

A new notion of equivalence for discrete time AR representations

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We present a new equivalence transformation termed *divisor equivalence*, that has the property of preserving both the finite and the infinite elementary divisor structures of a square non-singular polynomial matrix. This equivalence relation extends the known notion of strict equivalence, which dealt only with matrix pencils, to the general polynomial matrix case. It is proved that divisor equivalence characterizes in a closed form relation the equivalence classes of polynomial matrices that give rise to fundamentally equivalent discrete time auto-regressive representations.

1. Introduction

The problem of equivalence of polynomial matrices has been studied extensively. The primary target of these studies was the preservation of the finite elementary divisors structure of the polynomial matrices involved in the equivalence transformations examined. An equivalence relation for matrix pencils, termed *strict equivalence*, was initially introduced in Gantmacher (1959) where it was shown to have the property of preserving the finite elementary divisor structure of (strictly) equivalent pencils. Strict equivalence of matrix pencils has been extended to the general polynomial matrix case, in Rosenbrock (1970) by *unimodular equivalence*. However, both *strict equivalence* and *unimodular equivalence* can only be applied to matrices of the same dimension. In Pugh and Shelton (1978), *extended unimodular equivalence* was introduced as a closed form relation between polynomial matrices of possibly different dimensions, preserving the finite elementary divisor structure of the polynomial matrices involved.

Infinite elementary divisors (IEDs) of matrix pencils were initially defined in Gantmacher (1959) where it was shown that IEDs remain invariant under the transformation of strict equivalence. Furthermore, existing work on strict equivalence of matrix pencils makes no distinction between the preservation of their infinite zeros and infinite elementary divisors structures, since in the case of matrix pencils, the orders of the IEDs are related to the orders of zeros at infinity by the ‘plus one property’ (Vardulakis and Karcanias 1983). *Complete equivalence* is proposed in Pugh *et al.* (1987) for matrix pencils of possibly different dimensions, which preserves both finite and infinite zero (elementary divisor) structures of matrix pencils. Hayton *et al.* (1988) extended the

definition of the IEDs to the polynomial matrix case and showed that the IED structure gives a complete description of the total pole-zero structure at infinity of a polynomial matrix and not simply that associated with the zeros at infinity. Thus, although *full equivalence* presented later in Hayton *et al.* (1990) preserves the infinite zero structure of polynomial matrices, it does not preserve the IED structure, apart from the special class of polynomial matrices having the same order. *Strict equivalence* transformation of matrix pencils was generalized for the case of regular, i.e. square and non-singular, polynomial matrices in Vardulakis and Antoniou (2003) where it was shown to preserve both the FED and IED structures of the polynomial matrices involved. This notion of equivalence characterizes strict equivalent regular polynomial matrices through their first-order representations using strict equivalence of matrix pencils (Gantmacher 1959), providing no closed form equivalence relation, something that is its main drawback. Many authors applied the algebraic results of the above studies in order to define equivalence relations between state space and descriptor systems in both the continuous and discrete time cases. In this paper we present a *closed form equivalence relation* between polynomial matrices of possibly different degrees and dimensions, which leaves invariants both the finite and infinite elementary divisor structures, thus extending in these terms the notion of strict equivalence of matrix pencils. This work was motivated by the study of discrete time autoregressive representations (DTARR), where the finite and infinite elementary divisor structure of the polynomial matrix describing a DTARR was proved to be of crucial importance (Lewis 1984, Antoniou *et al.* 1998, Karampetakis 2004).

Consider a linear, homogeneous, matrix difference equation

$$A(\sigma)\beta(k) = 0, \quad k \in [0, N] \quad (1)$$

$$A(\sigma) = A_q\sigma^q + A_{q-1}\sigma^{q-1} + \dots + A_0 \in \mathbb{R}[\sigma]^{r \times r} \quad (2)$$

where $A_i \in \mathbb{R}^{r \times r}$, $i = 0, 1, \dots, q$, σ denotes the forward shift operator: $\sigma\beta(k) = \beta(k + 1)$ and $\beta(k): [0, N] \rightarrow \mathbb{R}^r$ is

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a vector valued sequence. Following the terminology of Willems (1986, 1989, 1991), we call the set of equations (1) an AR-representation of $B_{A(\sigma)}$ (behaviour) where $B_{A(\sigma)}$ is defined as

$$B_{A(\sigma)} := \{ \beta(k): [0, N] \rightarrow \mathbb{R}^r: (1) \text{ is satisfied } \forall k \in [0, N] \}.$$

In the particular case when the polynomial matrix $A(\sigma)$ is a matrix pencil $A(\sigma) = \sigma E - A$, the discrete time AR representations exhibit non-causal behaviour and the natural framework for their study appears to be over a finite time interval $[0, N]$ where the system can be essentially decomposed into a purely causal and a purely anticausal part (see Luenberger 1977, 1989, Lewis 1984, 1986, Nikoukhah *et al.* 1987, Lewis and Mertzios 1990). The causal and anticausal behaviours of such systems have been proved to be associated respectively with the finite and infinite elementary divisor structure of the polynomial matrix $A(\sigma)$ describing the system (see Lewis (1984) for descriptor systems or Antoniou *et al.* (1998) for higher-order systems). In the more general case when the matrix pencil is singular, then its causal and anticausal behaviour is due to the FED, IED and the right minimal indices of the pencil (Antoniou *et al.* 1997). The causal behaviour of (1) for the case when $A(\sigma)$ is a regular polynomial matrix has been extensively studied in Gohberg *et al.* (1982). These results have been extended in Antoniou *et al.* (1998), where both causal and anticausal behaviour has been studied. The more general case, where $A(\sigma)$ is a non-regular polynomial matrix, has been studied in Karampetakis (2004), where it is shown that additionally the right null space of the polynomial matrix $A(\sigma)$ plays a crucial role in the causal and anticausal behaviour of (1).

Vardulakis and Antoniou (2003) defined two AR representations as *fundamentally equivalent* (FE) if their solution spaces or behaviours (both causal and anticausal) are isomorphic in a particular way. Motivated by the fact that the behaviour of the AR representation (1), when considered over a *finite* time interval $[0, N]$, depends on the algebraic structure of both the *finite* and the *infinite elementary divisors* of the polynomial matrix $A(\sigma)$ associated with (1), they showed that this structure is identical to the corresponding structure of a block companion matrix pencil $\sigma \bar{E} - \bar{A} \in \mathbb{R}[\sigma]^{r_q \times r_q}$ which constitutes a *linearization* of the polynomial matrix (Antoniou *et al.* 1998) and consequently the AR associated with $\sigma \bar{E} - \bar{A}$ constitutes the natural first-order *fundamentally equivalent* representation (realization) of (1). Thus, they proposed a generalization of the concept of *strict equivalence* (SE) of regular matrix pencils (Gantmacher 1959) to the case of general non-singular polynomial matrices and showed that two AR representations described by regular polynomial matrices of possibly different degrees

and dimensions are *fundamentally equivalent* if and only if their fundamental equivalent matrix pencils are strict equivalent. This *fundamental equivalence* transformation proposed in Vardulakis and Antoniou (2003) generalizes the behaviour homomorphism presented in Fuhrmann (2002), for non-proper discrete time AR representations of the form (1) over a finite time interval. However the main disadvantage of this transformation is that it does not provide a closed formula relating the polynomial matrices that describe the *fundamentally equivalent* discrete time AR representations. Our main interest in this paper is to propose that missing closed formula by presenting a new equivalence transformation named *divisor equivalence*. We shall prove that divisor equivalence preserves both the finite and infinite elementary divisor structure of equivalent polynomial matrices and thus extends the *strict equivalence* presented in Gantmacher (1959) for matrix pencils to the general polynomial matrix case.

The paper is organized as follows. Section 2 provides the necessary mathematical background for the subsequent sections. In § 3 a generalization of strict equivalence to the polynomial matrix case, named divisor equivalence, is introduced and certain properties of divisor equivalence are obtained. In § 4 we show the connection between the divisor equivalence transformation and the strict equivalence transformation that has been presented by Gantmacher (1959) and Vardulakis and Antoniou (2003). Finally, in § 5 we summarize our results and propose directions for further research on the subject.

2. Preliminary results

In what follows, \mathbb{R}, \mathbb{C} denote respectively the fields of real and complex numbers and \mathbb{Z}, \mathbb{Z}^+ denote respectively the integers and non-negative integers. By $\mathbb{R}[s]$ and $\mathbb{R}[s]^{p \times m}$ we denote the sets of polynomials and $p \times m$ polynomial matrices respectively with real coefficients and indeterminate $s \in \mathbb{C}$.

