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Equivalence of AR-representations in the light of the impulsive-smooth behaviour

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SUMMARY

The paper presents a new notion of equivalence of non-regular AR-representations, based on the coincidence of the impulsive-smooth behaviours of the underlying systems. The proposed equivalence is characterized by a special case of the usual unimodular equivalence and a restriction of the matrix transformation of full equivalence (*Int. J. Control* 1988; **47**(1):53–64). Copyright © 2006 John Wiley & Sons, Ltd.

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1. INTRODUCTION

An equivalence relation preserving the structures of matrices, in a systems theory context, first appeared in [1], as *strict system equivalence* and its modification known as *Fuhrmann system equivalence* in [2]. These equivalences guarantee that the systems have the same finite frequency structure. Thus, systems with the same finite structure exhibit the same smooth behaviour, while in [3] it is shown that strict system equivalence implies the existence of an isomorphism between the smooth solution spaces of the systems. To analyse simultaneously the finite and infinite structures of the system matrices Verghese [4] proposed, in the case of generalized state space systems, the notion of *strong equivalence* which took on a closed-form description in [5] as *complete system equivalence*. In [6, 7] an interpretation of these equivalences as an isomorphism of the corresponding behaviours was given.

Behaviours were introduced in [8, 9] and have since then been extensively studied. In this context, two AR-representations are equivalent if they represent the same smooth behaviour [10] or if they

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represent isomorphic smooth behaviours [11–13]. In the first case, the polynomial matrices that describe the AR-representations are shown to be *unimodular equivalent*, whereas in the second case they are *Fuhrmann system equivalent* [12]. The behavioural approach, at least in its original form, is not concerned with the infinite frequency (impulsive) behaviour. During recent studies (see [14–18]) the importance of impulsive behaviour in 'switched' or 'multimode' systems is recognized, and relevant questions about minimality and equivalence, in a behavioural framework, are addressed. The following example, comes from [14], and demonstrates the limitation of unimodular equivalence in preserving the 'impulsive-smooth' solution space of AR-representations.

Example 1 (Bourles [14])
Consider the following systems:

$$\Sigma_1 : \underbrace{\begin{bmatrix} -1 & \partial+1 & 0 & 0 \\ 0 & 0 & \partial & -1 \\ 0 & 1 & -1 & 0 \end{bmatrix}}_{A_1(\partial)} \underbrace{\begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \end{bmatrix}}_{\xi^1} = 0$$

where $\partial := \mathrm{d}/\mathrm{d}t$, and the system

$$\Sigma_2 : \underbrace{\begin{bmatrix} -1 & \partial+1 & 0 & 0 \\ 1 & -1 & 0 & -1 \\ 0 & 1 & -1 & 0 \end{bmatrix}}_{A_2(\partial)} \underbrace{\begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \end{bmatrix}}_{\xi^2} = 0$$

arising from unimodular equivalent transformations on Σ_1

$$\underbrace{\begin{bmatrix} -1 & \partial+1 & 0 & 0 \\ 1 & -1 & 0 & -1 \\ 0 & 1 & -1 & 0 \end{bmatrix}}_{A_2(\partial)} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & \partial \\ 0 & 0 & 1 \end{bmatrix}}_{U(\partial)} \underbrace{\begin{bmatrix} -1 & \partial+1 & 0 & 0 \\ 0 & 0 & \partial & -1 \\ 0 & 1 & -1 & 0 \end{bmatrix}}_{A_1(\partial)}$$

It is explained in [14] that although this equivalence is valid in case when ξ^i are C^∞ (or at least 'smooth enough') functions from $\mathbb{R}$ to $\mathbb{R}$, it is not valid in the case of 'smooth-impulsive' behaviour. The first system has one input-decoupling zero at infinity, while the second system has none. Note that the smooth-impulsive solution sets of the above systems are given by

$$\mathscr{B}_1 = \left\{ \xi^1 = c_1 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \delta + c_2 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \delta + \begin{pmatrix} u+pu \\ u \\ u \\ pu \end{pmatrix} c_i \in \mathbb{R}, \quad i = 1,2, u \in \ell_{\mathrm{f}} \right\}$$

