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COMPUTING GENERALIZED INVERSES OF A RATIONAL MATRIX AND APPLICATIONS

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Abstract. In this paper we investigate symbolic implementation of two modifications of the Leverrier-Faddeev algorithm, which are applicable in computation of the Moore-Penrose and the Drazin inverse of rational matrices. We introduce an algorithm for computation of the Drazin inverse of rational matrices. This algorithm represents an extension of the papers [11] and [14]. and a continuation of the papers [15, 16]. The symbolic implementation of these algorithms in the package MATHEMATICA is developed. A few matrix equations are solved by means of the Drazin inverse and the Moore-Penrose inverse of rational matrices.

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1. Introduction

Let $\mathbf{C}$ be the set of complex numbers, $\mathbf{C}^{m\times n}$ be the set of $m \times n$ complex matrices, and $\mathbf{C}_r^{m\times n} = \{X \in \mathbf{C}^{m\times n} \ : \ \operatorname{rank}(X) = r\}$. As usual, $\mathbf{C}(s)$ denotes rational functions with complex coefficients in the indeterminate s. The $m \times n$ matrices with elements in $\mathbf{C}(s)$ are denoted by $\mathbf{C}(s)^{m\times n}$. By I_r and I we denote the identity matrix of the order r, and an appropriate identity matrix, respectively. By $\mathbf{O}$ is denoted an appropriate null matrix and by $\operatorname{Tr}(A)$ is denoted the trace of A.

For any matrix A of the order $m \times n$ consider the following equations in X, where $*$ denotes conjugate and transpose:

$$(1) \quad AXA=A \quad (2) \quad XAX=X \quad (3) \quad (AX)^*=AX \quad (4) \quad (XA)^*=XA,$$

and if $m = n$, also

$$(5) \quad AX = XA \quad (1^k) \quad A^{k+1}X = A^k.$$

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If X satisfies (1) and (2), it is said to be a reflexive g-inverse of A, whereas $X = A^{\dagger}$ is said to be the Moore-Penrose inverse of A if it satisfies (1)–(4). The group inverse $A^{\#}$ is the unique $\{1, 2, 5\}$ inverse of A, and exists if and only if $\mathrm{ind}(A) = \min_{k}\{k : \mathrm{rank}(A^{k+1}) = \mathrm{rank}(A^k)\} = 1$. A matrix $X = A^D$ is said to be the Drazin inverse of A if (1^k) (for some positive integer k), (2) and (5) are satisfied. In the case $\mathrm{ind}(A) = 1$, the Drazin inverse of A is equal to the group inverse of A.

Computation of generalized inverses of a constant complex matrix $A(s) \equiv A \in \mathbf{C}^{m\times n}$ by means of the Leverrier-Faddeev algorithm (also called the Souriau-Frame algorithm) has been investigated in many papers [1, 6, 9, 10, 11, 12, 14]. Also, computation of generalized inverses of polynomial matrices, based on the Leverrier-Faddeev algorithm, has been investigated in [1, 7, 8, 15, 16, 17, 18]. Moreover, in the literature it is known a large number of applications of the Leverrier-Faddeev algorithm and generalized inverses of polynomial matrices [15, 16, 17, 19, 20]. In [15] is described an implementation of the algorithm for computing the Moore-Penrose inverse of a singular rational matrix in the symbolic computational language MAPLE.

A modification of the Leverrier-Faddeev algorithm for obtaining the Drazin inverse of a constant square was given in [11]. Hartwig in [12] continues investigation of this algorithm. In [14] is presented a new proof for Grevile's finite algorithm. In [1] is presented a new derivation of the Leverrier-Faddeev algorithm. Also, in [1], the method is extended to produce a corresponding computational scheme for the inverse of the polynomial matrix $s^2 I_n - sA_1 - A_2$. In [21] is introduced a new extension of Leverrier's algorithm for computing the inverse of a matrix polynomial of arbitrary degree: $s^n I - s^{n-1}A_1 - \cdots sA_{n-1} - A_n$. In [16] is utilized a representation and two algorithms for computation of the Moore-Penrose inverse of a nonregular polynomial matrix of an arbitrary degree.

In [2] Bu and Wei are proposed a finite algorithm for computing Drazin inverse of two-variable polynomial matrices based on Grevile's finite algorithm for computing Drazin inverse of a constant matrix. Also, in [2] the implementation of introduced algorithms is developed in the symbolic package MATLAB.