Definition 1: Let $A(s) \in \mathbb{R}[s]^{p \times m}$ with $\text{rank}_{\mathbb{R}(s)} A(s) = r \leq \min(p, m)$. The values $\lambda_i \in \mathbb{C}$ that satisfy the condition $\text{rank}_{\mathbb{C}} A(\lambda_i) < r$ are called finite zeros of $A(s)$. Assume that $A(s)$ has l distinct zeros $\lambda_1, \lambda_2, \dots, \lambda_l \in \mathbb{C}$, and let

$$S_{A(s)}^{\lambda_i}(s) = \begin{bmatrix} \text{diag}\{(s - \lambda_i)^{m_{i1}}, \dots, (s - \lambda_i)^{m_{ir}}\} & 0_{r, m-r} \\ 0_{p-r, r} & 0_{p-r, m-r} \end{bmatrix}$$

be the local Smith form of $A(s)$ at $s = \lambda_i, i = 1, 2, \dots, l$ where $m_{ij} \in \mathbb{Z}^+$ and $0 \leq m_{i1} \leq m_{i2} \leq \dots \leq m_{ir}$. The terms $(s - \lambda_i)^{m_{ij}}$ are called the finite elementary divisors (f.e.d.) of $A(s)$ at $s = \lambda_i$. The total number n of the f.e.d. of $A(s)$ is $n := \sum_{i=1}^l \sum_{j=1}^r m_{ij}$.

Definition 2 (Gohberg *et al.* 1982, Vardulakis 1991): The dual matrix of $A(s) = A_q s^q + A_{q-1} s^{q-1} + \dots + A_0 \in \mathbb{R}[\sigma]^{p \times m}$, $A_q \neq 0_{p,m}$, is defined as $\tilde{A}(s) := s^q A(1/s) = A_0 s^q + A_1 s^{q-1} + \dots + A_q$. Since $\text{rank } \tilde{A}(0) = \text{rank } A_q$ the dual matrix $\tilde{A}(s)$ of $A(s)$ has zeros at $s=0$ iff $\text{rank } A_q < r$. Let $\text{rank } A_q < r$ and let

$$S_{\tilde{A}(s)}^0(s) = \begin{bmatrix} \text{diag}\{s^{\mu_1}, \dots, s^{\mu_r}\} & 0_{r, m-r} \\ 0_{p-r, r} & 0_{p-r, m-r} \end{bmatrix} \quad (3)$$

be the local Smith form of $\tilde{A}(s)$ at $s=0$ where $\mu_j \in \mathbb{Z}^+$ and $0 \leq \mu_1 \leq \mu_2 \leq \dots \leq \mu_r$. The infinite elementary divisors (i.e.d.) of $A(s)$ are defined as the f.e.d. σ^{μ_j} of its dual $\tilde{A}(s)$ at $s=0$. The total number μ of the i.e.d. of $A(s)$ is $\mu = \sum_{i=1}^r \mu_i$.

It is easily seen from Definition 2, that $A(s) \in \mathbb{R}[s]^{p \times m}$ has no i.e.d. iff $\text{rank } A_q = r$. The total number of f.e.d. and i.e.d. is connected with the dimension and the highest degree of all entries of a polynomial matrix as follows.

Theorem 1 (Gohberg *et al.* 1982, Vardulakis and Antoniou 2003): *The total number $n + \mu$ of f.e.d. and i.e.d. (orders accounted for) of a square polynomial matrix $A(s) \in \mathbb{R}[s]^{r \times r}$ is equal to rq , where q is the highest degree of all the entries of $A(s)$.*

Define by $P(m, l)$ the class of $(r + m) \times (r + l)$ polynomial matrices where l and m are fixed integers and r ranges over all integers which are greater than $\max(-m, -l)$. In the following, we will introduce some essential transformations between polynomial matrices.

Definition 3 (Pugh and Shelton 1978): $A_1(s), A_2(s) \in P(m, l)$ are said to be *extended unimodular equivalent* (e.u.e.) if there exist polynomial matrices $M(s), N(s)$ such that

$$\begin{bmatrix} M(s) & A_2(s) \end{bmatrix} \begin{bmatrix} A_1(s) \\ -N(s) \end{bmatrix} = 0 \quad (4)$$

where the compound matrices

$$\begin{bmatrix} M(s) & A_2(s) \end{bmatrix}; \quad \begin{bmatrix} A_1(s) \\ -N(s) \end{bmatrix} \quad (5)$$

have full rank $\forall s \in \mathbb{C}$.

In Pugh and Shelton (1978), it was shown that the transformations of strict system equivalence (s.s.e.) (Rosenbrock 1970), and Furhmann system equivalence (f.s.e.) (Fuhrmann 1976, 1977), is a specialization to system matrices of the extended unimodular equivalence. E.u.e. allows matrices of different dimensions to be related and preserves the f.e.d. of the polynomial matrices involved. Extending the notion of *strict equivalence* for matrix pencils, defined in Gantmacher (1959), to the case of square polynomial matrices we propose the following definition.

Definition 4: Let $A_1(s), A_2(s) \in \mathbb{R}[s]^{m \times l}$ be two polynomial matrices of the same degree. They are said to be *strictly equivalent* (s.e.) if there exist non-singular $M \in \mathbb{R}^{m \times m}, N \in \mathbb{R}^{l \times l}$ such that $MA_1(s)N = A_2(s)$.

S.e. is actually a unimodular and bicausal equivalence transformation and thus has the property of preserving both the f.e.d. and i.e.d. of polynomial matrices. Note however that it relates matrices of the same dimensions and degree.

Definition 5 (Karampetakis *et al.* 1994): $A_1(s), A_2(s) \in P(m, l)$ are said to be *{0}-equivalent* if there exist rational matrices $M(s), N(s)$, having no poles at $s=0$, such that (4) is satisfied and where the compound matrices in (5) have full rank at $s=0$.

{0}-equivalence preserves only the f.e.d. of $A_1(s), A_2(s) \in P(m, l)$ of the form $s^i, i > 0$.

3. A new notion of equivalence between square and regular polynomial matrices

Considering Theorem 1, a necessary condition for two polynomial matrices $A_1(s) \in \mathbb{R}[\sigma]^{r_1 \times r_1}, A_2(s) \in \mathbb{R}[\sigma]^{r_2 \times r_2}$ of degrees q_1, q_2 respectively, to have the same f.e.d. and i.e.d., is that $r_1 q_1 = r_2 q_2$. Therefore we define the following set of polynomial matrices

$$\mathbb{R}_c[s] := \{A(s) \text{ defined in (2) with } c = rq, r \geq 2\}. \quad (6)$$

To clarify the definition of the set $\mathbb{R}_c[s]$, we give the following example.

Example 1:

$$A_1(s) = \begin{bmatrix} 1 & s^2 \\ 0 & s+1 \end{bmatrix} \in \mathbb{R}_4[s]$$

$$A_2(s) = \begin{bmatrix} 1 & s^3 \\ 0 & s+1 \end{bmatrix} \in \mathbb{R}_6[s]$$

$$A_3(s) = \begin{bmatrix} s & 0 & -1 & 0 \\ 0 & s & 0 & -1 \\ 1 & 0 & 0 & s \\ 0 & 1 & 0 & 1 \end{bmatrix} \in \mathbb{R}_4[s].$$

Before we continue to the main results of this section, we will present a simple example to demonstrate the limitations of the extended unimodular transformation to take into account both finite and infinite elementary divisor structures.

Example 2: Consider the polynomial matrices

$$A_1(s) = \begin{bmatrix} 1 & s^2 \\ 0 & s+1 \end{bmatrix} \in \mathbb{R}_4[s]; \quad A_2(s) = \begin{bmatrix} 1 & s^3 \\ 0 & s+1 \end{bmatrix} \in \mathbb{R}_6[s]$$

and the following e.u.e. transformation between them

$$\begin{aligned} & \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{M(s)} \underbrace{\begin{bmatrix} 1 & s^2 \\ 0 & s+1 \end{bmatrix}}_{A_1(s)} \\ &= \underbrace{\begin{bmatrix} 1 & s^3 \\ 0 & s+1 \end{bmatrix}}_{A_2(s)} \underbrace{\begin{bmatrix} 1 & s^2 - s^3 \\ 0 & 1 \end{bmatrix}}_{N(s)}. \end{aligned}$$

As the Smith forms of $A_1(s), A_2(s)$ are $S_{A_2(s)}^{\mathbb{C}}(s) = S_{A_1(s)}^{\mathbb{C}}(s) = \text{diag}[1 \ s+1]$, both $A_1(s), A_2(s)$ have the same f.e.d. ($s+1$). As the Smith forms of the duals of $A_1(s), A_2(s)$ at $s=0$ are $S_{A_1(s)}^0(s) = \text{diag}[1 \ s^3]$ and $S_{A_2(s)}^0(s) = \text{diag}[1 \ s^5]$ respectively, they have a different i.e.d. (s^3 and s^5 respectively).

The above example indicates that in order to ensure that the associated transformation will preserve both the f.e.d. and i.e.d. of $A_1(s)$ and $A_2(s)$, further restrictions have to be imposed on the compound matrices defined in (5).

In this respect we propose the following transformation.

Definition 6: $A_1(s), A_2(s) \in \mathbb{R}_c[s]$ are said to be *divisor equivalent* (d.e.) if there exist polynomial matrices $M(s), N(s)$, such that (4) is satisfied and the compound matrices in (5) satisfy the two conditions

- (i) they have full rank over $\mathbb{R}(s)$ and no f.e.ds.,
- (ii) they have no i.e.ds.