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$$\mathscr{B}_2 = \left\{ \xi^2 = a \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \delta + \begin{pmatrix} u + pu \\ u \\ u \\ pu \end{pmatrix}, a \in \mathbb{R},\ u \in \ell_{\mathrm{f}} \right\}$$

where $p = \delta^{(1)}$ and ℓ_{f} is defined in Section 2. Obviously, therefore, unimodular equivalence does not preserve the 'impulsive' behaviour of the systems and so a restriction of it is required.

Our approach to the problem of equivalence of non-regular AR-representations has, as a starting point, the relation between the smooth-impulsive behaviours of such systems. We should note that the notion of *fundamental equivalence* proposed here ensures the smooth-impulsive behaviours of the two systems are identical. We establish that the matrix conditions guaranteeing fundamental equivalence coincide with those of *full unimodular equivalence* (presented in Section 3), which is a special case of *full matrix equivalence* presented in [19]. This provides a natural connection between the behavioural setting and the theory of polynomial matrix transformations.

2. PRELIMINARY RESULTS

In what follows $\mathbb{R}, \mathbb{C}$ denote the fields of real and complex numbers, respectively, $\mathbb{R}[s]$ the ring of polynomials with real coefficients and $\mathbb{R}(s)$ the field of real rational functions. Following [14–18] we adopt the class of *impulsive-smooth* distributions (ℓ_{imp}), as the 'function space' for our purposes. We review some of the key points of this theory.

The class ℓ_{imp} consists of distributions that are linear combinations of a smooth function and a purely impulsive distribution. The purely impulsive part is essentially any finite linear combination of the Dirac delta distribution δ, and its (distributional) derivatives $\delta^{(i)}$, $i \geqslant 1$. A smooth distribution corresponds to a function that is smooth on $\mathbb{R}^+$ and 0 elsewhere. A smooth function $f(t)$ on $\mathbb{R}^+$ is arbitrarily often differentiable on $(0, +\infty)$ and the limit $\lim_{t \downarrow 0} f^{(j)}(t)$ exists and it is finite for every $j \geqslant 0$. The class ℓ_{imp}, equipped with distributional convolution $(*)$ as multiplication, is a commutative algebra over $\mathbb{R}$ with δ as the unit element and thus closed under differentiation (convolution with $\delta^{(1)}$) and integration (convolution with $h = \delta^{(-1)}$, the Heaviside unit step distribution). Furthermore, ℓ_{imp} can be decomposed to a direct sum of two subalgebras $\ell_{p-\mathrm{imp}}$ the purely impulsive distributions (linear combinations of δ and its distributional derivatives) and ℓ_{sm} the smooth distributions. $f \in \ell_{\mathrm{imp}}$ can be written uniquely as $f = f_1 + f_2$ where its impulsive part is f_1 and its smooth part is f_2. Then $f(0+) := \lim_{t \downarrow 0} f_2(t) = f_2(0+)$. If $f \in \ell_{\mathrm{sm}}$, then the distributional derivative of f, $f * \delta^{(1)}$, equals $f^{(1)} + f(0+)$ (with $f(0+) = f(0+)\delta$), where $f^{(i)}$ denotes the distribution that corresponds to the ordinary ith derivative of f on $\mathbb{R}^+$. For more on the properties of ℓ_{imp} see [20, 21].

Consider a *non-regular* linear time-invariant system described by the AR-representation

$$\Sigma : A(\partial)\xi(t) = 0, \quad t \in [0, +\infty) \tag{1}$$

where $\partial = \mathrm{d}/\mathrm{d}t$ is the differential operator (interpreted as right-hand differentiation at the origin), $\xi(t) \in \mathbb{R}^m$ and $A(\partial) = A_q\partial^q + \cdots + A_1\partial + A_0 \in \mathbb{R}^{k \times m}[\partial]$ has $\mathrm{rank}_{\mathbb{R}(s)} A(s) = r$ and $A_q \neq 0$. *Non-regular* is used here either for non-square, or square but not invertible, polynomial matrices (and AR-representations accordingly).