Recently in [4] are investigated some interesting properties of weighted Moore-Penrose inverse of partitioned matrices in Banachiewicz-Schur form. Also, in [5] are investigated a condition number for the W-Weighted Drazin inverse and its applications in the solution of rectangular linear system.

Main disadvantages of the described symbolic implementation in the package MATHEMATICA are similar with the corresponding one, arising from the symbolic implementation in the package MAPLE. As known that the formula manipulation by a computer requires much more time and memory space with respect to the numerical implementation. Also, symbolic packages are in general "slow". Sometimes, the explosion of the size of generated symbolic expressions is actually far more terrible than we might naively anticipate [13].

Main advantages of the symbolic implementation in MATHEMATICA [22, 23] are as follows. Variables can be stored without loss of accuracy during calculations. Symbolic computation is free from the truncation error. Moreover, algorithms presented for rational matrices are applicable to significantly wider classes of matrices and to a wider set of problems, with respect to algorithms intended for constant matrices. Also, algorithms applicable to polynomial or rational matrices can be used in the construction of test matrices and in the verification of some hypothesis. Finally, these algorithms can be verified directly on test matrices, as it is shown in *Example 1*.

The paper is organized as follows. In the second section we investigate an extension of the Grevile's modification of the Leverrier-Faddeev algorithm, to the set of one-variable nonregular rational matrices. There proposed results can be considered as a continuation of the papers [15, 16], and a generalization of the results from [11] and [14]. In the third section we investigate solutions of a few system of matrix equations, by means of the Drazin inverse and the Moore-Penrose inverse of rational matrices. In Appendix we describe implementation details of introduced algorithms in the package MATHEMATICA. Also, for the sake of completeness, we describe the implementation of the algorithm for computing the Moore-Penrose and the Drazin inverse of rational matrices, which is introduced in [15] and [16].

2. Computing generalized inverses

We consider rational complex matrices $A(s) \in \mathbf{C}(s)^{n\times n}$, where the variable s is an indeterminate. The proof of the following theorem is in [11] and [14].

Theorem 1. *Consider a nonregular one-variable rational matrix $A(s)$. Assume that*

$$\begin{aligned} a(z,s) &= \det\left[zI_n - A(s)\right] \\ &= a_0(s)z^n + a_1(s)z^{n-1} + \cdots a_{n-1}(s)z + a_n(s), \quad a_0(s) \equiv 1, \quad z \in \mathbf{C} \end{aligned} \tag{1}$$

is the characteristic polynomial of $A(s)$. Also, consider the following sequence of $n \times n$ rational matrices

$$\begin{aligned} &B_j(s) = a_0(s)A(s)^j + a_1(s)A(s)^{j-1} + \cdots a_{j-1}(s)A(s) + a_j(s)I_n, \\ &a_0(s) = 1, \quad j = 0, \dots, n \end{aligned} \tag{2}$$

Let $a_n(s) \equiv 0, \dots, a_{t+1}(s) \equiv 0$, and $a_t(s) \neq 0$. Define the following set: $\Lambda = \{s_i \in \mathbf{C} : \ a_t(s_i) = 0\}$.
Also, assume $B_n(s) \equiv \mathbf{O}, \dots, B_r(s) \equiv \mathbf{O}, \quad B_{r-1}(s) \neq \mathbf{O}$ and $k = r - t$.

In the case $s \in \mathbf{C}\backslash\Lambda$ and $k > 0$, the Drazin inverse of $A(s)$ is given by

$$\begin{aligned} A(s)^D &= a_t(s)^{-k-1}A(s)^k B_{t-1}(s)^{k+1} \\ &= a_t(s)^{-k-1}A(s)^k \left[a_0(s)A(s)^{t-1} + \cdots + a_{t-2}(s)A(s) + a_{t-1}(s)I_n\right]^{k+1}. \end{aligned} \tag{3}$$

In the case $s \in \mathbf{C}\backslash\Lambda$ and $k = 0$, we get $A(s)^D = \mathbf{O}$.

For $s_i \in \Lambda$ denote by t_i the largest integer satisfying $a_{t_i}(s_i) \neq 0$, and by r_i the smallest integer satisfying $B_{r_i}(s_i) \equiv \mathbf{O}$. Then the Drazin inverse of $A(s_i)$ is equal to

$$\begin{aligned} A(s_i)^D &= a_{t_i}(s_i)^{-k_i-1} A(s_i)^{k_i} B_{t_i-1}(s_i)^{k_i+1} \\ &= a_{t_i}(s_i)^{-k_i-1} A(s_i)^{k_i} \Big[a_0(s_i) A(s_i)^{t_i-1} + \cdots + a_{t_i-2}(s_i) A(s_i) \\ &\qquad + a_{t_i-1}(s_i) I_n \Big]^{k_i+1} \end{aligned} \tag{4}$$

where $k_i = r_i - t_i$.

