$, are also linearly independent. Therefore the leading coefficient matrices of the vectors $\tilde{u}_i(\sigma)$ of the dual polynomial basis are not zero and the vectors $\tilde{u}_i(\sigma)$ have the same degrees as the ones of the right minimal polynomial basis of $A(\sigma)$ i.e $u_i(\sigma)$.

Let

$$\Psi_j^B(k) := \left[\xi_{j,0}^B(k) \quad \xi_{j,1}^B(k) \quad \cdots \quad \xi_{j,\varepsilon_j-1}^B(k)\right],$$

$$C_j^B := \left[x_{j,0} \quad x_{j,1} \quad \cdots \quad x_{j,\varepsilon_j-1}\right],$$

$$J_j^B := \begin{bmatrix} 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix} \in R^{\varepsilon_j \times \varepsilon_j} \quad j = r+1, r+2, \ldots, m$$

and

$$\Psi_B^\varepsilon(k) := \left[\Psi_{r+1}^B(k) \quad \Psi_{r+2}^B(k) \quad \cdots \quad \Psi_m^B(k)\right],$$

$$C_B^\varepsilon := \left[C_{r+1}^B(k) \quad C_{r+2}^B(k) \quad \cdots \quad C_m^B(k)\right],$$

$$J_B^\varepsilon := \text{blockdiag}\left[J_{r+1}^B(k) \quad J_{r+2}^B(k) \quad \cdots \quad J_m^B(k)\right].$$

Denote the space B_B^ε that is spanned by the columns of the matrix $\Psi_B^\varepsilon(k)$ i.e.

$$B_B^\varepsilon = \left\langle \Psi_B^\varepsilon(k) \right\rangle = \left\langle C_B^\varepsilon (J_B^\varepsilon)^{N-k} \right\rangle.$$

Then, following similar lines to the proof of Theorem 15, we have the following.

Theorem 22. *The generators of B_B^ε lie in $B_D - X$ and*

$$\dim B_B^\varepsilon = \varepsilon := \textit{total sum of the right minimal indices.}$$

Example 23. Consider the AR-representation of Example 16. Then, from Example 7 we have that the vector

$$\varepsilon_3(\sigma) = \begin{bmatrix} -1-\sigma \\ 0 \\ 1 \end{bmatrix}$$

constitutes a minimal bases of the right kernel of $A(\sigma)$ and $\varepsilon_3 = 1$. Let

$$x_{3,0} := \frac{1}{0!}\varepsilon_3(0) = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

Define the vector valued function

$$\xi_{3,0}^F(k) := \delta(k)x_{3,0} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \delta(k).$$

The columns of the following matrix

$$\Psi_F^\varepsilon(k) := [\xi_{3,0}^F(k)] = \left[\underbrace{\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}}_{C_F^\varepsilon} \underbrace{[0]}_{J_F^\varepsilon}{}^{k} \right]$$

produce the vector space B_F^ε with dimension $\dim B_F^\varepsilon = 1$. Consider the dual minimal polynomial base

$$\left[\underbrace{\begin{bmatrix} -\sigma - 1 \\ 0 \\ \sigma \end{bmatrix}}_{\tilde{\varepsilon}_3(\sigma)} \right]$$

of $\tilde{A}(\sigma)$. Let also

$$x_{3,0} := \frac{1}{0!}\tilde{\varepsilon}_3(0) = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}.$$

Define the vector valued function

$$\xi_{3,0}^B(k) := \delta(N-k)x_{3,0} = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} \delta(N-k).$$

The columns of the following matrix

$$\Psi_B^\varepsilon(k) := [\xi_{3,0}^B(k)] = \left[\underbrace{\begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}}_{C_B^\varepsilon} \underbrace{[0]}_{J_B^\varepsilon}{}^{N-k} \right]$$

produce the vector space B_B^ε with dimension $\dim B_B^\varepsilon = 1$. Note also that the columns of the matrix

$$U := \begin{bmatrix} \varepsilon_{3,0} & 0 & \cdots & 0 \\ \varepsilon_{3,1} & \varepsilon_{3,0} & \cdots & 0 \\ 0 & \varepsilon_{3,1} & \ddots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \varepsilon_{3,1} \end{bmatrix} = \begin{bmatrix} -1 & 0 & & 0 \\ 0 & 0 & \cdots & 0 \\ 1 & 0 & & 0 \\ -1 & -1 & & 0 \\ 0 & 0 & \cdots & 0 \\ 0 & 1 & & 0 \\ 0 & -1 & & \\ 0 & 0 & \ddots & \vdots \\ 0 & 0 & & \\ & & & -1 \\ \vdots & \vdots & \ddots & 0 \\ & & & 1 \\ 0 & 0 & & -1 \\ 0 & 0 & \cdots & 0 \\ 0 & 0 & & 0 \end{bmatrix} \in R^{(N+1)m \times N}$$

constitute a basis of the space X.

Therefore, we conclude that the right minimal indices give rise to ε linearly independent forward solutions and ε linearly independent backward solutions which do not belong to X.