Note that condition (i) is equivalent to the relative primeness requirements of e.u.e. while (from the remark after Definition 2) condition (ii) amounts to the requirement that the coefficient matrices corresponding to the highest degree terms in each of the compound matrices in (5) have respectively full row (column) rank over \mathbb{R} . The two conditions taken together are equivalent to the requirement that the compound matrices in (5) are *minimal bases* (Vardulakis 1991) of the rational vector spaces spanned by their respective rows (columns). D.e. is a special case of e.u.e. Our main interest is to prove that d.e. leaves invariant the finite and infinite elementary divisor structure of the polynomial matrices involved. However, before we proceed with the main theorem of this paper, we need to prove some preliminary results. Firstly we need the following partial transitivity result.

Lemma 1: *If $A_1(s), A_2(s) \in \mathbb{R}_c[s]$ are d.e. and $A_2(s), A_3(s) \in \mathbb{R}_c[s]$ are s.e., then $A_1(s), A_3(s) \in \mathbb{R}_c[s]$ are d.e.*

Proof: See Appendix 1. □

Given a polynomial matrix $A(s)$, we can construct a matrix pencil $sE - A$ as in (7) (Gohberg *et al.* 1982, p. 186) which possesses the same f.e.d. (Gohberg *et al.*

1982) and i.e.d. (Praagman 1991, Vardulakis and Antoniou 2003) with $A(s)$. In the following lemma it is shown that the proposed pencil $sE - A$ is actually d.e. to the polynomial matrix $A(s)$.

Lemma 2: *With $c = rq$, $r \geq 2$ the polynomial matrix $A(s) \in \mathbb{R}_c[s]$ defined in (2) and the matrix pencil*

$$sE - A := \begin{bmatrix} sI_r & -I_r & 0 & \cdots & 0 & 0 \\ 0 & sI_r & -I_r & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & sI_r & -I_r \\ A_0 & A_1 & A_2 & \cdots & A_{q-2} & sA_q + A_{q-1} \end{bmatrix} \in \mathbb{R}_c[s] \tag{7}$$

are d.e.

Proof: See Appendix 2. □

The required conditions of d.e. give rise to certain degree conditions on the left and right transforming matrices as we can see in the following lemma.

Lemma 3:

- (a) *Let $A_1(s), A_2(s) \in \mathbb{R}_c[s]$ with dimensions $m \times m$ and $(m+r) \times (m+r)$ respectively where $r \neq 0$. Then the two conditions of d.e. imply the following degree conditions: $\text{deg } M(s) \leq \text{deg } A_2(s)$ and $\text{deg } N(s) \leq \text{deg } A_1(s)$, where $\text{deg}[\cdot]$ denotes the highest degree among the polynomial entries of the indicated matrix.*
- (b) *Let $A_1(s), A_2(s) \in \mathbb{R}_c[s]$ have the same dimensions $r \times r$ and therefore, since they belong to the same class $\mathbb{R}_c[s]$, the same degree q . If $A_1(s), A_2(s)$ are d.e. then $\text{deg } M(s) = \text{deg } N(s)$.*

Proof: See Appendix 3. □

We can always increase the degree of the polynomial matrices $A_1(s), A_2(s)$ involved in the d.e. transformation (4), without changing the initial f.e.d. and i.e.d. of $A_1(s), A_2(s)$, by multiplying both of them by certain polynomial terms, as we can easily see in the following lemma.

Lemma 4: *If $A_1(s)(s - s_0)^k \in \mathbb{R}^{m \times m}[s]$ has the same f.e.d. and i.e.d. as $A_2(s)(s - s_0)^k \in \mathbb{R}^{m \times m}[s]$, where $s_0 \neq 0$ is not a zero of either $A_1(s)$ or $A_2(s)$, then $A_1(s)$ and $A_2(s)$ have the same f.e.d. and i.e.d.*

Proof: See Appendix 4. □

Now, using the above lemmas we are in a position to prove that the d.e. transformation leaves invariant both the finite and infinite elementary divisors of the polynomial matrices involved in the transformation.

Theorem 2: *$A_1(s), A_2(s) \in \mathbb{R}_c[s]$ are d.e. iff they have the same finite and infinite elementary divisors.*

Proof: Sufficiency: (a) According to condition (i) of d.e., $A_1(s)$ and $A_2(s)$ are also e.u.e. and thus have the same f.e.d.

(b) In the sequel we prove that the second condition of d.e. implies that the two polynomial matrices possess the same i.e.d. In the following proof we distinguish two cases: (i) $A_1(s)$ and $A_2(s) \in \mathbb{R}_c[s]$ have different dimensions and (ii) $A_1(s)$ and $A_2(s) \in \mathbb{R}_c[s]$ have the same dimensions.

(i) Let $A_1(s)$ and $A_2(s) \in \mathbb{R}_c[s]$ with dimensions $m \times m$ and $(m+r) \times (m+r)$ respectively where $r \neq 0$. Then, from Lemma 3(a) the first two conditions of d.e. imply the following degree conditions $\deg M(s) \leq \deg A_2(s)$ and $\deg N(s) \leq \deg A_1(s)$. Let also $d_{MA_2} = \deg [M(s) \ A_2(s)]$ and $d_{A_1N} = \deg [A_1(s)^T \ -N(s)^T]^T$. Then, by setting $s = 1/w$, equation (4) can be rewritten as

$$\begin{bmatrix} M(1/w) & A_2(1/w) \end{bmatrix} \begin{bmatrix} A_1(1/w) \\ -N(1/w) \end{bmatrix} = 0 \tag{8}$$

and then premultiplying and postmultiplying (8) by $w^{d_{MA_2}}$ and $w^{d_{A_1N}}$ respectively as

$$\begin{aligned} w^{d_{MA_2}} \begin{bmatrix} M(1/w) & A_2(1/w) \end{bmatrix} \begin{bmatrix} A_1(1/w) \\ -N(1/w) \end{bmatrix} w^{d_{A_1N}} &= 0 \\ \iff \begin{bmatrix} M(w) & \widetilde{A_2(w)} \end{bmatrix} \begin{bmatrix} \widetilde{A_1(w)} \\ -N(w) \end{bmatrix} &= 0 \end{aligned} \tag{9}$$

where with $\widetilde{}$ we denote the dual matrix. Since $d_{MA_2} \leq \deg [A_2(s)]$ and $d_{A_1N} \leq \deg [A_1(s)]$ equation (9) can be rewritten as

$$\begin{bmatrix} M'(w) & \widetilde{A_2(w)} \end{bmatrix} \begin{bmatrix} \widetilde{A_1(w)} \\ -N'(w) \end{bmatrix} = 0. \tag{10}$$

The compound matrix $[M(s) \ A_2(s)]$ (respectively $[\widetilde{M(s)} \ \widetilde{A_2(s)}]$) has no i.e.d. and therefore its dual $[M'(w) \ A_2(w)]$ (respectively $[\widetilde{M'(w)} \ \widetilde{A_2(w)}]$) has no finite zeros at $w=0$. Therefore relation (10) is a $\{0\}$ -equivalence relation which preserves the f.e.d. of $\widetilde{A_1(w)}, \widetilde{A_2(w)}$ at $w=0$ or otherwise the i.e.d. of $A_1(s), A_2(s)$.

(ii) Let $A_1(s), A_2(s) \in \mathbb{R}[s]^{m \times m}$ with the same degree d . Then according to Lemma 3(b) $d_t = \deg M(s) = \deg N(s)$. In the case where $d_t \leq d$ then the proof is the same as the one presented in (i) above. Consider now the case where $d_t > d$. Let $s_0 \neq 0$ not a zero of either $A_1(s)$ or $A_2(s)$. Then

$$\begin{bmatrix} M(s) & A_2(s)(s-s_0)^{d_t-d} \end{bmatrix} \begin{bmatrix} A_1(s)(s-s_0)^{d_t-d} \\ -N(s) \end{bmatrix} = 0 \tag{11}$$

is an e.u.e. relation which implies that $A_2(s)(s-s_0)^{d_t-d}$ and $A_1(s)(s-s_0)^{d_t-d}$ have the same f.e.d. Therefore according to Lemma 4, $A_1(s)$ and $A_2(s)$ have the same f.e.d. Taking the duals of (11) we have

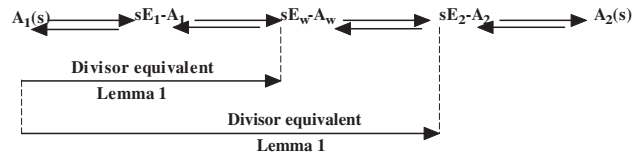
$$\begin{bmatrix} \widetilde{M}(s) & A_2(s)(s-s_0)^{d_t-d} \end{bmatrix} \begin{bmatrix} A_1(s)(s-s_0)^{d_t-d} \\ -\widetilde{N}(s) \end{bmatrix} = 0. \tag{12}$$

Equation (12) is a $\{0\}$ -equivalence relation, so

$$A_2(s)(s-s_0)^{d_t-d} \text{ and } A_1(s)(s-s_0)^{d_t-d}$$

have the same f.e.d. at 0 and thus $A_2(s)(s-s_0)^{d_t-d}$ and $A_1(s)(s-s_0)^{d_t-d}$ have the same i.e.d., i.e. $A_2(s)$ and $A_1(s)$ have the same i.e.d. (Lemma 4).