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The aim is to obtain a fully algebraic treatment of the above systems so we denote $1=\delta$, $p=\delta^{(1)}$ and in general, $p^i=\delta^{(i)}, i\in\mathbb{Z}$ while convolution will be implied by juxtaposition of distributions and the distributional derivative of $f\in\ell_{\rm imp}, f*\delta^{(1)}$, is thus written as pf. Hence $pf=f^{(1)}+f(0+)$ if $f\in\ell_{\rm sm}$ and $p^if=f^{(i)}+\sum_{j=0}^{i-1}p^jf^{(i-1-j)}(0+)$, $i\geqslant 1$. It should be mentioned that the operator p is not equal to the Heaviside operator $\partial=\mathrm{d}/\mathrm{d}t$. In order to conform with the above distributional framework we introduce as in [18, 14] the distributional version of (1)

$$A(p)\xi=S_{q-1}(p)X_A\bar{\xi}_0 \tag{2}$$

where $S_{q-1}(p)=[Ip^{q-1}\ \cdots\ Ip\ I]$

$$\bar{\xi}_0=\begin{bmatrix}\xi_{0,0}\\ \vdots\\ \xi_{0,q-1}\end{bmatrix}\quad\text{and}\quad X_A=\begin{bmatrix}A_q & \cdots & 0\\ \vdots & \ddots & \vdots\\ A_1 & \cdots & A_q\end{bmatrix} \tag{3}$$

while $\xi\in\ell_{\rm imp}^m$ are vector distributions in $\ell_{\rm imp}$ and $\xi_{0,j}\in\mathbb{R}^m$ are arbitrary real vectors which have to be interpreted as the initial values of Equation (2), i.e. the values of the jth derivative of $\xi(t)$ 'at $t=0^-$', immediately before starting the dynamic process given by (1). The vector $\bar{\xi}_0$ will be termed the *initial value* of ξ, while $X_A\bar{\xi}_0$ will be termed the *initial condition* for ξ, for reasons which will become apparent subsequently. Accordingly the real vector space $\mathcal{X}=\mathcal{R}(X_A)$ will be termed the *initial condition space* of (2). For each $\bar{\xi}_0\in\mathbb{R}^{\rm qm}$ we define the solution set

$$B(\bar{\xi}_0):=\{\xi\in\ell_{\rm imp}^m : A(p)\xi=S_{q-1}(p)X_A\bar{\xi}_0\}$$

and every $\xi\in B(\bar{\xi}_0)$ is called a solution of (2) for $\bar{\xi}_0$. An important feature of non-regular systems of the form (2) is that they are not in general *solvable* for every initial value (and thus initial condition) of ξ. This can be seen in the following example.

Example 2
Let

$$\begin{bmatrix}\partial\\ 1\end{bmatrix}\xi(t)=0,\quad t\in[0,+\infty) \tag{4}$$

The distributional version of (4) is then

$$\begin{bmatrix}p\\ 1\end{bmatrix}\xi=\begin{bmatrix}1\\ 0\end{bmatrix}\xi_{00} \tag{5}$$

The first equation gives $\xi=h\xi_{00}$ where h is the Heaviside unit step, while the second one gives $\xi=0$. Obviously, the two equations are incompatible for $\xi_{00}\neq 0$. Thus, system (5) does not always possess a solution for every initial value ξ_{00}.

Definition 3 (Geerts [17, 18])
(1) is C-solvable (control-solvable) for $\xi\in\ell_{\rm imp}^m$ if

$$\forall\bar{\xi}_0\in\mathbb{R}^{\rm qm}\ (\text{and thus } X_A\bar{\xi}_0): B(\bar{\xi}_0)\neq\emptyset$$

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The solvability requirement is reasonable in our case where we study equivalence of systems through their solution spaces (behaviours).