In view of the results of *Theorem 1* we present the following algorithm for computation of the Drazin inverse.

Algorithm 1. *It is assumed that $A(s) \in \mathbf{C}(s)^{n\times n}$ is a given rational matrix.*

- *Step 1. Construct the sequence of numbers $\{a_0(s), a_1(s), \dots, a_n(s)\}$ and the sequence of rational matrices $\{B_0(s), B_1(s), \dots, B_n(s)\}$ as in the following:*

$$\begin{aligned} &A_0(s) = \mathbf{O}, \ \ a_0(s) = 1, \ \ B_0(s) = I_n \\ &A_1(s) = A(s)B_0(s), \ \ a_1(s) = -\frac{Tr(A_1(s))}{1}, \ \ B_1(s) = A_1(s) + a_1(s) I_n \\ &\qquad \dots \qquad \dots \qquad \dots \\ &A_n(s) = A(s)B_{n-1}(s), \ \ a_n(s) = -\frac{Tr(A_n(s))}{n}, \ \ B_n(s) = A_n(s) + a_n(s) I_n. \end{aligned}$$

- *Step 2. Let $t = \max\{l : a_l(s) \neq 0\}$, $r = \min\{l : B_l(s) = \mathbf{O}\}$, $k = r - t$. For $s \in \mathbf{C}\backslash\Lambda$ the Drazin inverse $A(s)^D$ is given by*

$$A(s)^D = a_t(s)^{-k-1} A(s)^k B_{t-1}(s)^{k+1}.$$

  For those $s_i \in \Lambda$, denote by t_i the largest integer satisfying $a_{t_i}(s_i) \neq 0$, and by r_i the smallest integer satisfying $B_{r_i}(s_i) \equiv \mathbf{O}$. For the integer $k_i = r_i - t_i$, the Drazin inverse of $A(s_i)$ is equal to

$$A(s_i)^D = a_{t_i}(s_i)^{-k_i-1} A(s_i)^{k_i} B_{t_i-1}(s_i)^{k_i+1}.$$

3. Application

In this section we solve a few systems of matrix equations, using the Moore-Penrose and the Drazin inverse of a given rational matrix.

The matrix equation $A(s)X(s)B(s) = C(s)$ is investigated in [16]. We restate here this result for the sake of completeness.

Lemma 1 ([16])**.** *The matrix equation $A(s)X(s)B(s) = C(s)$ in $X(s)$, where $A(s) \in \mathbf{C}(s)^{m\times n}$, $B(s) \in \mathbf{C}(s)^{p\times q}$ and $C(s) \in \mathbf{C}(s)^{m\times q}$, has a solution if and only if*

$$A(s)A(s)^{\dagger} C(s) B(s)^{\dagger} B(s) = C(s). \tag{5}$$

In this case, all the solutions are given by the formula

$$X(s) = A(s)^{\dagger}C(s)B(s)^{\dagger} + Y(s) - A(s)^{\dagger}A(s)Y(s)B(s)B(s)^{\dagger} \tag{6}$$

where $Y(s)$ is arbitrary and has the dimensions of $X(s)$. As it is in [16] mentioned, (5) *and* (6) *hold when $A(s)^{\dagger}$ and $B(s)^{\dagger}$ are replaced by specific $\{1\}$-inverses $A(s)^{(1)}$ and $B(s)^{(1)}$, respectively.*

We also use the following result.

Lemma 2. *System of rational matrix equations*

$$A(s)X(s) = B(s), \qquad X(s)D(s) = E(s) \tag{7}$$

has a common solution if and only if each of these equations has a solution and

$$A(s)E(s) = B(s)D(s). \tag{8}$$

In the following definition we introduce a notion of the index of rational square matrices.

Definition 1. The *index* of a given square rational matrix $A(s) \in \mathbf{C}[s]^{n\times n}$, denoted by $\operatorname{ind}(A(s))$, is equal to

$$\operatorname{ind}(A(s)) = \left\{k : \operatorname{rank}(A(s)^k) = \operatorname{rank}(A(s)^{k+1})\right\}.$$

In *Lemma 3* and *Lemma 4* we generalize known representations of the Drazin inverse and weak Drazin inverse of a given constant matrix from [3, p. 205].