Necessity: Assume that $A_1(s), A_2(s) \in \mathbb{R}_c[s]$ have identical f.e.d. and i.e.d. Then according to Lemma 2 $A_1(s)$ and $A_2(s)$ are d.e. to matrix pencils $sE_1 - A_1$ and $sE_2 - A_2$ of the form (7) and vice versa. The pencils $sE_1 - A_1$ and $sE_2 - A_2$ are also strictly equivalent to their respective Weierstrass forms denoted thereafter by $W(sE_1 - A_1)$ and $W(sE_2 - A_2)$ (Gantmacher 1959). Since $sE_1 - A_1, sE_2 - A_2 \in \mathbb{R}_c[s]$ and share the same f.e.d. and i.e.d. they have the same Weierstrass form: $sE_w - A_w \equiv W(sE_1 - A_1) \equiv W(sE_2 - A_2)$. Repeated use of the transitivity property proved in Lemma 10 shows that $A_1(s)$ is divisor equivalent to $sE_2 - A_2$. This argument is summarized in the figure.



Therefore there exist polynomial matrices $A(s), B(s)$ such that the transformation

$$A(s)(sE_2 - A_2) = A_1(s)B(s) \tag{13}$$

is a d.e. transformation. The polynomial matrices $sE_2 - A_2$ and $A_2(s)$ are also connected, according to Lemma 3, through the following divisor equivalent transformation

$$\begin{pmatrix} 0 \\ sI - A_0 \end{pmatrix} A_2(s) = (sE_2 - A_2) \begin{pmatrix} sI - A_0 \\ (sI - A_0)s \\ \vdots \\ (sI - A_0)s^{q_2-1} \end{pmatrix} \tag{14}$$

where $q_2 = \deg [A_2(s)]$ and $sI - A_0$ as defined in Lemma 2. Premultiplying (14) by $A(s)$ and then using (13) we get the transformation

$$\left\{ A(s) \begin{pmatrix} 0 \\ (sI - A_0) \end{pmatrix} \right\} A_2(s) = A_1(s) \left\{ B(s) \begin{pmatrix} sI - A_0 \\ (sI - A_0)s \\ \vdots \\ (sI - A_0)s^{q_2-1} \end{pmatrix} \right\}. \tag{15}$$

We divide both sides with $(s-s_0)$ and thus we have the transformation

$$\left\{ A(s) \begin{pmatrix} 0 \\ I \end{pmatrix} \right\} A_2(s) = A_1(s) \left\{ B(s) \begin{pmatrix} I \\ sI \\ \vdots \\ s^{q_2-1}I \end{pmatrix} \right\}$$

$$\iff \begin{bmatrix} A(s) \begin{pmatrix} 0 \\ I \end{pmatrix} & A_1(s) \end{bmatrix} \begin{bmatrix} A_2(s) \\ -B(s) \begin{pmatrix} I \\ sI \\ \vdots \\ s^{q_2-1}I \end{pmatrix} \end{bmatrix} = 0. \tag{16}$$

We shall show in the sequel that (16) is a d.e. transformation. In order to prove the absence of finite and infinite elementary divisors of the compound matrices

$$\begin{bmatrix} A(s) \begin{pmatrix} 0 \\ I \end{pmatrix} & A_1(s) \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} A_2(s) \\ -B(s) \begin{pmatrix} I \\ sI \\ \vdots \\ s^{q_2-1}I \end{pmatrix} \end{bmatrix} \tag{17}$$

it is enough to prove that (17) possesses the same finite and infinite elementary divisors respectively as

$$\begin{bmatrix} A(s) & A_1(s) \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} sE_2 - A_2 \\ -B(s) \end{bmatrix} \tag{18}$$

since according to the d.e. transformation (13), the matrices in (18) have neither finite nor infinite elementary divisors.

(i) Our first goal is to prove that the matrices

$$\begin{bmatrix} sE_2 - A_2 \\ -B(s) \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} A_2(s) \\ -B(s) \begin{pmatrix} I \\ sI \\ \vdots \\ s^{q_2-1}I \end{pmatrix} \end{bmatrix} \tag{19}$$

possess the same finite and infinite elementary divisors.

(i-a) *Same finite elementary divisors.*

Consider the transformation

$$\begin{bmatrix} \begin{pmatrix} 0 \\ I \\ 0 \end{pmatrix} & 0 & sE_2 - A_2 \end{bmatrix} \begin{bmatrix} A_2(s) \\ -B(s) \begin{pmatrix} I \\ sI \\ \vdots \\ s^{q_2-1}I \end{pmatrix} \\ - \begin{pmatrix} I \\ sI \\ \vdots \\ s^{q_2-1}I \end{pmatrix} \end{bmatrix} = 0.$$

It is easily seen (see also (7)) that both compound matrices of the above transformation include the unit matrix and therefore do not possess finite elementary

divisors. Therefore the matrices (19) are e.u.e. and thus have the same finite elementary divisors.

(i-b) *Same infinite elementary divisors.*

Consider now the transformation

$$\begin{bmatrix} \begin{pmatrix} 0 \\ (sI - A_0) \\ 0 \end{pmatrix} & 0 & sE_2 - A_2 \end{bmatrix} \times \begin{bmatrix} A_2(s) \\ -B(s) \begin{pmatrix} I \\ sI \\ \vdots \\ s^{q_2-1}I \end{pmatrix} \\ - \begin{pmatrix} I \\ sI \\ \vdots \\ s^{q_2-1}I \end{pmatrix} (sI - A_0) \end{bmatrix} = 0. \tag{20}$$

The highest coefficient matrices of the above compound matrices are

$$\begin{bmatrix} 0 & 0 & I & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & I & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & I & 0 \\ I & 0 & 0 & 0 & \dots & 0 & A_{q_2} \\ 0 & I & & & & & -B_1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} A_{2,q_2} \\ -B_1 \begin{pmatrix} 0 \\ 0 \\ \vdots \\ I \end{pmatrix} \\ - \begin{pmatrix} 0 \\ 0 \\ \vdots \\ I \end{pmatrix} \end{bmatrix}$$

where $B(s) = B_0 + B_1s$ (Lemma 3(a)) and $A_2(s) = A_{2,0} + A_{2,1}s + \dots + A_{2,q_2}s^{q_2}$ with $A_{2,q_2} \neq 0$. The above highest coefficient matrices have full row and column rank respectively and therefore the compound matrices of (20) have no i.e.d. Consequently the matrices in (19) are d.e. and thus possess the same i.e.d.

(ii) Our second goal is to prove that the matrices

$$\begin{bmatrix} A(s) \begin{pmatrix} 0 \\ I \end{pmatrix} & A_1(s) \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} A(s) & A_1(s) \end{bmatrix} \tag{21}$$

possess the same finite and infinite elementary divisors.

(ii-a) *Same finite elementary divisors.*

Consider the transformation

$$I \begin{bmatrix} A(s) \begin{pmatrix} 0 \\ I \end{pmatrix} & A_1(s) \end{bmatrix} = \begin{bmatrix} A(s) & A_1(s) \end{bmatrix} \begin{bmatrix} \begin{pmatrix} 0 \\ I \end{pmatrix} & 0 \\ 0 & I \end{bmatrix}.$$

We observe that both compound matrices of the above transformation include the unit matrix and therefore do not possess finite elementary divisors.

Therefore the matrices (21) are e.u.e. and thus have the same finite elementary divisors.

(ii-b) *Same infinite elementary divisors.*

Consider the transformation

$$[(sI - A_0)^{q_1} \begin{bmatrix} A(s) \begin{pmatrix} 0 \\ I \end{pmatrix} & A_1(s) \end{bmatrix} \\ = \begin{bmatrix} A(s) & A_1(s) \end{bmatrix} \begin{bmatrix} \begin{pmatrix} 0 \\ (sI - A_0)^{q_1} \end{pmatrix} & 0 \\ 0 & (sI - A_0)^{q_1} \end{bmatrix} \quad (22)$$

$$\iff \begin{bmatrix} (sI - A_0)^{q_1} & A(s) & A_1(s) \end{bmatrix} \\ \times \begin{bmatrix} A(s) \begin{pmatrix} 0 \\ I \end{pmatrix} & A_1(s) \\ -\begin{pmatrix} 0 \\ (sI - A_0)^{q_1} \end{pmatrix} & 0 \\ 0 & -(sI - A_0)^{q_1} \end{bmatrix} = 0$$

where $A(s) = A_0 + A_1s + \dots + A_{q_1}s^{q_1}$ and $A_1(s) = A_{1,0} + A_{1,1}s + \dots + A_{1,q_1}s^{q_1}$ with $A_{1,q_1} \neq 0$. The highest coefficient matrices of the above compound matrices are

$$\begin{bmatrix} I & A_{q_1} & A_{1,q_1} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} A_{q_1} \begin{pmatrix} 0 \\ I \end{pmatrix} & A_{1,q_1} \\ -\begin{pmatrix} 0 \\ I \end{pmatrix} & 0 \\ 0 & -I \end{bmatrix}.$$

The above highest coefficient matrices have full row and column rank respectively and therefore the compound matrices in (20) have no i.e.d. Therefore the matrices in (21) are d.e. and thus possess the same i.e.d.