Let

$$\ell_{\mathrm{f}} := \{f \in \ell_{\mathrm{imp}} : f = f_1 f_2^{-1}, f_1, f_2 \in \ell_{p-\mathrm{imp}}, f_2 \neq 0\}$$

be the subalgebra of *fractional impulses*. Then we have the following basic result [22]:

Lemma 4

Let $T(s) \in \mathbb{R}^{k_1 \times k_2}(s), \eta(s) \in \mathbb{R}^{1 \times k_1}(s), w(s) \in \mathbb{R}^{k_2 \times 1}(s)$ and let $T(p), \eta(p), w(p)$ be the corresponding distributional matrices, then

$$\eta(s)T(s) = 0 \quad \Leftrightarrow \quad \eta(p)T(p) = 0$$

$$T(s)w(s) = 0 \quad \Leftrightarrow \quad T(p)w(p) = 0$$

The characterization of C-solvable systems is,

Theorem 5

The non-regular AR-representation (2) is C-solvable iff all the left minimal indices of $A(s)$ are zero.

Proof

(if) Assume that (2) is *C-solvable* and there exists a left minimal polynomial basis of $A(s)$, $\{v_1(s), v_2(s), \ldots, v_{k-r}(s)\}$, where the row vectors $v_j(s) = v_{j\eta_j} s^{\eta_j} + \cdots + v_{j0} \in \mathbb{R}^{1 \times k}[s]$ and η_j are the left minimal indices of $A(s)$. Furthermore, we assume that the vectors $v_j(s)$ are ordered in descending order, i.e. $\eta_1 \geqslant \eta_2 \geqslant \cdots \geqslant \eta_{k-r}$. Then the polynomial matrix

$$V(s) = \begin{bmatrix} v_1(s) \\ \vdots \\ v_{k-r}(s) \end{bmatrix}$$

has full normal row rank, no finite zeros, is row proper and

$$V(s)A(s) = 0 \tag{6}$$

Assume now that $\eta_1 > 0$, i.e. that there exists at least one left minimal index of $A(s)$ of order greater than zero. We look for a contradiction. Since (2) is C-solvable for every initial condition it holds for every $X_A \bar{\xi}_0$. From Lemma 4, we have that $V(s)A(s) = 0$ is equivalent to $V(p)A(p) = 0$. Premultiplying (2) by $V(p)$ we have

$$V(p)S_{q-1}(p)X_{A_i}\bar{\xi}_0 = 0 \quad \forall \bar{\xi}_0$$

or equivalently by Lemma 4,

$$V(s)S_{q-1}(s)X_{A_i} = 0 \tag{7}$$

Equating the coefficients of s^0 in (6) and (7) implies $V_0 A_0 = 0$ and $V_0 A_j = 0$, $j = 1, 2, \ldots, q$. Thus $V_0 A_j = 0$, for all $j = 0, 1, 2, \ldots, q$. Equating successively the coefficients of $s^1, s^2, \ldots, s^{\eta_1}$

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in (6), (7) and making use of the corresponding relations for coefficients of $V(s)$ of lower order, we get

$$V_i A_j = 0 \quad \forall i = 0, 1, \ldots, \eta_1 \quad \forall j = 0, 1, \ldots, q \tag{8}$$

Now since $V(s)$ is row proper we can write

$$V(s) = \operatorname{diag}\{s^{\eta_1}, s^{\eta_2}, \ldots, s^{\eta_{k-r}}\}[V]_r^h + \{\text{lower order terms}\}$$

where $[V]_r^h$ is a constant full row rank matrix, with its ith row being the ith row of V_{η_i}. In view of (8) it is obvious that

$$[V]_r^h A(s) = 0$$

The matrix $[V]_r^h$ thus satisfies all the properties of a left minimal polynomial basis of $A(s)$ and its row orders are obviously less than the corresponding ones of $V(s)$. This is a contradiction since $V(s)$ is assumed to be minimal. Thus $\eta_1 = 0$ and $\eta_j = 0$ for every $j = 0, 1, \ldots, k-r$, which completes the proof of the (if) part.