Remark 24. It is easily seen, due to (20), that the right minimal indices that are less than q produce forward solutions that belong to the space of initial conditions $[0, q-1]$ and backward solutions that belong to the space of final conditions $[N-q+1, N]$, while the right minimal indices that are greater than q produce forward and backward solutions that excess the discrete time space of initial conditions $[0, q-1]$ or final conditions $[N-q+1, N]$.

3.4. Construction of the whole behavior space

We have proved in Theorem 13 that the dimension of $\hat{B}_D$ is equal to $n+\mu+2\varepsilon$. In the last three subsections we have constructed the solution vector spaces B_F^D, B_B^D, B_F^ε and B_B^ε which are due to the finite and infinite elementary divisors and the right kernel of $A(\sigma)$. The total number of the generators of the above spaces is equal to $n+\mu+2\varepsilon$ which is the same as the dimension of $\hat{B}_D$. The next theorem shows that the solution vector spaces B_F^D, B_B^D, B_F^ε and B_B^ε are not interconnected and, therefore, their direct sum is equal to the space $\hat{B}_D$.

Theorem 25. *The vector space $\hat{B}_D := B_D/X$ coincides with the direct sum of the vector spaces B_F^D, B_F^D, B_B^D and B_B^ε i.e.*

$$\hat{B}_D := B_D/X = B_F^D \oplus B_F^D \oplus B_B^D \oplus B_B^\varepsilon.$$

Proof. In the previous three subsections we have constructed a set of $n+\mu+2\varepsilon$ solution vectors of the space $B_D - X$ let $\xi_i(k)$. Due to the above construction we can show (following similar lines to [7]) that these vectors are linearly independent. The proof is based on the fact that the proposed solution vectors are constructed from the linearly independent columns of the unimodular matrix $U_R(\sigma)$ and thus the values of $U_R(\sigma)$ at $\sigma=0$ are linearly independent. However, in order to avoid all these technicalities we leave the proof. Therefore, the vectors $[\xi_i(k)] \in B_D/X$ span the vector space $\hat{B}_D$ which verifies the proof. □

Let also

$$C_F := \begin{bmatrix} C_F^D & C_F^\varepsilon \end{bmatrix}; \quad J_F := \begin{bmatrix} J_F^D & 0 \\ 0 & J_F^\varepsilon \end{bmatrix},$$

$$C_B := \begin{bmatrix} C_B^D & C_B^\varepsilon \end{bmatrix}; \quad J_B := \begin{bmatrix} J_B^D & 0 \\ 0 & J_B^\varepsilon \end{bmatrix}.$$

Then, an interesting conclusion of the above theorem is that

$$B_D = \left\{ \begin{array}{c} \xi(k) := \begin{bmatrix} C_F & C_B \end{bmatrix} \begin{bmatrix} J_F^k & 0 \\ 0 & J_B^{N-k} \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} + x(k) \\ x(k) := \sum_{i=0}^{k} \hat{\varepsilon}_{r+1}(i) z_1(k-i) + \sum_{i=0}^{k} \hat{\varepsilon}_{r+2}(i) z_2(k-i) \\ + \cdots + \sum_{i=0}^{k} \hat{\varepsilon}_m(i) z_{m-r}(k-i) \end{array} \right\}. \tag{21}$$

Example 26. Consider the AR-representation of Example 16. Then

$$C_F = \begin{bmatrix} C_F^D & C_F^\varepsilon \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad J_F = \begin{bmatrix} J_F^D & 0 \\ 0 & J_F^\varepsilon \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix},$$

$$C_B = \begin{bmatrix} C_B^D & C_B^\varepsilon \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & -2 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

and

$$J_B = \begin{bmatrix} J_B^D & 0 \\ 0 & J_B^\varepsilon \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Thus

$$
\begin{aligned}
B_D &= \left\{ \begin{array}{l} \xi(k) : \xi(k) := \begin{bmatrix} C_F & C_B \end{bmatrix} \begin{bmatrix} J_F^k & 0 \\ 0 & J_B^{N-k} \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} \\ \qquad + \sum_{i=0}^{k} \begin{bmatrix} -\delta(i-1) - \delta(i) \\ 0 \\ \delta(i-1) \end{bmatrix} z(k-i) \end{array} \right\} \\
&= \left\{ \begin{array}{l} \xi(k) : \xi(k) := \begin{bmatrix} 1 & -1 & -1 & 0 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & -2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\
\times \begin{bmatrix} (-1)^k & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \delta(k) & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \delta(N-k) & \delta(N-(k-1)) & \delta(N-(k-2)) & \delta(N-(k-3)) & \delta(N-(k-4)) & 0 \\ 0 & 0 & 0 & \delta(N-k) & \delta(N-(k-1)) & \delta(N-(k-2)) & \delta(N-(k-3)) & 0 \\ 0 & 0 & 0 & 0 & \delta(N-k) & \delta(N-(k-1)) & \delta(N-(k-2)) & 0 \\ 0 & 0 & 0 & 0 & 0 & \delta(N-k) & \delta(N-(k-1)) & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \delta(N-k) & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \delta(N-k) \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \\ b_6 \end{bmatrix} + \sum_{i=0}^{k} \begin{bmatrix} -\delta(i-1) - \delta(i) \\ 0 \\ \delta(i-1) \end{bmatrix} z(k-i) \end{array} \right\} \\
&= \left\{ \begin{array}{l} \begin{bmatrix} (-1)^k a_1 - \delta(k) a_2 - \delta(N-k) b_1 - \delta(N-k+1) b_2 - \delta(N-k+2) b_3 \\ (-1)^k a_1 - 2\delta(N-k) b_3 - 2\delta(N-k+1) b_4 - 2\delta(N-k+2) b_5 \\ \delta(k) a_2 + \delta(N-k) b_1 + \delta(N-k+1) b_2 + \delta(N-k+2) b_3 \end{bmatrix} \\ \quad + \begin{bmatrix} -\delta(N-k+3) b_4 - \delta(N-k+4) b_5 - \delta(N-k) b_6 \\ 0 \\ \delta(N-k+3) b_4 + \delta(N-k+4) b_5 \end{bmatrix} \\ \qquad + \begin{bmatrix} -z(k-1) - z(k) \\ 0 \\ z(k-1) \end{bmatrix} \end{array} \right\}
\end{aligned}
$$