Note: In the proof of the necessity of Theorem 2, it is possible to start with the matrix $A_2(s)$ and follow identical arguments to yield a transformation of divisor equivalence between $A_2(s)$ and $A_1(s)$, i.e. $M(s)A_2(s) = A_1(s)N(s)$. \square

Remark 1: Examining closely the constructive proof of Theorem 2, we can conclude that if $A_1(s), A_2(s) \in \mathbb{R}_c[s]$ have the same f.e.d. and i.e.d. then we can find

$$M(s) := A(s) \begin{pmatrix} 0 \\ I \end{pmatrix}; \quad N(s) := B(s) \begin{pmatrix} I \\ sI \\ \vdots \\ s^{q_2-1}I \end{pmatrix}$$

such that $M(s)A_1(s) = A_2(s)N(s)$ is a d.e. transformation with $\deg[M(s)] \leq \deg[A_2(s)]$ and $\deg[N(s)] \leq \deg[A_1(s)]$.

Remark 2: It is easily seen from the proof of the above theorem that condition (i) of the d.e. guarantees the invariance of the f.e.d., while condition (ii) guarantees the invariance of the i.e.d. of the involved matrices. Therefore in order to prove that two polynomial matrices $A_1(s), A_2(s) \in \mathbb{R}_c[s]$ possess the same finite

(resp. infinite) elementary divisors it is enough to prove that there exist polynomial matrices $M(s), N(s)$ of appropriate dimensions that satisfy $M(s)A_1(s) = A_2(s)N(s)$ and condition (i) (resp. (ii)).

Corollary 1: *The matrix pencil $sE - A$ defined in (7) is d.e. to the square polynomial matrix $A(s)$ defined in (2) and thus according to Theorem 2, $A(s)$ and $sE - A$ possess the same f.e.d. and i.e.d. This has already been proved in a different way in Praagman (1991) and Vardulakis and Antoniou (2003).*

Example 3: Consider the transformation

$$\underbrace{\begin{bmatrix} s+1 & 0 \\ s^2 & 0 \\ 0 & -1 \end{bmatrix}}_{M(s)} \underbrace{\begin{bmatrix} s^2 & 1 \\ 0 & s^3 \end{bmatrix}}_{A_1(s)} \\ = \underbrace{\begin{bmatrix} s^2 & 1 & 0 \\ 0 & s & 1 \\ 0 & 0 & s^2 \end{bmatrix}}_{A_2(s)} \underbrace{\begin{bmatrix} 1 & 0 \\ s^3 & s+1 \\ 0 & -s \end{bmatrix}}_{N(s)}.$$

Note that $A_1(s), A_2(s) \in \mathbb{R}_6[s]$. The Smith forms of the compound matrix $\begin{bmatrix} M(s) & A_2(s) \end{bmatrix}$ and of its dual are

$$S^{\mathbb{C}} \begin{bmatrix} M(s) & A_2(s) \end{bmatrix} (s) = \begin{bmatrix} I_4 & 0 \end{bmatrix}$$

and

$$M_1(s) := \begin{bmatrix} M(s) & \widetilde{A_2(s)} \end{bmatrix} = \begin{bmatrix} s+s^2 & 0 & 1 & s^2 & 0 \\ 1 & 0 & 0 & s & s^2 \\ 0 & -s^2 & 0 & 0 & 1 \end{bmatrix}$$

with $S^0_{M_1(s)}(s) = \begin{bmatrix} I_3 & 0 \end{bmatrix}$ and thus the compound matrix $\begin{bmatrix} M(s) & A_2(s) \end{bmatrix}$ has no f.e.d. nor i.e.d. Also since

$$S^{\mathbb{C}} \begin{bmatrix} A_1(s) \\ -N(s) \end{bmatrix} (s) = \begin{bmatrix} I_2 \\ 0 \end{bmatrix}$$

and

$$N_1(s) := \begin{bmatrix} \widetilde{A_1(s)} \\ -N(s) \end{bmatrix} = \begin{bmatrix} s & s^3 \\ 0 & 1 \\ -s^3 & 0 \\ -1 & -s^2 - s^3 \\ 0 & s^2 \end{bmatrix}$$

with

$$S^0_{N_1(s)}(s) = \begin{bmatrix} I_2 \\ 0 \end{bmatrix}$$

the compound matrix $\begin{bmatrix} A_1(s)^T & -N(s)^T \end{bmatrix}^T$ possesses no f.e.d. or i.e.d. Therefore $A_1(s)$ and $A_2(s)$ are divisor equivalent and thus according to Theorem 2 they should have the same f.e.d. and i.e.d. Checking their Smith forms

$$S_{A_1(s)}^{\mathbb{C}} = \begin{bmatrix} 1 & 0 \\ 0 & s^5 \end{bmatrix}; \quad S_{A_2(s)}^{\mathbb{C}} = \begin{bmatrix} I_3 & 0 \\ 0 & s^5 \end{bmatrix}$$

and

$$S_{\tilde{A}_1(s)}^0 = \begin{bmatrix} 1 & 0 \\ 0 & s \end{bmatrix}; \quad S_{\tilde{A}_2(s)}^0 = \begin{bmatrix} I_3 & 0 \\ 0 & s \end{bmatrix}$$

this is indeed confirmed.

As is shown in the following theorem regarding the special case of matrix pencils with the same dimension, strict equivalence and d.e. define the same equivalence class.

Theorem 3: *Let $sE_1 - A_1, sE_2 - A_2 \in \mathbb{R}[s]^{m \times m}$ with $\det[sE_i - A_i] \neq 0, i = 1, 2$. Then $sE_1 - A_1, sE_2 - A_2$ are strictly equivalent iff they are d.e.*

Proof: Sufficiency: If $sE_1 - A_1, sE_2 - A_2$ are strictly equivalent then there exist non-singular matrices $M, N \in \mathbb{R}^{m \times m}$ such that

$$\begin{aligned} M[sE_1 - A_1]N &= [sE_2 - A_2] \\ \implies M[sE_1 - A_1] &= [sE_2 - A_2]N^{-1}. \end{aligned}$$

Select s_0 such that $\{\det[s_0E_i - A_i] \neq 0, i = 1, 2\}$ and construct the transformation

$$[(s - s_0)M][sE_1 - A_1] = [sE_2 - A_2][(s - s_0)N^{-1}].$$

It is easily seen that the above transformation is a d.e. transformation.

Necessity: Suppose that $sE_1 - A_1, sE_2 - A_2$ are d.e. Then according to Theorem 2, $sE_1 - A_1$ and $sE_2 - A_2$ possess the same f.e.d. and i.e.d. Therefore (Gantmacher 1959) the pencils $sE_1 - A_1$ and $sE_2 - A_2$ are strictly equivalent. \square

Strict equivalence defines an equivalence class on the set of square and non-singular pencils, and thus according to the above theorem defines the same equivalence class with d.e. However d.e. defines an equivalence class in the more general set of polynomial matrices that belong to $\mathbb{R}_c[s]$ as can be seen in the following theorem.

Theorem 4: *Divisor equivalence is an equivalence relation on $\mathbb{R}_c[s]$.*

Proof: (i) *Reflexivity.*

Let $A(s) \in \mathbb{R}_c[s]$ and consider the relation

$$[(s - s_0)^{\deg[A]}I \quad A(s)] \begin{bmatrix} A(s) \\ -(s - s_0)^{\deg[A]}I \end{bmatrix} = 0$$

where s_0 is not a zero of $A(s)$. It is easily proved that the above transformation is a d.e. transformation.

(ii) *Symmetry.*

Let $A_1(s), A_2(s) \in \mathbb{R}_c[s]$ be related by a d.e. transformation of the form

$$M(s)A_1(s) = A_2(s)N(s). \tag{23}$$

Then from Theorem 2, $A_1(s)$ and $A_2(s)$ have identical f.e.d. and i.e.d. Hence from the converse of Theorem 2 (see note in the proof) there exists a d.e. relation between $A_2(s)$ and $A_1(s)$ of the form (4).

(iii) *Transitivity.*

Suppose that $A_1(s), A_2(s) \in \mathbb{R}_c[s]$ are d.e. and that $A_2(s), A_3(s) \in \mathbb{R}_c[s]$ are also d.e. Then from Theorem 2 $A_1(s)$ and $A_3(s)$ have identical f.e.d. and i.e.d. Hence, from the converse of the Theorem 2, $A_1(s)$ and $A_3(s)$ are divisor equivalent. \square

4. On the connection of d.e. and strict equivalence

In this section we make clear the connection between the proposed transformation of d.e. and the transformations of *strict equivalence* and *fundamental equivalence* defined in Vardulakis and Antoniou (2003).