(only if) Assume that $\eta_j = 0$ for every $j = 0, 1, \ldots, k-r$. Then $\exists$ a constant left minimal polynomial basis of $A(s)$, $V \in \mathbb{R}^{(k-r)\times k}$. Thus, $\exists$ a constant square, invertible matrix W having as its last $k-r$ rows the rows of V such that

$$WA(s) = \begin{bmatrix} \bar{A}(s) \\ 0 \end{bmatrix} \tag{9}$$

where $\bar{A}(s) \in \mathbb{R}^{r\times m}[s]$ is a full row rank rational matrix. We can also write

$$A(s) = \bar{W}\bar{A}(s) \iff A_j = \bar{W}\bar{A}_j, \; j = 0, 1, \ldots, q \tag{10}$$

where $\bar{W} \in \mathbb{R}^{p\times r}$ is the matrix consisting of the first r columns of W^{-1}. Consider now the equation

$$\bar{A}(p)\xi = \bar{S}_{q-1}(p)X_{\bar{A}}\bar{\bar{\xi}}_0 \tag{11}$$

where $\bar{S}_{q-1}(p) = [p^{q-1}I_r, \ldots, I_r]$ and $X_{\bar{A}}, \bar{\bar{\xi}}_0$ as in (3). The above equation is C-solvable simply because $\bar{A}(p)$ has full row rank and thus the fractional impulse space (rational vector space) spanned by $\bar{S}_{q-1}X_{\bar{A}}$ is always contained in the corresponding space spanned by $\bar{A}(p)$ (see Proposition 2.7, [18]). Premultiplying both sides of (11) by $\bar{W}$ and using (10), we get

$$A(p)\xi = \bar{W}\bar{S}_{q-1}(p)X_{\bar{A}}\bar{\bar{\xi}}_0$$

Now $\bar{W}\bar{S}_{q-1}(p) = S_{q-1}(p)\operatorname{diag}\{\bar{W}, \bar{W}, \ldots, \bar{W}\}$, where $S_{q-1}(p) = [p^{q-1}I_p, \ldots, I_p]$. Using again (10) we have

$$A(p)\xi = S_{q-1}(p)X_A\bar{\bar{\xi}}_0 \tag{12}$$

Thus, every solution of (11) is also a solution of (12). Since (11) is solvable for every initial condition so is (12). $\square$

The above result gives a characterization of solvability of non-regular AR-representations in terms of the structural invariants of the polynomial matrix $A(s)$, and is a generalization of the corresponding conditions appearing in [17, 18].

As it is clear from Theorem 5, a special case of solvable systems is described by (2), with rank $A(s) = k = r$, i.e. full row rank. Actually, as the following corollary suggests, the class of full row rank systems is wide enough for our purposes.

Corollary 6
Every C-solvable system of the form (2) can be replaced to an equivalent full row rank system, which has the same solution space as (2).

Proof
The proof is straightforward in view of the proof of the 'only if' part of Theorem 5. □

It follows that the full row rank assumption can be made without loss of generality for solvable systems. Thus, we restrict ourselves in the sequel to the case, rank $A(s) = k = r$.