Lemma 3. *Consider a square rational matrix $A(s) \in \mathbf{C}(s)^{n\times n}$ satisfying the condition ind$(A(s)) = k$. Let the rational matrix $A_c(s)$ be any solution of the system of matrix equations*

$$A(s)^l = X(s)A(s)^{l+1} \tag{9}$$

$$A(s)X(s) = X(s)A(s), \tag{10}$$

where $l \geq k$ is an arbitrary integer. Then $A_c(s)$ satisfies the following equalities:

$$A(s)^l(A_c(s))^l = (A_c(s))^l A(s)^l, \tag{11}$$

$$A(s)^l(A_c(s))^l A(s)^l = A(s)^l. \tag{12}$$

and the solution $A_c(s)$ always exists.

Proof. The equality in (11) follows from $A(s)A_c(s) = A_c(s)A(s)$. We verify the equality in (12). Because of $\operatorname{rank}(A^{k+1}) = \operatorname{rank}(A^k)$, it is easy to conclude the existence of the matrix $A_c(s)$. Using (9), (10) and (11), we get

$$\begin{aligned}
A(s)^l(A_c(s))^l A(s)^l &= A(s)^{2l}(A_c(s))^l \\
&= A(s)^{l-1}A(s)^{l+1}(A_c(s))^l \\
&= A(s)^{l-1}(A_c(s))^{l-1}A_c(s)A(s)^{l+1} \\
&= A(s)^{l-1}A(s)^l(A_c(s))^{l-1}
\end{aligned}$$

$$= A(s)^{2l-1}(A_c(s))^{l-1}.$$

Continuing in the same way, we obtain

$$A(s)^l(A_c(s))^l A(s)^l = A(s)^{l+1}A_c(s) = A(s)^l.$$

Hence, (12) is verified. □

Lemma 4. *Let $A(s) \in \mathbf{C}(s)^{n\times n}$, $ind(A(s)) = k$ and $l \geq k$. Let $A_c(s)$ be any solution of the system of matrix equations* (9) *and* (10). *Then*

$$A(s)^D = A(s)^l(A_c(s))^{l+1} = (A_c(s))^{l+1}A(s)^l. \tag{13}$$

Proof. Using the known representation of the Drazin inverse from [3], it is not difficult to verify the following representation of the Drazin inverse $A(s)^D$:

$$A(s)^D = A(s)^l(A(s)^{2l+1})^{(1)}A(s)^l.$$

Also, in view of (12), we get $(A_c(s))^l \in A(s)^l\{1\}$ and

$$A(s)^D = A(s)^l(A_c(s))^{2l+1}A(s)^l$$

Now, using equalities in (11) and (12) it follows that

$$A(s)^D = A(s)^l(A_c(s))^l A(s)^l(A_c(s))^{l+1} = A(s)^l(A_c(s))^{l+1}. \quad \square$$

Theorem 2. *Let $A(s) \in \mathbf{C}(s)^{n\times n}$ and $ind(A(s)) = k$. For arbitrary integers $l \geq k$ and $m \geq k$ each of the following matrix equations*

$$A(s)^l = X(s)A(s)^{l+1} \tag{14}$$

$$A(s)^m = A(s)^{m+1}X(s) \tag{15}$$

has the general solution, which are represented by the following expressions, respectively:

$$X(s) = A(s)^D + Y(s)\Big(I - A(s)A(s)^D\Big), \tag{16}$$

$$X(s) = A(s)^D + \Big(I - A(s)A(s)^D\Big)W(s), \tag{17}$$

where $Y(s)$ and $W(s)$ are appropriate rational matrices of the order $n \times n$. Also, the system of matrix equations (14) *and* (15) *has the following general solution:*

$$X(s) = A(s)^D + \Big(I - A(s)A(s)^D\Big)Z(s)\Big(I - A(s)A(s)^D\Big), \tag{18}$$

where $Z(s)$ is arbitrary $n \times n$ rational matrix.