while

$$
\hat{B}_D = \left\langle \begin{array}{c} \left[\begin{bmatrix} (-1)^k \\ (-1)^k \\ 0 \end{bmatrix}\right], \left[\begin{bmatrix} -\delta(k) \\ 0 \\ \delta(k) \end{bmatrix}\right], \left[\begin{bmatrix} -\delta(N-k) \\ 0 \\ \delta(N-k) \end{bmatrix}\right], \\ \left[\begin{bmatrix} -\delta(N-k+1) \\ 0 \\ \delta(N-k+1) \end{bmatrix}\right], \left[\begin{bmatrix} -\delta(N-k+2) \\ -2\delta(N-k) \\ \delta(N-k+2) \end{bmatrix}\right], \\ \left[\begin{bmatrix} -\delta(N-k+3) \\ -2\delta(N-k+1) \\ \delta(N-k+3) \end{bmatrix}\right], \left[\begin{bmatrix} -\delta(N-k+4) \\ -2\delta(N-k+2) \\ \delta(N-k+4) \end{bmatrix}\right], \left[\begin{bmatrix} -\delta(N-k) \\ 0 \\ 0 \end{bmatrix}\right] \end{array} \right\rangle
$$

4. Admissible boundary values of discrete time AR-representations

In the case where $A(\sigma)$ is regular, [1] has shown that although the initial–final condition vector space has dimension $2qr$, the *admissible* initial–final condition vector space has dimension qr and thus qr is the total number of arbitrary assigned values, distributed at both end points of the time interval. In this section, we study the nonregular case and more explicitly the role that the left null space plays to the boundary conditions of the system. More analytically, let a minimal polynomial basis of the left kernel of $A(\sigma)$ be

$$\begin{bmatrix} \eta_{r+1}(\sigma) & \eta_{r+2}(\sigma) & \cdots & \eta_p(\sigma) \end{bmatrix},$$

where

$$\eta_i(\sigma) = \eta_{i,0} + \eta_{i,1}\sigma + \cdots + \eta_{i,\eta_i}\sigma^{\eta_i}$$

and $\{\eta_{r+1}, \eta_{r+2}, \ldots, \eta_p\}$ be the left minimal indices of $A(\sigma)$. Firstly we give some necessary conditions for the existence of a solution due to the left kernel of $A(\sigma)$.

Theorem 27. *A necessary condition for the AR-representation* (1), *with given initial–final conditions, to have a solution is that the following* $\eta = \eta_{r+1} + \eta_{r+2} + \cdots + \eta_p$ *linearly independent conditions between the initial conditions*

$$\begin{bmatrix} \eta_{i,\eta_i} & 0 & \cdots & \vdots \\ \eta_{i,\eta_i-1} & \eta_{i,\eta_i} & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ \eta_{i,1} & \eta_{i,2} & \cdots & \eta_{i,\eta_i} \end{bmatrix} \begin{bmatrix} A_0 & A_1 & \cdots & A_{q-1} \\ 0 & A_0 & \cdots & A_{q-2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_0 \end{bmatrix} \begin{bmatrix} \xi(0) \\ \xi(1) \\ \vdots \\ \xi(q-1) \end{bmatrix} = 0 \tag{22}$$

and η *linearly independent conditions between the final conditions*

$$\begin{bmatrix} \eta_{i,0} & 0 & \cdots & 0 \\ \eta_{i,1} & \eta_{i,0} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \eta_{i,\eta_i-1} & \eta_{i,\eta_i-2} & \cdots & \eta_{i,0} \end{bmatrix} \begin{bmatrix} A_q & A_{q-1} & \cdots & A_1 \\ 0 & A_q & \cdots & A_2 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_q \end{bmatrix} \begin{bmatrix} \xi(N) \\ \xi(N-1) \\ \vdots \\ \xi(N-q+1) \end{bmatrix} = 0 \tag{23}$$

are satisfied.

Proof. The proof is divided into two parts. In the first part we show that the AR-representation (1) has a solution if the constraints in (22) and (23) are satisfied, while in the second part we show that all these constraints are linearly independent.