Following the terminology of [18] we denote by $\mathscr{B}$ the *solution space* or *behaviour* of Σ, i.e.

$$\mathscr{B} = \{\xi \in \ell^m_{\text{imp}} : A(p)\xi = S_{q-1}(p)X_A\bar{\bar{\xi}}_0 \tag{13}$$

$$\forall \bar{\bar{\xi}}_0 = (\xi^{\mathrm{T}}_{00} \quad \xi^{\mathrm{T}}_{01} \quad \ldots \quad \xi^{\mathrm{T}}_{0(q-1)})^{\mathrm{T}} \in \mathbb{R}^{\mathrm{qm}}\} \tag{14}$$

In the case where $A(p)$ is square and non-singular, $\mathscr{B}$ is finite dimensional and its dimension is equal to the total number of finite (n) and infinite zeros ($\hat{q}$) of $A(p)$ (multiplicities accounted for) [23] i.e.

$$\dim \mathscr{B} = n + \hat{q}$$

In the more general case, where $A(p)$ contains a right null space structure, $\mathscr{B}$ is infinite dimensional. This is easy to see if we consider any *fractional impulse* lying in $\ker A(p)$. Obviously, the *fractional impulse* vector distribution satisfies (2) and $\ker A(p)$ is an infinite dimensional vector space over ℓ_{f}. Let

$$\mathscr{L} = \{\xi \in \mathscr{B} : X_A\bar{\bar{\xi}}_0 = 0\} \tag{15}$$

i.e. the subspace of $\mathscr{B}$ which contains the solutions having zero *initial conditions*. The fact that there are solutions corresponding to zero initial conditions is somehow unnatural, since what is usually expected from a system of homogeneous differential equations is its non-trivial solutions to be triggered by non-zero initial conditions. An alternative interpretation to the question of what constitutes the solution space of non-regular systems, which overcomes this problem, has been proposed in [24]. According to this approach, the trajectory space $\mathscr{B}$ can be partitioned according to the relation

$$\xi \sim \xi' \quad \Leftrightarrow \quad X_A\bar{\bar{\xi}}_0 = X_A\bar{\bar{\xi}}'_0 \tag{16}$$

It is easy to see that '$\sim$' is an equivalence relation and the resulting equivalence classes consist of distributional solutions of (2) that correspond to the same initial condition vector $X_A\bar{\bar{\xi}}_0$. If $\xi \in \mathscr{B}$, write

$$[\xi] = \xi + \mathscr{L} \tag{17}$$

where $\mathscr{L} = [0_{\mathscr{B}}]$ and $[\xi]$ is the equivalence class of ξ. By $\mathscr{B}/\mathscr{L}$ we denote the quotient space of $\mathscr{B}$ over $\mathscr{L}$, i.e. the set of all equivalence classes of $\mathscr{B}$. It can be proved [24], that $\mathscr{B}/\mathscr{L}$ is a finite dimensional vector space over $\mathbb{R}$ which can be decomposed as follows:

$$\mathscr{B}/\mathscr{L} = (\mathscr{B}^{\mathbb{C}} \oplus \mathscr{B}^{\infty} \oplus \mathscr{B}^{\varepsilon})/\mathscr{L} = \mathscr{B}^{\mathbb{C}}/\mathscr{L} \oplus \mathscr{B}^{\infty}/\mathscr{L} \oplus \mathscr{B}^{\varepsilon}/\mathscr{L} \tag{18}$$

where $\mathscr{B}^{\mathbb{C}}, \mathscr{B}^{\infty}, \mathscr{B}^{\varepsilon}$ are finite dimensional distributional spaces corresponding to the finite zero structure, the infinite zero structure and the right minimal indices of $A(s)$. Actually $\mathscr{B}/\mathscr{L}$, is the finite dimensional sectional of the infinite dimensional vector space $\mathscr{B}$. The dimensions of the