Proof. According to *Lemma 1*, it is not difficult to verify consistency of the matrix equation (14). Indeed, using the results of *Lemma 3* and *Lemma 4*, we get

$$A(s)^l(A(s)^{l+1})^{(1)}A(s)^{l+1} = A(s)^l(A_c(s))^{l+1}A(s)^{l+1} = A(s)^D A(s)^{l+1}$$
$$= A(s)^l$$

which is a verification of *Lemma 1* for this case. Also, according to *Lemma 1*, *Lemma 3* and *Lemma 4*, the general solution of (16) is equal to

$$\begin{aligned}X(s) &= A(s)^l (A(s)^{l+1})^{(1)} + Y(s) - Y(s)A(s)^{l+1}(A(s)^{l+1})^{(1)} \\ &= A(s)^l (A_c(s))^{l+1} + Y(s)\left(I - A(s)A(s)^l (A_c(s))^{l+1}\right) \\ &= A(s)^D + Y(s)(I - A(s)A(s)^D).\end{aligned}$$

In a similar way one can verify that (17) is a general solution of (15). Now we derive a general solution of the system of matrix equations (14) and (15). Since $A(s)^{m+1}A(s)^l = A(s)^m A(s)^{l+1}$, according to *Lemma 2* we conclude consistency of the system of matrix equations (14) and (15). After a substitution of (16) in (15) we get

$$A(s)^m = A(s)^{m+1}\left[A(s)^D + Y(s)(I - A(s)A(s)^D)\right]$$

and later

$$A(s)^{m+1}\left[Y(s)(I - A(s)A(s)^D)\right] = \mathbf{O}.$$

Applying again the result of *Lemma 1*, we verify consistency of this equation and get the following general solution:

$$\begin{aligned}Y(s) = Z(s) &- (A(s)^{m+1})^{(1)} A(s)^{m+1} \\ &\times Z(s)(I - A(s)A(s)^D)(I - A(s)A(s)^D)^{(1)},\end{aligned}$$

where $Z(s)$ is an arbitrary rational matrix of appropriate dimensions. Since the matrix $I - A(s)A(s)^D$ is idempotent, it follows that $I \in (I - A(s)A(s)^D)\{1\}$. Also, according to *Lemma 3*, we get

$$Y(s) = Z(s) - (A_c(s))^{m+1}A(s)^m A(s)Z(s)(I - A(s)A(s)^D).$$

Using (13) we obtain

$$\begin{aligned}Y(s) &= Z(s) - A(s)^D A(s)Z(s)(I - A(s)A(s)^D) \\ &= (I - A(s)A(s)^D)Z(s)(I - A(s)A(s)^D).\end{aligned}$$

A substitution of $Y(s)$ in (16) leads to

$$\begin{aligned}X(s) &= A(s)^D + (I - A(s)A(s)^D)Z(s)(I - A(s)A(s)^D)(I - A(s)A(s)^D) \\ &= A(s)^D + (I - A(s)A(s)^D)Z(s)(I - A(s)A(s)^D).\end{aligned}$$

□

Theorem 3. *Let $A(s) \in \mathbf{C}(s)^{n\times n}$ satisfy ind$(A(s)) = k$. For arbitrary integers $l \geq k$ the following system of matrix equations*

$$A(s)X(s)A(s) = A(s) \qquad (19)$$

$$A(s)^l = X(s)A(s)^{l+1} \qquad (20)$$

has the general solution

$$X(s) = A(s)^D + Z(s)(I - A(s)A(s)^D) \qquad (21)$$

$$+ A(s)^{\dagger}(I - A(s)Z(s))(I - A(s)A(s)^{D})A(s)A(s)^{\dagger}(I - A(s)A(s)^{D})$$

where $Z(s)$ is an appropriate $n \times n$ rational matrix.

Proof. A substitution of the general solution (16) of the equation (14) in (19) produces the following matrix equation

$$A(s)\left[A(s)^{D} + Y(s)(I - A(s)A(s)^{D}\right]A(s) = A(s),$$

which is equivalent to

$$A(s)Y(s)(I - A(s)A(s)^{D})A(s) = (I - A(s)A(s)^{D})A(s).$$

From *Lemma 1*, it is easy to verify the consistency of this equation, and derive its general solution in the following form:

$$\begin{aligned} Y(s) = {} & A(s)^{\dagger}(I - A(s)A(s)^{D})A(s)\left[(I - A(s)A(s)^{D})A(s)\right]^{(1)} + Z(s) \\ & - A(s)^{\dagger}A(s)Z(s)(I - A(s)A(s)^{D})A(s)\left[(I - A(s)A(s)^{D})A(s)\right]^{(1)} \end{aligned}$$

where $Z(s)$ is an appropriate rational matrix. It is not difficult to verify

$$A(s)^{\dagger} \in \left((I - A(s)A(s)^{D})A(s)\right)\{1\}.$$

Also, using $I \in (I - A(s)A(s)^{D})\{1\}$ we get

$$\begin{aligned} Y(s) = {} & A(s)^{\dagger}(I - A(s)A(s)^{D})A(s)A(s)^{\dagger} + Z(s) \\ & - A(s)^{\dagger}A(s)Z(s)(I - A(s)A(s)^{D})A(s)A(s)^{\dagger} \end{aligned}$$

and later

$$Y(s) = Z(s) + A(s)^{\dagger}(I - A(s)Z(s))(I - A(s)A(s)^{D})A(s)A(s)^{\dagger}.$$