Definition 7 (Vardulakis and Antoniou 2003): $A_1(s), A_2(s) \in \mathbb{R}_c[s]$ are called *strictly equivalent* iff their equivalent matrix pencils $sE_1 - A_1 \in \mathbb{R}[s]^{c \times c}$ and $sE_2 - A_2 \in \mathbb{R}[s]^{c \times c}$ proposed in (7) are strictly equivalent according to Definition 5.

The coincidence of divisor equivalence and strict equivalence (Gantmacher 1959 and Definition 7) is proved in the following theorem.

Theorem 5: *Strict equivalence (Definition 7) belongs to the same equivalence class as d.e.*

Proof: Necessity: Suppose that $A_1(s), A_2(s)$ are d.e. Then

$$sE_1 - A_1 \stackrel{d.e.}{\sim} A_1(s) \stackrel{d.e.}{\sim} A_2(s) \stackrel{d.e.}{\sim} sE_2 - A_2.$$

Therefore from the transitivity property of d.e. $sE_1 - A_1 \stackrel{d.e.}{\sim} sE_2 - A_2$. However d.e. coincides with strict equivalence in matrix pencils as has already been proved in Theorem 3 and thus the two pencils are also strictly equivalent.

Sufficiency: Suppose that $A_1(s)$ and $A_2(s)$ are strictly equivalent according to Definition 7. Then

$$A_1(s) \stackrel{d.e.}{\sim} sE_1 - A_1 \stackrel{d.e.}{\sim}_{s.e.} sE_2 - A_2 \stackrel{d.e.}{\sim} A_2(s).$$

Then from the transitivity property of d.e. $A_1(s) \stackrel{d.e.}{\sim} A_2(s)$. \square

The geometrical meaning of strict equivalence, presented in Vardulakis and Antoniou (2003), is given in the sequel.

Definition 8 (Vardulakis and Antoniou 2003): Two discrete time AR representations of the form

$$A_i(\sigma)\beta_i(k) = 0, \quad k = 0, 1, 2, \dots, N - \deg[A_i(s)] \geq 0, \\ i = 1, 2$$

where σ is the shift operator, $A_i(\sigma) \in \mathbb{R}[\sigma]^{r_i \times r_i}$, $\det[A_i(s)] \neq 0, i = 1, 2$ will be called *fundamentally equivalent* (f.e.) over the finite time interval $k = 0, 1, 2, \dots, N \geq \max\{\deg[A_i(s)]\}$ iff there exists a bijective polynomial map between their respective behaviours $\mathcal{B}_{A_1(\sigma)}, \mathcal{B}_{A_2(\sigma)}$.

Note that fundamental equivalence constitutes an extension of the behaviour homomorphism presented in Fuhrmann (2001, 2002, 2003), for the case of non-proper discrete time AR representations, when studied in a closed time interval. Fundamental equivalence (Definition 8) is the geometric interpretation of strict equivalence (Definition 7) as can be easily seen in the following theorem.

Theorem 6: (Vardulakis and Antoniou 2003): *Two discrete time AR representations of the form (1) are strict equivalent iff they are fundamentally equivalent.*

Based on the above theorem, we can now extend the results presented in Theorem 5.

Theorem 7: *Two discrete time AR representations of the form (1) are d.e. iff they are fundamentally equivalent.*

Proof: The theorem can be proved using Theorem 5 and Theorem 6. \square

Therefore, strict equivalence, fundamental equivalence and d.e. define the same equivalence classes. In the following remark we give a geometrical meaning of the right transforming matrix $N(s)$ involved in the d.e. transformation (4).

Remark 3: Suppose that $A_1(s), A_2(s) \in \mathbb{R}_c[s]$ are d.e. Then there exist polynomial matrices $M(s), N(s)$ such that

$$M(\sigma)A_1(\sigma) = A_2(\sigma)N(\sigma). \quad (24)$$

By multiplying (24) on the right by $\beta_1(k)$ we get

$$M(\sigma)A_1(\sigma)\beta_1(k) = A_2(\sigma)N(\sigma)\beta_1(k) \\ \implies 0 = A_2(\sigma)N(\sigma)\beta_1(k) \\ \implies \exists \beta_2(k) \in \mathcal{B}_{A_2(\sigma)} \\ \text{such that } \beta_2(k) = N(\sigma)\beta_1(k). \quad (25)$$

According to the conditions of d.e., $[A_1(\sigma)^T - N(\sigma)^T]^T$ has full rank and no f.e.d. or i.e.d. This implies (Karampetakis 2004) that $\beta_1(k) = 0$. Therefore the map defined by the polynomial matrix

$N(\sigma): \mathcal{B}_{A_1(s)} \rightarrow \mathcal{B}_{A_2(s)} | \beta_1(k) \mapsto \beta_2(k)$ is injective. Using the symmetry property of d.e. we can find polynomial matrices $\hat{M}(\sigma), \hat{N}(\sigma)$ such that $\hat{M}(\sigma)A_2(\sigma) = A_1(\sigma)\hat{N}(\sigma)$ is a d.e. relation. Then in a similar manner we get that the map defined by the polynomial matrix $\hat{N}(\sigma): \mathcal{B}_{A_2(s)} \rightarrow \mathcal{B}_{A_1(s)} | \beta_2(k) \mapsto \beta_1(k)$ is also injective. Therefore both maps are bijections between $\mathcal{B}_{A_1(\sigma)}, \mathcal{B}_{A_2(\sigma)}$.

Note that the results of the above remark, concerning the right transforming matrix $N(s)$, come in full accordance with the ones presented by Fuhrmann (2002, Theorem 4.3). The only difference is that here we are referring both to the purely causal and purely anticausal part of the behaviour of non-proper discrete time AR representation over a closed time interval, while the result in Fuhrmann (2002) refers only to the causal part of the behaviour, but to a more general class of discrete time AR representations (where the polynomial matrices are not necessarily square and non-singular) and with time axis equal to \mathbb{Z}^+ (and not a closed time interval). However, since the study of the causal part of the behaviour over the closed time interval comes in accordance with the study of the causal part of the behaviour according to Fuhrmann on \mathbb{Z}^+ , half of our results coincide with the ones of Fuhrmann (2002), i.e. full rank and coprimeness of the compound matrices in divisor equivalence ensures the isomorphism between the causal parts of the behaviours. The extra condition arising in divisor equivalence, i.e. the absence of infinite elementary divisors, ensures that an isomorphism also exists, between the anticausal parts of the behaviours of the equivalent systems.

5. Conclusions

It is known (Antoniou *et al.* 1998) that linear homogeneous matrix difference equations of the form (1) exhibit a forward in time behaviour which is due to the finite elementary divisors of $A(\sigma)$ and a backward in time behaviour which is due to the infinite elementary divisors of $A(\sigma)$. On the other hand continuous time systems of the respective form, i.e. $A(\rho)\beta(t) = 0$ with $\rho := d/dt$, exhibit smooth and impulsive behaviour (Vardulakis 1991) due respectively to the finite and infinite zero structure of $A(\sigma)$. Since the behaviour of continuous and discrete time systems depends on different structural invariants of the polynomial matrix describing such systems, a new transformation, between square and non-singular polynomial matrices of different degrees and dimensions, termed *divisor equivalence*, has been introduced. It is shown that *divisor equivalence* is an equivalence relation on a specific set of polynomial matrices and has the special property of preserving both the finite

and the infinite elementary divisors of the polynomial matrices involved in the transformation. For the special case of matrix pencils with the same dimension, *divisor equivalence* gives rise to the same equivalence class defined through the transformation of *strict equivalence* and extends it in the case of general polynomial matrices. Furthermore, it is shown that *divisor equivalence* constitutes a closed form to test fundamental equivalence of discrete time AR representations.

Although a study of the backward behaviour of regular discrete time systems of the form (1) has been carried out by Antoniou *et al.* (1998), there are still no results concerning the connection of the backward in time behaviour of regular discrete time polynomial matrix descriptions driven by some non-zero inputs (ARMA models) with the structural invariants of the system. Further research will establish the connection between properties of discrete time polynomial matrix descriptions and the structural invariants of the systems related with both finite and infinite elementary divisors of certain polynomial matrices. Divisor equivalence will then play a key role in the study of equivalence between system matrices as *full equivalence* (Hayton *et al.* 1990) has played in the study of equivalence between continuous time polynomial matrix descriptions.