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$\mathscr{B}^{\mathbb{C}}/\mathscr{Z}, \mathscr{B}^{\infty}/\mathscr{Z}, \mathscr{B}^{\varepsilon}/\mathscr{Z}$ spaces are $n, \hat{q}$ and ε, respectively, where $n, \hat{q}$ and ε are the total number of finite zeros, infinite zeros and right minimal indices of $A(s)$ (multiplicities accounted for). Furthermore, it can be shown that the dimension of $\mathscr{B}/\mathscr{Z}$ is

$$\dim \mathscr{B}/\mathscr{Z} \overset{[24]}{=} n + \hat{q} + \varepsilon \overset{[4]}{=} \delta_M(A(s)) \tag{19}$$

where $\delta_M(A(s)) = \mathrm{rank}_{\mathbb{R}} X_A$ denotes the McMillan degree of $A(s)$. In the general case, where $A(s)$ has not necessarily full row rank then it is known [4] that $\delta_M(A(s)) = n + \hat{q} + \varepsilon + \eta$ where η denotes the total number of left minimal indices (multiplicities accounted for). As a result of this discussion, we call $\mathscr{B}/\mathscr{Z} \triangleq \bar{\mathscr{B}}$ the *quotient solution space* of (2).

Note that according to the above definitions, there is a one-to-one correspondence between the initial condition vectors $X_A \bar{\xi}_0$ and the elements of $\bar{\mathscr{B}}$, and thus an isomorphism between the initial conditions space $\mathscr{X}$ and the quotient solution space. On this basis the quotient solution space is a finite dimensional view of the actual solution space (behaviour) of the AR-representation (2). The behaviour itself is of course, an infinite dimensional and complete view of the solution space.

3. FUNDAMENTAL EQUIVALENCE

The aim of this section is to establish a connection between the existing theory of matrix equivalence and the behavioural framework. In the case where both the *smooth-impulsive* solution set is of interest, we need a restriction of the unimodular equivalence transformation.

Definition 7
$P_1(s), P_2(s) \in \mathbb{R}[s]^{r\times m}$ are said to be *fully unimodular equivalent* (FE) if $\exists$ a unimodular matrix $U(s) \in \mathbb{R}[s]^{r\times r}$ such that

$$U(s)P_1(s) = P_2(s)$$

where the compound matrix $[U(s)\;\; P_2(s)]$

(i) has no infinite zeros
(ii) $\delta_M[U(s)\;\; P_2(s)] = \delta_M[P_2(s)]$ where $\delta_M(\cdot)$ indicates the McMillan degree of the indicated matrix.

Full unimodular equivalence is a special case of *full system equivalence* [25] and thus has the nice property of preserving the finite and infinite zero structure of polynomial matrices (see [19]) in contrast to unimodular equivalence which preserves only the finite aspects.

We now introduce a notion of equivalence using a solution space approach, and in this way we provide a direct dynamical interpretation of the conditions appearing in Definition 7. We give the following definition.

Definition 8
Let the systems be described by

$$\Sigma_i : A_i(\partial)\xi_i(t) = 0, \quad t \in [0, +\infty), \;\; i = 1, 2$$

where $\partial = \mathrm{d}/\mathrm{d}t$ is the differential operator, $\xi_i(t) \in \mathbb{R}^m$ and $A_i(\partial) = A_{iq}\partial^q + \cdots + A_{i1}\partial + A_{i0} \in \mathbb{R}^{r\times m}[\partial], i = 1, 2$ is a polynomial matrix with $\mathrm{rank}_{\mathbb{R}(s)} A_i(s) = r$ and A_{iq} not both identically zero,

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and let the distributional version of the above systems be the following [18]:

$$\Sigma_i : A_i(p)\xi_i = S_{q-1}(p)X_{A_i}\bar{\xi}_0^i$$

where $\xi_i \in \ell_{\text{imp}}^{m_i}$ are vector distributions in ℓ_{imp}. Define also as

$$\mathscr{B}_i := \{\xi_i \in \ell_{\text{imp}}^{m_i} : A_i(p)\xi_i = S_{q-1}(p)X_{A_i}\bar{\xi}_0^i\}, \quad i = 1, 2$$

The systems Σ_i are *fundamentally equivalent* iff $\mathscr{B}_1 = \mathscr{B}_2$.