(a) It is easily seen from (4) that a necessary and sufficient condition so that the system (1) would have a solution is the following:

$$\operatorname{rank}_{R(z)}\left[A(z) \mid \mathscr{Z}_q\Re_q\tilde{\xi}(0) + \mathscr{Z}_N\Re_0\tilde{\xi}(N)\right] = \operatorname{rank}_{R(z)}[A(z)]. \tag{24}$$

The geometrical meaning of the condition (24) is that the space W which is spanned by the columns of $A(z)$ is exactly the same as the space Z which is spanned by the columns of $A(z)$ and the initial value vector $\mathscr{Z}_q\Re_q\tilde{\xi}(0) + \mathscr{Z}_N\Re_0\tilde{\xi}(N)$. An equivalent condition is that their orthogonal spaces W^{T} and Z^{T} respectively are the same. Thus if

$$\eta_i(z) := \eta_{i,0} + \eta_{i,1}z + \cdots + \eta_{i,\eta_i}z^{\eta_i} \in R[z]^{1\times p}$$

is a polynomial vector of the left kernel of $A(z)$, then by premultiplying both sides of (4) with $\eta_i(z) \in R[z]^{1\times p}$ we obtain that

$$\begin{aligned}
&\eta_i(z)A(z)\xi(z)\\
&\quad= \left[\eta_{i,0} + \eta_{i,1}z + \cdots + \eta_{i,\eta_i}z^{\eta_i}\right]\left[z^qI_p \quad \cdots \quad zI_p \quad I_p\right]\\
&\qquad\times \begin{bmatrix} A_q & 0 & \cdots & 0\\ A_{q-1} & A_q & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ A_1 & A_2 & \cdots & A_q \end{bmatrix}\begin{bmatrix} \xi(0)\\ \xi(1)\\ \vdots\\ \xi(q-1) \end{bmatrix}\\
&\qquad+ \left[\eta_{i,0} + \eta_{i,1}z + \cdots + \eta_{i,\eta_i}z^{\eta_i}\right]\left[z^{-N}I_p \;\cdots\; z^{-N+q-2}I_p \;\; z^{-N+q-1}I_p\right]\\
&\qquad\times \begin{bmatrix} A_0 & 0 & \cdots & 0\\ A_1 & A_0 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ A_{q-1} & A_{q-2} & \cdots & A_0 \end{bmatrix}\begin{bmatrix} \xi(N)\\ \xi(N-1)\\ \vdots\\ \xi(N-q+1) \end{bmatrix}.
\end{aligned}$$

We can easily see that the left part of the above equation is zero. Taking into account that $-N + q - 1 + \max\{\eta_i\} < 0 \Longrightarrow N > q - 1 + \max\{\eta_i\}$, we get

$$\begin{aligned}
&\left[\eta_{i,0} + \eta_{i,1}z + \cdots + \eta_{i,\eta_i}z^{\eta_i}\right]\left[z^qI_p \quad \cdots \quad z^2I_p \quad zI_p\right]\\
&\qquad\times \begin{bmatrix} A_q & 0 & \cdots & 0\\ A_{q-1} & A_q & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ A_1 & A_2 & \cdots & A_q \end{bmatrix}\begin{bmatrix} \xi(0)\\ \xi(1)\\ \vdots\\ \xi(q-1) \end{bmatrix}\\
&\quad= \left[\eta_{i,0}z^q + \eta_{i,1}z^{q+1} + \cdots + \eta_{i,\eta_i}z^{\eta_i+q} \;\cdots\; \eta_{i,0}z + \eta_{i,1}z^2 + \cdots + \eta_{i,\eta_i}z^{\eta_i+1}\right]\\
&\qquad\times \begin{bmatrix} A_q & 0 & \cdots & 0\\ A_{q-1} & A_q & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ A_1 & A_2 & \cdots & A_q \end{bmatrix}\begin{bmatrix} \xi(0)\\ \xi(1)\\ \vdots\\ \xi(q-1) \end{bmatrix}
\end{aligned}$$

$$= -\begin{bmatrix} z^{\eta_i} I_p & \cdots & z^2 I_p & z I_p \end{bmatrix} \begin{bmatrix} \eta_{i,\eta_i} & 0 & \cdots & \vdots \\ \eta_{i,\eta_i-1} & \eta_{i,\eta_i} & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ \eta_{i,1} & \eta_{i,2} & \cdots & \eta_{i,\eta_i} \end{bmatrix}$$

$$\times \begin{bmatrix} A_0 & A_1 & \cdots & A_{q-1} \\ 0 & A_0 & \cdots & A_{q-2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_0 \end{bmatrix} \begin{bmatrix} \xi(0) \\ \xi(1) \\ \vdots \\ \xi(q-1) \end{bmatrix}.$$