A substitution of $Y(s)$ in (16) produces (21) immediately. □

Theorem 4. *Let $A(s) \in \mathbf{C}(s)^{n\times n}$, $ind(A(s)) = k$. For arbitrary integers $l \geq k$ the following system of matrix equations*

$$A(s)X(s)A(s) = A(s) \tag{22}$$

$$A(s)^{m} = A(s)^{m+1}X(s) \tag{23}$$

has the following general solution

$$\begin{aligned} X(s) = {} & A(s)^{D} + (I - A(s)A(s)^{D})Z(s) + \\ & + (I - A(s)A(s)^{D})A(s)^{\dagger}A(s)(I - A(s)A(s)^{D})(I - Z(s)A(s))A(s)^{\dagger}, \end{aligned} \tag{24}$$

where $Z(s)$ is an arbitrary $n \times n$ rational matrix.

Proof. Similar with the proof of the previous theorem. □

Theorem 5. *Let* $A(s) \in \mathbf{C}(s)^{n\times n}$*,* $ind(A(s)) = k$*. For an arbitrary integer* $l \geq k$ *the system of matrix equations* (19)*,* (14) *and* (15) *has the following general solution*

$$\begin{aligned} X(s) = A(s)^D &+ (I - A(s)A(s)^D)Y(s)(I - A(s)A(s)^D) \\ &+ (I - A(s)A(s)^D)A(s)^\dagger A(s)(I - A(s)A(s)^D) \\ &\times \left[I - Y(s)(I - A(s)A(s)^D)A(s)\right] A(s)^\dagger(I - A(s)A(s)^D), \end{aligned}$$

where $Y(s)$ *is an arbitrary rational matrix.*

Proof. Substituting the general solution (16) of the system (14), (15) in the equation (22), we get the following equation in $Z(s)$:

$$A(s)\left[A(s)^D + (I - A(s)A(s)^D)Z(s)(I - A(s)A(s)^D)\right] A(s) = A(s)$$

or

$$A(s)(I - A(s)A(s)^D)Z(s)(I - A(s)A(s)^D)A(s) = A(s)(I - A(s)A(s)^D).$$

Consistency of this equation can be easily verified, and its general solution is equal to

$$\begin{aligned} Z(s) = &\left[A(s)(I - A(s)A(s)^D)\right]^{(1)} A(s)(I - A(s)A(s)^D) \\ &\times \left[(I - A(s)A(s)^D)A(s)\right]^{(1)} + Y(s) \\ &- \left[A(s)(I - A(s)A(s)^D)\right]^{(1)} A(s)(I - A(s)A(s)^D)Y(s) \\ &\times (I - A(s)A(s)^D)A(s)\left[(I - A(s)A(s)^D)A(s)\right]^{(1)}, \end{aligned}$$

where $Y(s)$ is an appropriate rational matrix. Since

$$A(s)(I - A(s)A(s)^D) = (I - A(s)A(s)^D)A(s)$$

and $A(s)^\dagger \in \left[A(s)(I - A(s)A(s)^D)\right]\{1\}$, it follows that

$$\begin{aligned} Z(s) &= Y(s) + A(s)^\dagger A(s)(I - A(s)A(s)^D)A(s)^\dagger \\ &\quad - A(s)^\dagger A(s)(I - A(s)A(s)^D)Y(s)(I - A(s)A(s)^D)A(s)A(s)^\dagger \\ &= Y(s) + A(s)^\dagger A(s)(I - A(s)A(s)^D) \\ &\quad \times \left[(I - Y(s)(I - A(s)A(s)^D)A(s)\right]A(s)^\dagger. \end{aligned}$$

The proof can be completed after the substitution of $Z(s)$ in (18). □

Remark 1. Theorems 3–5 are valid when $A(s)^\dagger$ is replaced by a specific $\{1\}$-inverse $A(s)^{(1)}$ of $A(s)$.