A new approach of the past decade has been the so-called ‘behavioural approach’ introduced by Willems, where the behaviour of a linear, time invariant, discrete time dynamical system is defined as the set of solutions of a system of the form (1) where now the polynomial matrix $A(\sigma)$ is non-regular, i.e. $A(\sigma) \in \mathbb{R}[\sigma]^{r \times m}$ with $r \neq m$ or $A(\sigma) \in \mathbb{R}[\sigma]^{r \times r}$ with $\text{rank}_{\mathbb{R}(\sigma)} A(\sigma) < r$. In that case, not only the finite and infinite elementary divisors but also the right minimal indices of $A(\sigma)$ play a crucial role in both the forward and the backward behaviour of the AR-representation, while the left minimal indices of $A(\sigma)$ influence the existence of a solution of the AR representation under specific initial conditions (Karampetakis 2004). In order to treat equivalence between non-regular polynomial matrices in a manner similar to that of regular polynomial matrices, an extension of the notion of divisor equivalence is needed. The authors have presented in Karampetakis and Vologiannidis (2003) a new equivalence transformation between non-regular polynomial matrices by using a twofold approach: (a) the ‘homogeneous polynomial matrix approach’ where in place of polynomial matrices we have studied their homogeneous polynomial matrix forms and use 2-D equivalence transformations in order to preserve their infinite elementary divisor structure, and (b) the ‘polynomial matrix approach’ where an additional condition has

been added on divisor equivalence in order to treat non-regular polynomial matrices. However, in contrast to divisor equivalence presented in this work, both equivalence transformations presented in Karampetakis and Vologiannidis (2003) give necessary (and not sufficient) conditions in order for two non-regular polynomial matrices to possess the same finite and infinite elementary divisor structure. Further research will be carried out on finding sufficient conditions for two non-regular polynomial matrices to possess the same finite and infinite elementary divisor structure. Since the results concerning the fundamental equivalence of AR representations presented in Vardulakis and Antoniou (2003) were applied only to regular polynomial matrices, an extension of fundamental equivalence to non-regular polynomial matrices and its connection with divisor equivalence has to be established. Additional invariants, apart from the finite and infinite elementary divisors, may also be included in the extension of divisor equivalence since the right and left null structure of non-regular polynomial matrices play a key role in the behaviour of non-regular AR representations. Another application of divisor equivalence could be on the study of the equivalence of AR representations according to the ‘behavioural approach’ of Willems. According to Willems, two AR representations are equivalent if and only if they share the same behaviour. In the particular case where only the forward behaviour of the AR representations is of interest, then ‘unimodular equivalence’ between the corresponding polynomial matrices is the transformation that we are looking for. However, in the case where both forward and backward behaviour is under research, an extension of unimodular equivalence is needed, following the lines of divisor equivalence. Similar approaches can be applied to the wider class of normalized autoregressive moving average (NARMA) representations presented in Fuhrmann (2002).

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Appendix 1. Proof of Lemma 1

Let $A_1(s), A_2(s) \in \mathbb{R}_c[s]$ be related by the divisor equivalence transformation

$$M_1(s)A_1(s) = A_2(s)N_1(s) \quad (26)$$

while $A_2(s), A_3(s) \in \mathbb{R}_c[s]$ are related by the strict equivalence transformation

$$M_2A_2(s) = A_3(s)N_2 \quad (27)$$

where M_2, N_2 are constant and non-singular matrices (therefore $A_2(s)$ and $A_3(s)$ have the same dimensions). Premultiplying relation (26) by M_2 and using relation (27) we get the transformation

$$\begin{aligned} [M_2M_1(s)]A_1(s) &= A_3(s)[N_2N_1(s)] \\ \iff [M_2M_1(s) \quad A_3(s)] \begin{bmatrix} A_1(s) \\ -N_2N_1(s) \end{bmatrix} &= 0. \end{aligned} \quad (28)$$

Our main goal is to prove that the compound matrices involved in (28) satisfy the properties of divisor equivalence. We observe that

$$\begin{bmatrix} I & 0 \\ 0 & N_2 \end{bmatrix} \begin{bmatrix} A_1(s) \\ -N_1(s) \end{bmatrix} = \begin{bmatrix} A_1(s) \\ -N_2N_1(s) \end{bmatrix} I$$

and

$$M_2 \begin{bmatrix} M_1(s) & A_2(s) \end{bmatrix} = \begin{bmatrix} M_2M_1(s) & A_3(s) \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & N_2 \end{bmatrix}$$

are strict equivalent transformations and thus the compound matrices

$$\begin{bmatrix} M_2M_1(s) & A_3(s) \end{bmatrix}; \quad \begin{bmatrix} A_1(s) \\ -N_2N_1(s) \end{bmatrix} \quad (29)$$

possess the same f.e.d. and i.e.d. with the respective compound matrices

$$\begin{bmatrix} M_1(s) & A_2(s) \end{bmatrix}; \quad \begin{bmatrix} A_1(s) \\ -N_1(s) \end{bmatrix}. \quad (30)$$

However, (26) is a divisor equivalence transformation and thus the compound matrices (30) or equivalently (29) do not possess any f.e.d. or i.e.d. Therefore, (28) is a divisor equivalence transformation.

Appendix 2. Proof of Lemma 2

Let $s_0 \in \mathbb{R}$ such that $\det[A(s_0)] \neq 0$, $\Lambda_0 := \text{diag}[s_0, s_0, \dots, s_0] \in \mathbb{R}^{r \times r}$ so that $sI_r - \Lambda_0, A(s)$ are coprime. Then consider the identity

$$\begin{aligned} & \underbrace{\begin{bmatrix} 0_{(q-1)r,r} \\ sI_r - \Lambda_0 \end{bmatrix} A(s)}_{M(s)} \\ &= \begin{bmatrix} sI_r & -I_r & 0 & \cdots & 0 & 0 \\ 0 & sI_r & -I_r & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & sI_r & -I_r \\ A_0 & A_1 & A_2 & \cdots & A_{q-2} & sA_q + A_{q-1} \end{bmatrix} \\ & \quad \underbrace{\times \begin{bmatrix} I_r \\ sI_r \\ \vdots \\ s^{q-1}I_r \end{bmatrix}}_{N(s)} (sI_r - \Lambda_0). \end{aligned} \tag{31}$$

In the compound matrix

$$\begin{aligned} & [M(s) \quad sE - A] \\ &= \begin{bmatrix} 0 & sI_r & -I_r & 0 & \cdots & 0 & 0 \\ 0 & 0 & sI_r & -I_r & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & sI_r & -I_r \\ (sI_r - \Lambda_0) & A_0 & A_1 & A_2 & \cdots & A_{q-2} & sA_q + A_{q-1} \end{bmatrix} \end{aligned}$$

firstly the rq -order minors $\det[sE - A]$ and $\det[L(s)]$ where

$$L(s) := \begin{bmatrix} 0 & -I_r & 0 & \cdots & 0 & 0 \\ 0 & sI_r & -I_r & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & sI_r & -I_r \\ (sI_r - \Lambda_0) & A_1 & A_2 & \cdots & A_{q-2} & sA_q + A_{q-1} \end{bmatrix}$$

are coprime since $A(s)$ and $sI_r - \Lambda_0$ are coprime.

Secondly it is easily seen that the rank of the highest degree coefficient matrix of $[M(s) \quad sE - A]$ is

$$\begin{aligned} & \text{rank}_{\mathbb{R}} [\tilde{M}(0) \quad E] \\ &= \text{rank}_{\mathbb{R}} \begin{bmatrix} 0 & I_r & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & I_r & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & I_r & 0 \\ I_r & 0 & 0 & 0 & \cdots & 0 & A_q \end{bmatrix} = rq \end{aligned}$$

and therefore the compound matrix $[M(s) \quad sE - A]$ has no i.e.d.

Now consider the second compound matrix

$$\begin{bmatrix} A(s) \\ -N(s) \end{bmatrix} = \begin{bmatrix} A(s) \\ (sI_r - \Lambda_0) \\ (sI_r - \Lambda_0)s \\ \vdots \\ (sI_r - \Lambda_0)s^{q-1} \end{bmatrix}.$$

First we can easily find two greatest order minors, i.e.

$$Q_1(s) = A(s) \quad \text{and} \quad Q_2(s) = sI_r - \Lambda_0$$

with

$$\det[Q_1(s)] = \det[A(s)] \quad \text{and} \quad \det[Q_2(s)] = \det[sI_r - \Lambda_0].$$

However Λ_0 is selected such that the greatest common divisor of $\det[Q_1(s)]$ and $\det[Q_2(s)]$ is 1. Therefore the compound matrix $[\underbrace{A(s)}_{-N(s)}]$ has no f.e.d.

Second the highest degree coefficient matrix of $[\underbrace{A(s)}_{-N(s)}]$ i.e. $[A_q^T \quad 0 \quad \cdots \quad I_r]^T$ has full column rank and thus the compound matrix $[\underbrace{A(s)}_{-N(s)}]$ has no i.e.d.

Thus, we have proved that the compound matrices $[M(s) \quad sE - A]$ and $[\underbrace{A(s)}_{-N(s)}]$ involved in (31) satisfy the conditions of d.e., and therefore $A(s)$ and $sE - A$ are d.e.