Similarly

$$\begin{bmatrix} \eta_{i,0} + \eta_{i,1} z + \cdots + \eta_{i,\eta_i} z^{\eta_i} \end{bmatrix} \begin{bmatrix} z^{-N} I_p & \cdots & z^{-N+q-2} I_p & z^{-N+q-1} I_p \end{bmatrix}$$

$$\times \begin{bmatrix} A_0 & 0 & \cdots & 0 \\ A_1 & A_0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_{q-1} & A_{q-2} & \cdots & A_0 \end{bmatrix} \begin{bmatrix} \xi(N) \\ \xi(N-1) \\ \vdots \\ \xi(N-q+1) \end{bmatrix}$$

$$= \big[\eta_{i,0} z^{-N} + \eta_{i,1} z^{-N+1} + \cdots + \eta_{i,\eta_i} z^{-N+\eta_i} \quad \cdots \quad \eta_{i,0} z^{-N+q-1} + \eta_{i,1} z^{-N+q} + \cdots + \eta_{i,\eta_i} z^{-N+q-1+\eta_i} \big]$$

$$\times \begin{bmatrix} A_0 & 0 & \cdots & 0 \\ A_1 & A_0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_{q-1} & A_{q-2} & \cdots & A_0 \end{bmatrix} \begin{bmatrix} \xi(N) \\ \xi(N-1) \\ \vdots \\ \xi(N-q+1) \end{bmatrix}$$

$$= \begin{bmatrix} z^{-N} & z^{-N+1} & \cdots & z^{-N+q-1+\eta_i} \end{bmatrix} \begin{bmatrix} \eta_{i,0} & 0 & \cdots & 0 \\ \eta_{i,1} & \eta_{i,0} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \eta_{i,\eta_i} & \eta_{i,\eta_i-1} & \cdots & \vdots \\ 0 & \eta_{i,\eta_i} & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \eta_{i,\eta_i} \end{bmatrix}$$

$$\times \begin{bmatrix} A_0 & 0 & \cdots & 0 \\ A_1 & A_0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_{q-1} & A_{q-2} & \cdots & A_0 \end{bmatrix} \begin{bmatrix} \xi(N) \\ \xi(N-1) \\ \vdots \\ \xi(N-q+1) \end{bmatrix}$$

$$
= -\begin{bmatrix} z^{-N} & z^{-N+1} & \cdots & z^{-N+q-1+\eta_i} \end{bmatrix} \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \eta_{i,0} & 0 & \cdots & \vdots \\ \eta_{i,1} & \eta_{i,0} & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ \eta_{i,\eta_i-1} & \eta_{i,\eta_i-2} & \cdots & \eta_{i,0} \end{bmatrix}
$$

$$
\times \begin{bmatrix} A_q & A_{q-1} & \cdots & A_1 \\ 0 & A_q & \cdots & A_2 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_q \end{bmatrix} \begin{bmatrix} \xi(N) \\ \xi(N-1) \\ \vdots \\ \xi(N-q+1) \end{bmatrix}
$$

$$
= -\begin{bmatrix} z^{-N+q-1} & z^{-N+q} & \cdots & z^{-N+q-1+\eta_i} \end{bmatrix} \begin{bmatrix} \eta_{i,0} & 0 & \cdots & 0 \\ \eta_{i,1} & \eta_{i,0} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \eta_{i,\eta_i-1} & \eta_{i,\eta_i-2} & \cdots & \eta_{i,0} \end{bmatrix}
$$

$$
\times \begin{bmatrix} A_q & A_{q-1} & \cdots & A_1 \\ 0 & A_q & \cdots & A_2 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_q \end{bmatrix} \begin{bmatrix} \xi(N) \\ \xi(N-1) \\ \vdots \\ \xi(N-q+1) \end{bmatrix}.
$$

Equating the powers of z to zero we get the necessary boundary conditions of the theorem.

(b) Following similar lines to the ones in [7] and with the help of the next corollary, we prove that the above conditions are linearly independent. □

Corollary 28. *Note that the condition* (23) *between the final conditions may be rewritten as follows:*

$$
\begin{bmatrix} \eta_{i,0} & 0 & \cdots & 0 \\ \eta_{i,1} & \eta_{i,0} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \eta_{i,\eta_i-1} & \eta_{i,\eta_i-2} & \cdots & \eta_{i,0} \end{bmatrix} \begin{bmatrix} A_q & A_{q-1} & \cdots & A_1 \\ 0 & A_q & \cdots & A_2 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_q \end{bmatrix} \begin{bmatrix} \xi(N) \\ \xi(N-1) \\ \vdots \\ \xi(N-q+1) \end{bmatrix} = 0
$$

$$
\Leftrightarrow \begin{bmatrix} \eta_{i,0} & 0 & \cdots & 0 \\ \eta_{i,1} & \eta_{i,0} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \eta_{i,\eta_i-1} & \eta_{i,\eta_i-2} & \cdots & \eta_{i,0} \end{bmatrix} \begin{bmatrix} A_1 & A_2 & \cdots & A_q \\ A_2 & A_3 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_q & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} \xi(N-q+1) \\ \xi(N-q+2) \\ \vdots \\ \xi(N) \end{bmatrix} = 0
$$