Example 1. Consider the matrix $A = \{\{s, 2, 3\}, \{3, 2, s\}, \{s, 2, 3\}\}$. The index of A is 2. Using the function call *Th31a*[A,2] we obtain the following expression which represents the solution (16) of the matrix equation (14).

$$\Big\{\Big\{\frac{1}{(5+s)^3}(-6+155y[1,1]+30y[1,2]+30y[1,3]+s^3(2y[1,1]+y[1,2]$$
$$+y[1,3])+3s(-1+32y[1,1]+7y[1,2]+7y[1,3]),$$
$$s^2(-1+23y[1,1]+8y[1,2]+8y[1,3])),$$
$$\frac{2(5+s)y[1,1]+(35+12s+s^2)y[1,2]+2(-1+5y[1,3]+sy[1,3])}{(5+s)^2},$$
$$\frac{1}{(5+s)^3}(-9+45y[1,1]+45y[1,2]+170y[1,3]+s^3y[1,3]$$
$$+5s^2(y[1,1]+y[1,2]+4y[1,3])+s(-5+34y[1,1]+34y[1,2]+109y[1,3]))\Big\},$$
$$\Big\{\frac{1}{(5+s)^3}(-6+155y[2,1]+30y[2,2]+30y[2,3]+s^3(2y[2,1]+y[2,2]+y[2,3])$$
$$+3s(-1+32y[2,1]+7y[2,2]+7y[2,3])+s^2(-1+23y[2,1]+8y[2,2]+8y[2,3])),$$
$$\frac{2(5+s)y[2,1]+(35+12s+s^2)y[2,2]+2(-1+5y[2,3]+sy[2,3])}{(5+s)^2},$$
$$\frac{1}{(5+s)^3}(-9+45y[2,1]+45y[2,2]+170y[2,3]+s^3y[2,3]$$
$$+5s^2(y[2,1]+y[2,2]+4y[2,3])+s(-5+34y[2,1]+34y[2,2]+109y[2,3]))\Big\},$$
$$\frac{1}{(5+s)^3}(-6+155y[3,1]+30y[3,2]+30y[3,3]+s^3(2y[3,1]+y[3,2]+y[3,3])$$
$$+3s(-1+32y[3,1]+7y[3,2]+7y[3,3])+s^2(-1+23y[3,1]+8y[3,2]+8y[3,3])),$$
$$\frac{2(5+s)y[3,1]+(35+12s+s^2)y[3,2]+2(-1+5y[3,3]+sy[3,3])}{(5+s)^2},$$
$$\Big\{\frac{1}{(5+s)^3}(-9+45y[3,1]+45y[3,2]+170y[3,3]+s^3y[3,3]$$
$$+5s^2(y[3,1]+y[3,2]+4y[3,3])+s(-5+34y[3,1]+34y[3,2]+109y[3,3]))\Big\}\Big\}.$$

In this expression $y[i,j]$, $i,j=1,2,3$ are arbitrary rational functions of s.

4. Conclusion

We introduce an algorithm for computation of the Drazin inverse of singular rational matrices. This algorithm is an extension of the modification of the Leverrier-Faddeev algorithm, introduced in [11, 14]. We develop an implementation of this algorithm in the package MATHEMATICA. This algorithm and programs can be considered as continuations of the papers [11, 14, 15, 16, 18]. Programs developed in MATHEMATICA are universal, and can be applied for computation of the Drazin inverse of rational, polynomial matrices, or constant complex matrices. An illustrative example is presented. Several systems of matrix equations are solved by means of the Drazin and the Moore-Penrose inverse for rational matrices. A symbolic computation of these solutions is presented.

Appendix

For the sake of completeness we present the following code for the computation of the Moore-Penrose inverse of rational matrices, based on the algorithm which is introduced in [15] and [16].

```
 Index[a_]:= Block[{b=a,c=IdentityMatrix[Length[a]],d=a,k=0},
    While[rank[c]=!=rank[d],d=d.b;c=c.b;k += 1];k]

rank[a_]:= Module[{b=a,i,m,n,r,c},
   {m,n}=Dimensions[b];
   b=RowReduce[b];r=Sum[zeros[b[[i]]],{i,m}]];MatrixQ[a]

SquareMatrixQ[a_]:=Length[a]==Length[a[[1]]];MatrixQ[a]

zeros[u_]:=Module[{v=u,n,i=1,lg=0},
      n=Length[v];While[i<=n,If[v[[i]]=!=0,lg=1];i++];lg]

MoorePenrose[t_]:=
   Block[{a=t,bj,j1,bj2,act,amp,e,c,d,j=0,m,n,pj=1},
    act=Conjugate[Transpose[a]];
    {m,n}=Dimensions[a];
    e=IdentityMatrix[m];
    c=a.act;bj=bj1=e;
    While[pj!=0,j++;d=c.bj;pj1=pj;pj=j^(-1)*Trg[d];
      bj2=bj1;bj1=bj;bj=d- pj*e];
    If[j==1,amp=Table[0,{m},{n}],amp=pj1^(-1)*act.bj2];
    Return[Simplify[amp]]]
```