Similarly we can show that the following identity between $sE - A$ and $A(s)$ is a d.e. transformation

$$\begin{aligned} & \underbrace{\begin{bmatrix} -(sI_r - \Lambda_0)s^{q-2}E_0(s) & -(sI_r - \Lambda_0)s^{q-3}E_1(s) \\ \cdots & -(sI_r - \Lambda_0)E_{q-2}(s) \end{bmatrix}}_{M(s)} \underbrace{\begin{bmatrix} (sI_r - \Lambda_0)s^{q-1} \\ \cdots \\ (sI_r - \Lambda_0)s^{q-1} \end{bmatrix}}_{N(s)} \\ & \times \begin{bmatrix} sI_r & -I_r & 0 & \cdots & 0 & 0 \\ 0 & sI_r & -I_r & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & sI_r & -I_r \\ A_0 & A_1 & A_2 & \cdots & A_{q-2} & sA_q + A_{q-1} \end{bmatrix} \\ & \quad \underbrace{\hspace{10em}}_{sE-A} \\ &= A(s) \underbrace{\begin{bmatrix} 0_{r,(q-1)r} & (sI_r - \Lambda_0) \end{bmatrix}}_{N(s)} \end{aligned} \tag{32}$$

where $E_i(s) = E_{i-1}(s) + A_i s$, $i = 0, 1, \dots, q - 2$ with $E_0 = A_0$ and Λ_0 as defined above.

Appendix 3. Proof of Lemma 3(a)

Assume that there exists $M(s)$ with $\deg M(s) > \deg A_2(s)$. In order for $[M(s) \ A_2(s)]$ to have no i.e.d. its highest coefficient matrix must be of full row rank, i.e. $\text{rank}[M_{hc} \ 0] = m + r$. This is impossible because of the dimension of the matrix $M_{hc} \in \mathbb{R}^{(m+r) \times m}$. So $\deg M(s) \leq \deg A_2(s)$.

Let $\deg N(s) > \deg A_1(s)$. Then there exists $d \neq 0$ such that $\deg A_1(s) + d = \deg N(s)$. Condition (i) of d.e. implies that the matrices $A_1(s)$ and $A_2(s)$ are e.u.e. so they have the same f.e.d. (and of course the same number of f.e.d., i.e. $S_{\mathbb{R}}(A_1(s)) = S_{\mathbb{R}}(A_2(s))$ where $S_{\mathbb{R}}(A(s))$ denotes the total number of f.e.d. of $A(s)$ (order accounted for). Taking the duals of the compound matrices in (4) we get

$$[M'(w) \ \tilde{A}_2(w)] \begin{bmatrix} w^d \tilde{A}_1(w) \\ -\tilde{N}(w) \end{bmatrix} = 0. \tag{33}$$

Since the compound matrices in (5) have no i.e.d., the compound matrices in (33) will have no f.e.d. at $s = 0$ and thus (33) is a $\{0\}$ -equivalence relation. Therefore, $\tilde{A}_2(w)$ and $w^d \tilde{A}_1(w)$ have the same f.e.d. at 0. Denoting by $S_l(A(w))$ the total number of f.e.d. (order accounted for) at l of $A(w)$ we have

$$\begin{aligned} S_0(\tilde{A}_2(w)) &= S_0(w^d \tilde{A}_1(w)) > S_0(\tilde{A}_1(w)) \\ \implies S_0(\tilde{A}_2(w)) &> S_0(\tilde{A}_1(w)) \\ \implies S_{\infty}(A_2(s)) &> S_{\infty}(A_1(s)). \end{aligned} \tag{34}$$

According to our assumption $A_1(s), A_2(s) \in \mathbb{R}_c[s]$. Thus $c = S_{\infty}(A_2(s)) + S_{\mathbb{R}}(A_2(s)) = S_{\infty}(A_1(s)) + S_{\mathbb{R}}(A_1(s))$ or equivalently since $S_{\mathbb{R}}(A_2(s)) = S_{\mathbb{R}}(A_1(s))$ we have that $S_{\infty}(A_2(s)) = S_{\infty}(A_1(s))$ which contradicts (34).

Proof of Lemma 3(b): First we shall prove that if one of the chosen transforming matrices has degree more than d then $\deg M(s) = \deg N(s)$. Let

$$\begin{aligned} d_M &= \deg M, & d_N &= \deg N \\ D_L &= \deg[M(s) \ A_2(s)]; & D_R &= \deg \begin{bmatrix} A_1(s) \\ -N(s) \end{bmatrix} \\ d &= \deg A_1(s) = \deg A_2(s). \end{aligned}$$

Then

$$\begin{aligned} [M(s) \ \tilde{A}_2(s)] &= [w^{-d_M+D_L} \tilde{M}(w) \quad w^{-d+D_L} \tilde{A}_2(w)] \\ \begin{bmatrix} \tilde{A}_1(s) \\ -N(s) \end{bmatrix} &= \begin{bmatrix} w^{-d+D_R} \tilde{A}_1(w) \\ -w^{-d_N+D_R} \tilde{N}(w) \end{bmatrix}. \end{aligned}$$

From (4) taking the duals gives

$$[w^{-d_M+D_L} \tilde{M}(w) \quad w^{-d+D_L} \tilde{A}_2(w)] \begin{bmatrix} w^{-d+D_R} \tilde{A}_1(w) \\ -w^{-d_N+D_R} \tilde{N}(w) \end{bmatrix} = 0. \tag{35}$$

It is easily seen for the reasons explained in the proof of the previous lemma that (35) is a $\{0\}$ -equivalence relation either between $w^{-d+D_L} \tilde{A}_2(w)$ and $w^{-d+D_R} \tilde{A}_1(w)$ or between $w^{-d_M+D_L} \tilde{M}(w)$ and $w^{-d_N+D_R} \tilde{N}(w)$ and thus

$$\begin{aligned} \begin{cases} S_0(w^{-d_M+D_L} \tilde{M}(w)) = S_0(w^{-d_N+D_R} \tilde{N}(w)) \\ S_0(w^{-d+D_L} \tilde{A}_2(w)) = S_0(w^{-d+D_R} \tilde{A}_1(w)) \end{cases} \\ \implies S_{\infty}(M(s)) + (-d_M + D_L)m \\ = S_{\infty}(N(s)) + (-d_N + D_R)m \end{aligned} \tag{36}$$

$$S_{\infty}(A_2(s)) + (-d + D_L)m = S_{\infty}(A_1(s)) + (-d + D_R)m. \tag{37}$$

Also (4) is an e.u.e. relation and

$$S_{\mathbb{R}}(M(s)) = S_{\mathbb{R}}N(s); \quad S_{\mathbb{R}}(A_2(s)) = S_{\mathbb{R}}(A_1(s)). \tag{38}$$

But since $A_1(s)$ and $A_2(s) \in \mathbb{R}_c[s]$ we have

$$\begin{aligned} S_{\mathbb{R}}(A_1(s)) + S_{\infty}(A_1(s)) &= S_{\mathbb{R}}(A_2(s)) + S_{\infty}(A_2(s)) \\ \stackrel{(37)}{\implies} S_{\mathbb{R}}A_1(s) + S_{\infty}A_2(s) + (-d + D_L)m \\ - (-d + D_R)m &= S_{\infty}A_2(s) + S_{\mathbb{R}}A_2(s) \\ \stackrel{(38)}{\implies} D_L &= D_R. \end{aligned} \tag{39}$$

Assume that $d_M > d$. Then

$$d_M = D_L \stackrel{(39)}{=} D_R \tag{40}$$

i.e. the degree of the compound matrix is equal to the maximum degree of $M(s)$ and $A_2(s)$ which is d_M . From (39) and (40) we have that

$$D_R = d_M > d.$$

The above inequality tells us that the degree of the right compound matrix in (4) (the maximum degree of $N(s)$ and $A_1(s)$) is more than $\deg(A_1(s))$ and thus $D_R = d_N$. Therefore we conclude that $d_M = D_L = D_R = d_N$. Similar proofs also apply in the case where we assume that $d_N > d$.

Second we shall show that if one of the chosen transforming matrices has degree less than d then $\deg M(s) = \deg N(s)$. Suppose that $N(s)$ has degree d_N such that $d_N(s) < d_M(s) \leq d$, i.e. note that from the first part of the proof if one of the matrices has degree

less than d then the other one cannot have degree more than d . Then by equating the coefficient matrices of the highest degrees of s in (4) we get that

$$M^{hc} A_1^{hc} = 0 \quad (41)$$

where A^{hc} denotes the highest column degree coefficient matrix of the polynomial matrix $A(s)$. Since $d_N(s) < d$ we have that A_1^{hc} has full column rank and therefore $\dim(\text{Ker}(M^{hc})) = m$. Thus $\text{rank}(M^{hc}) = 0$ and therefore

$\deg M(s) < d_M$ which contradicts our second assumption.

Appendix 4. Proof of Lemma 4

The proof is trivial having in mind that (a) the f.e.d. of $A_1(s)(s - s_0)^k$ is the f.e.d. of $A_1(s)$ plus m divisors of the form $(s - s_0)^k$ and (b) the i.e.d. of $A_1(s)(s - s_0)^k \in \mathbb{R}^{m \times m}[s]$, where $s_0 \neq 0$ is not a zero of $A_1(s)$, is exactly the i.e.d. of $A_1(s)$.