$$\Leftrightarrow \begin{bmatrix} \eta_{i,0}A_1 & \eta_{i,0}A_2 & \cdots & \eta_{i,0}A_q \\ \eta_{i,1}A_1 + \eta_{i,0}A_2 & \eta_{i,1}A_2 + \eta_{i,0}A_3 & \cdots & \eta_{i,1}A_q \\ \vdots & \vdots & \ddots & \vdots \\ \eta_{i,\eta_i-1}A_1 + \cdots + \eta_{i,0}A_{\eta_i} & \eta_{i,\eta_i-1}A_2 + \cdots + \eta_{i,1}A_{\eta_i} & \cdots & \eta_{i,\eta_i-1}A_q \end{bmatrix}$$

$$\times \begin{bmatrix} \xi(N-q+1) \\ \xi(N-q+2) \\ \vdots \\ \xi(N) \end{bmatrix} = 0$$

$$\Leftrightarrow - \begin{bmatrix} \eta_{i,1}A_0 & \eta_{i,1}A_1 + \eta_{i,2}A_0 & \cdots & \eta_{i,1}A_{q-1} + \eta_{i,2}A_{q-2} + \cdots + \eta_{i,\eta_i}A_{q-\eta_i} \\ \eta_{i,2}A_0 & \eta_{i,2}A_1 + \eta_{i,3}A_0 & \cdots & \eta_{i,2}A_{q-1} + \eta_{i,3}A_{q-2} + \cdots + \eta_{i,\eta_i}A_{q-\eta_i+1} \\ \vdots & \vdots & \ddots & \vdots \\ \eta_{i,\eta_i}A_0 & \eta_{i,\eta_i}A_1 & \cdots & \eta_{i,\eta_i}A_{q-1} \end{bmatrix}$$

$$\times \begin{bmatrix} \xi(N-q+1) \\ \xi(N-q+2) \\ \vdots \\ \xi(N) \end{bmatrix} = 0 \Leftrightarrow \begin{bmatrix} \eta_{i,1} & \eta_{i,2} & \cdots & \eta_{i,\eta_i} \\ \eta_{i,2} & \eta_{i,3} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \eta_{i,\eta_i} & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} A_0 & A_1 & \cdots & A_{q-1} \\ 0 & A_0 & \cdots & A_{q-2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_0 \end{bmatrix}$$

$$\times \begin{bmatrix} \xi(N-q+1) \\ \xi(N-q+2) \\ \vdots \\ \xi(N) \end{bmatrix} = 0 \Leftrightarrow \begin{bmatrix} \eta_{i,\eta_i} & 0 & \cdots & \vdots \\ \eta_{i,\eta_i-1} & \eta_{i,\eta_i} & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ \eta_{i,1} & \eta_{i,2} & \cdots & \eta_{i,\eta_i} \end{bmatrix}$$

$$\times \begin{bmatrix} A_0 & A_1 & \cdots & A_{q-1} \\ 0 & A_0 & \cdots & A_{q-2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_0 \end{bmatrix} \begin{bmatrix} \xi(N) \\ \xi(N-1) \\ \vdots \\ \xi(N-q+1) \end{bmatrix} = 0. \tag{25}$$

We can easily check the similarities between the relations (22) *and* (25) *and therefore, we conclude that the same conditions are applying both for the initial and final conditions. Therefore, it is enough to prove in Theorem 27 that the first η conditions between the initial conditions are linearly independent* (*using the proof in* [7]) *and the same will hold for the final conditions* (*since the conditions are the same but between the final conditions*).

Example 29. Consider the AR-representation of Example 16. Then from Example 7 the polynomial vector

$$\eta_3(\sigma) = \begin{bmatrix} 1 & -\sigma & 0 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \end{bmatrix}}_{\eta_{3,0}} + \underbrace{\begin{bmatrix} 0 & -1 & 0 \end{bmatrix}}_{\eta_{3,1}}\sigma$$

constitutes a polynomial basis of the left kernel of $A(\sigma)$. The AR-representation, according to the above theorem, must satisfy the following $2\eta = 2\eta_3 = 2$ conditions between the initial–final conditions:

$$\begin{bmatrix}\eta_{3,1} & 0 & 0 & 0\end{bmatrix}\begin{bmatrix}A_0 & A_1 & A_2 & A_3\\ 0 & A_0 & A_1 & A_2\\ 0 & 0 & A_0 & A_1\\ 0 & 0 & 0 & A_0\end{bmatrix}\begin{bmatrix}\xi(0)\\ \xi(1)\\ \xi(2)\\ \xi(3)\end{bmatrix} = 0$$

$$\Leftrightarrow \eta_{3,1}A_0\xi(0) + \eta_{3,1}A_1\xi(1) + \eta_{3,1}A_2\xi(2) + \eta_{3,1}A_3\xi(3) = 0$$

$$\Leftrightarrow \begin{bmatrix}0 & -1 & 0\end{bmatrix}\begin{bmatrix}0 & 0 & 0\\ 1 & 0 & 1\\ 0 & 1 & 0\end{bmatrix}\begin{bmatrix}\xi_1(0)\\ \xi_2(0)\\ \xi_3(0)\end{bmatrix} + \begin{bmatrix}0 & -1 & 0\end{bmatrix}\begin{bmatrix}1 & 0 & 1\\ 0 & 0 & 1\\ 0 & 1 & 0\end{bmatrix}$$