The method described in *Algorithm 1* is implemented by means of the following code.

```
DrazinInverse[a_]:=Module[{bj,bj1,bs1,drz,nul,e,c,j=0,m,n,
  aj=1,as,k,r,s,logr,logs},logr=logs=True;{m,n}=Dimensions[a];
  If[m =!= n,Print["Matrix is not square"],
    nul=Table[0,{m},{n}];e=IdentityMatrix[m];bj=bj1=e;
    While[(logr || logs),j++;c= a.bj;
      aj=-Simplify[j^(-1)*Tr[c]];
      bj1=bj; bj=Simplify[a.bj1+aj*e];
    If[aj=!=0,s=j;as=aj;bs1=bj1;logs=False];
    If[bj==nul,r=j;logr=False]];
    k=r-s;
    drz=Simplify[as^(-k-1)*MatrixPower[a,k].MatrixPower[bs1,k+1]];
 Return[Simplify[drz]]]]
```

In the rest of this section we present a symbolic implementation of the solutions given in *Theorem 2–5* in the package MATHEMATICA. Symbolic expressions (16) and (17) can be generated by means of the following functions *Th31a[a,l]* and *Th31b[a,l]*. The value of the formal parameter l in the function *Th31a[a,l]* represents the exponent l in (14). Similarly, the parameter l in the function *Th31b[a,l]* represents the exponent m in (15).

```
Th31a[a_,l_]:=Block[{d,x,m,n,e},
 {m,n}=Dimensions[a];
 e=IdentityMatrix[n];
 If[m=!=n,Print["Matrix is not square "],
   If[l<Index[a],Print["l is smaller than index"],
      d=DrazinInverse[a];
      x=Array[y,{n,n}];
 Return[Simplify[d + x.(e - a.d)]]]]]

Th31b[a_,l_]:=Block[{d,x,m,n,e},
 {m,n}=Dimensions[a];
 e=IdentityMatrix[n];
 If[m=!=n,Print["Matrix is not square "],
   If[l<Index[a],Print["m is smaller than index"],
     d=DrazinInverse[a]; x=Array[w,{n,n}];
 Return[Simplify[d+(e-a.d).x]]]]]
```

The general solution (21) is implemented in the following function.

```
Th32[a_,l_]:=Block[{d,g,x,m,n,e,d1},
  {m,n}=Dimensions[a];
  e=IdentityMatrix[n];
  If[m=!=n,Print["Matrix is not square "],
    If[l<Index[a],Print["m is smaller than index"],
     d=DrazinInverse[a];
     g=MoorePenrose[a];
     x=Array[z,{n,n}];
     d1=Simplify[e-a.d];
  Return[Simplify[d+x.d1+g.(e-a.x).d1.a.g.d1]]]]]
```

The general solution in (24) can be implemented applying the following function.

```
Th33[a_,l_]:=Block[{d,g,x,m,n,e,d1},
  {m,n}=Dimensions[a];
  e=IdentityMatrix[n];
  If[m=!=n,Print["Matrix is not square "],
   If[l<Index[a],Print["m is smaller than index"],
     d=DrazinInverse[a];
     g=MoorePenrose[a];
     x=Array[z,{n,n}];
     d1=Simplify[e-a.d];
  Return[Simplify[d+d1.x+d1.g.a.d1.(e-x.a).g]]]]]
```

The parameters l and $l1$ in the function *Th34[a,l,l1]* denote, respectively, values of the exponents l and m in the expressions (14) and (15).

```
Th34[a_,l_,l1_]:=Block[{d,g,x,m,n,e,d1},
  {m,n}=Dimensions[a];
  e=IdentityMatrix[n];
  If[m=!=n,Print["Matrix is not square "],
    If[l<Index[a] || l1 < Index[a],
     Print["l or m is smaller than index"],
```

```
    d=DrazinInverse[a];
    g=MoorePenrose[a];
    x=Array[y,{n,n}];
    d1=Simplify[e-a.d];
Return[Simplify[d+d1.x.d1+d1.g.a.d1.(e-x.d1.a).g.d1]]]]]
```

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