$$\times \begin{bmatrix}\xi_1(1)\\ \xi_2(1)\\ \xi_3(1)\end{bmatrix} + \begin{bmatrix}0 & -1 & 0\end{bmatrix}\begin{bmatrix}0 & 0 & 1\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}\xi_1(2)\\ \xi_2(2)\\ \xi_3(2)\end{bmatrix} + \begin{bmatrix}0 & -1 & 0\end{bmatrix}$$

$$\times \begin{bmatrix}0 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}\xi_1(3)\\ \xi_2(3)\\ \xi_3(3)\end{bmatrix} = 0$$

$$\Leftrightarrow -\xi_1(0) - \xi_3(0) - \xi_3(1) - \xi_2(3) = 0$$

$$\begin{bmatrix}\eta_{3,0} & 0 & 0 & 0\end{bmatrix}\begin{bmatrix}A_4 & A_3 & A_2 & A_1\\ 0 & A_4 & A_3 & A_2\\ 0 & 0 & A_4 & A_3\\ 0 & 0 & 0 & A_4\end{bmatrix}\begin{bmatrix}\xi(N)\\ \xi(N-1)\\ \xi(N-2)\\ \xi(N-3)\end{bmatrix} = 0$$

$$\Leftrightarrow \eta_{3,0}A_4\xi(N) + \eta_{3,0}A_3\xi(N-1) + \eta_{3,0}A_2\xi(N-2) + \eta_{3,0}A_1\xi(N-3) = 0$$

$$\Leftrightarrow \begin{bmatrix}1 & 0 & 0\end{bmatrix}\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}\xi_1(N)\\ \xi_2(N)\\ \xi_3(N)\end{bmatrix}$$

$$+\begin{bmatrix}1 & 0 & 0\end{bmatrix}\begin{bmatrix}0 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}\xi_1(N-1)\\ \xi_2(N-1)\\ \xi_3(N-1)\end{bmatrix}$$

$$+\begin{bmatrix}1 & 0 & 0\end{bmatrix}\begin{bmatrix}0 & 0 & 1\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}\xi_1(N-2)\\ \xi_2(N-2)\\ \xi_3(N-2)\end{bmatrix}$$

$$+\begin{bmatrix}1 & 0 & 0\end{bmatrix}\begin{bmatrix}1 & 0 & 1\\ 0 & 0 & 1\\ 0 & 1 & 0\end{bmatrix}\begin{bmatrix}\xi_1(N-3)\\ \xi_2(N-3)\\ \xi_3(N-3)\end{bmatrix} = 0$$

$$\Leftrightarrow \xi_2(N) + \xi_3(N-2) + \xi_1(N-3) + \xi_3(N-3) = 0.$$

We have shown in (21), that every solution $\xi(k)$, $k = 0, 1, \ldots, n$ of (1) is given by

$$\xi(k) := \begin{bmatrix} C_F & C_B \end{bmatrix} \begin{bmatrix} J_F^k & 0 \\ 0 & J_B^{N-k} \end{bmatrix} \begin{bmatrix} \zeta_F \\ \zeta_B \end{bmatrix} + x(k)$$

for $k = 0, 1, \ldots, N$, for a specific vector $\zeta = [\zeta_F^{\mathrm{T}} \quad \zeta_B^{\mathrm{T}}]^{\mathrm{T}} \in R^{(n+\mu+2\varepsilon)}$. The proposed solution $\xi(k)$ must satisfy also the initial–final conditions. The initial condition vector is given by

$$\underbrace{\begin{bmatrix} \xi(0) \\ \xi(1) \\ \vdots \\ \xi(q-1) \end{bmatrix}}_{\tilde{\xi}(0)} = \underbrace{\begin{bmatrix} C_F & C_B J_B^N \\ C_F J_F & C_B J_B^{N-1} \\ \vdots & \vdots \\ C_F J_F^{q-1} & C_B J_B^{N-q+1} \end{bmatrix}}_{Q_0} \begin{bmatrix} \zeta_F \\ \zeta_B \end{bmatrix} + \underbrace{\begin{bmatrix} x(0) \\ x(1) \\ \vdots \\ x(q-1) \end{bmatrix}}_{\tilde{x}(0)},$$

where $\tilde{x}(0) \in \mathrm{Kernel}[\Re_q]$

while the final condition vector is given by

$$\underbrace{\begin{bmatrix} \xi(N) \\ \xi(N-1) \\ \vdots \\ \xi(N-q+1) \end{bmatrix}}_{\tilde{\xi}(N)} = \underbrace{\begin{bmatrix} C_F J_F^N & C_B \\ C_F J_F^{N-1} & C_B J_B \\ \vdots & \vdots \\ C_F J_F^{N-q+1} & C_B J_B^{q-1} \end{bmatrix}}_{Q_1} \begin{bmatrix} \zeta_F \\ \zeta_B \end{bmatrix} + \underbrace{\begin{bmatrix} x(N) \\ x(N-1) \\ \vdots \\ x(N-q+1) \end{bmatrix}}_{\tilde{x}(N)},$$

where $\tilde{x}(N) \in \mathrm{Kernel}[\Re_0]$.

Combining the above two relations we get

$$\begin{bmatrix} \tilde{\xi}(0) \\ \tilde{\xi}(N) \end{bmatrix} = \begin{bmatrix} Q_0 & 0 \\ 0 & Q_1 \end{bmatrix} \begin{bmatrix} \zeta_F \\ \zeta_B \end{bmatrix} + \begin{bmatrix} \tilde{x}(0) \\ \tilde{x}(N) \end{bmatrix}. \tag{26}$$

Therefore, (22), (25) and (26) provide us with *necessary* and *sufficient* conditions in order for the AR-representation (1) to have a solution. Moreover, we can easily prove that the matrix $Q = [Q_0^{\mathrm{T}} \quad Q_1^{\mathrm{T}}]^{\mathrm{T}} \in R^{2qm \times (n+\mu+2\varepsilon)}$ has full column rank (based on the fact that the solutions $\xi(k)$ defined by the spectral pairs $\left\{[C_F \quad C_B], \begin{bmatrix} J_F \\ J_B \end{bmatrix}\right\}$ are linearly independent, and following similar reasoning to the ones in [6] or by taking large enough $N - q + 1 > \mu + \varepsilon$). Therefore, the vector space

$$V_0 = \left\{ \begin{array}{c} v = \begin{bmatrix} \tilde{\xi}(0) \\ \tilde{\xi}(N) \end{bmatrix} - \begin{bmatrix} \tilde{x}(0) \\ \tilde{x}(N) \end{bmatrix} \in \mathrm{Im}\left(\begin{bmatrix} Q_0 & 0 \\ 0 & Q_1 \end{bmatrix} \right) \\ \tilde{\xi}(0), \tilde{\xi}(N) \text{ satisfying (22), (25)} \end{array} \right\} := \Xi_0 \oplus X_0$$

$$\text{with } X_0 = \left\{ \begin{bmatrix} \tilde{x}(0) \\ \tilde{x}(N) \end{bmatrix}, \tilde{x}(0) \in \mathrm{Kernel}[\Re_q],\ \tilde{x}(N) \in \mathrm{Kernel}[\Re_0] \right\}$$

has dimension equal to

$$\dim V_0 = 2qm - (n + \mu + 2\varepsilon) = 2q(m - r) + 2qr - (n + \mu + 2\varepsilon) = \stackrel{\text{Lemma 6}}{=} 2\{q(m - r) - \varepsilon\} + n + \mu + 2\varepsilon + 2\eta$$

We can easily prove, according to Theorem 16, that

$$\dim X_0 = 2\{q(m - r) - \varepsilon\}$$

and therefore, the possible dimension of Ξ_0 is $n + \mu + 2\varepsilon + 2\eta$. Taking into account Theorem 27, 2η linearly independent conditions must be satisfied among the vectors of Ξ_0 and thus

$$\dim \Xi_0 = n + \mu + 2\varepsilon.$$

Therefore, the initial–final condition vectors are chosen from the $n + \mu + 2\varepsilon$ −dimensional vector space Ξ_0 modulo the space X_0 or otherwise both the initial–final condition vector $\begin{bmatrix}\tilde{\xi}(0)^{\mathrm{T}} & \tilde{\xi}(N)^{\mathrm{T}}\end{bmatrix}^{\mathrm{T}} \in \Xi_0$ and $\begin{bmatrix}\tilde{\xi}(0)^{\mathrm{T}} & \tilde{\xi}(N)^{\mathrm{T}}\end{bmatrix}^{\mathrm{T}} \oplus \begin{bmatrix}\tilde{x}(0)^{\mathrm{T}} & \tilde{x}(N)^{\mathrm{T}}\end{bmatrix}^{\mathrm{T}} \in \Xi_0 \oplus X_0$ gives rise to the same solution class. The connection between $\dim \Xi_0$ and $\dim \hat{B}_D$ is obvious.

5. Conclusions

In this paper we have studied the solution vector space of discrete time nonregular AR-representations over a finite time interval and, thus, extending the results presented in [1,6]. More specifically, we have shown that the solution space of discrete time nonregular AR-representations over a finite time interval is divided into equivalence classes, where each equivalence class represents the whole number of solutions of the AR-representation under certain boundary values (initial–final conditions). The dimension of the new equivalence class space has been determined and is shown to be equal to the sum of the degrees of the finite and infinite elementary divisors plus two times the sum of the right minimal indices (order accounted for) of the polynomial matrix that describes the AR-representation. An algorithm for the construction of the forward and backward solution space of a nonregular discrete time AR-representations over a finite time interval is given in terms of the structural invariants of the polynomial matrix which describe the system. Finally, we have shown that the left kernel of the polynomial matrix which describes the system plays a crucial role in the existence of solution on the boundary condition problem. The meaning of the algebraic structure of a polynomial matrix in relation to the solution vector space of nonregular discrete time AR-representations has thus been elucidated.

The investigation of the solution vector space of discrete nonregular AR-representations gives rise to numerous applications, such as the solution of the zeroing-output problem, the determination of the controllable or uncontrollable and observable or unobservable subspaces of discrete polynomial matrix descriptions etc.

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