Fundamental equivalence extends the notion of [10] to the 'smooth-impulsive' solution set. We are interested in the conditions under which $\mathscr{B}_1 \subseteq \mathscr{B}_2$. We need the following

Lemma 9
If $U(s), A_1(s), A_2(s)$ are polynomial matrices of appropriate dimensions such that

$$U(s)A_1(s) = A_2(s) \tag{20}$$

then

$$\begin{aligned} X_U X_{A_1} &= 0 \\ \bar{U} X_{A_1} &= X_{A_2} - X_U \bar{A}_1 \end{aligned} \tag{21}$$

where

$$X_P = \begin{bmatrix} P_q & \cdots & 0 \\ \vdots & \ddots & \vdots \\ P_1 & \cdots & P_q \end{bmatrix}, \quad \bar{P} = \begin{bmatrix} P_0 & \cdots & P_{q-1} \\ \vdots & \ddots & \vdots \\ 0 & \cdots & P_0 \end{bmatrix}$$

and $P(s) = P_q s^q + \cdots + P_0$ is any of the matrices $A_1(s), A_2(s), U(s)$ (assuming without loss of generality that all the above matrices have the same degree q).

Proof
The proof is straightforward by equating like powers of s in (20). □

Theorem 10
The following statements are equivalent:

(i) $\mathscr{B}_1 \subseteq \mathscr{B}_2$
(ii) $\exists U(s) \in \mathbb{R}^{r\times r}[s] : U(s)A_1(s) = A_2(s)$ and $\delta_M[U(s), A_2(s)] = \delta_M(A_2(s))$.

Proof
Assume $\mathscr{B}_1 \subseteq \mathscr{B}_2$. Then the same inclusion property will hold for $\hat{\mathscr{B}}_1 \subseteq \hat{\mathscr{B}}_2$ where

$$\hat{\mathscr{B}}_i := \{\xi_i \in \ell_{\text{sm}}^{m_i} : A_i(p)\xi_i = S_{q-1}(p)X_{A_i}\bar{\xi}_0^i\}, \quad i = 1, 2$$

Thus, from [26, p. 36, Lemma 3.7] we have

$$\exists U(s) \in \mathbb{R}^{r\times r}[s] : U(s)A_1(s) = A_2(s) \tag{22}$$

Furthermore, let $\xi_1 \in \mathscr{B}_1$. Then ξ_1 satisfies

$$A_1(p)\xi_1 = S_{q-1}(p)X_{A_1}\bar{\xi}_0^1 \tag{23}$$

Since $\mathscr{B}_1 \subseteq \mathscr{B}_2$ holds, ξ_1 must satisfy

$$A_2(p)\xi_2 = S_{q-1}(p)X_{A_2}\xi_0^2 \tag{24}$$

which from (23) gives

$$A_2(p)\xi_1 = S_{q-1}(p)X_{A_2}\xi_0^2 \tag{25}$$

or equivalently using (22)

$$U(p)A_1(p)\xi_1 = S_{q-1}(p)X_{A_2}\xi_p^2 \tag{26}$$

and finally using (23)

$$U(p)S_{q-1}(p)X_{A_1}\xi_0^1 = S_{q-1}(p)X_{A_2}\xi_0^2 \tag{27}$$

Now equating powers of p in the above equation we obtain

$$\begin{aligned}(X_U X_{A_1})\xi_0^1 &= 0\\ X_{A_2}\xi_0^2 &= (\bar{U}X_{A_1})\xi_0^1\end{aligned} \tag{28}$$

which in view of (21), reduce to

$$X_{A_2}\xi_0^2 = (X_{A_2} - X_U\bar{A}_1)\xi_0^1 \tag{29}$$

Since the above equation is solvable for ξ_0^2, we conclude that

$$\mathscr{R}(X_U\bar{A}_1) \subseteq \mathscr{R}(X_{A_2}) \tag{30}$$

Notice that since $A_1(s)$ has full row rank the same will hold for $[\bar{A}_1, X_{A_1}]$ (see [23]). Thus, $\mathscr{R}(X_U) = \mathscr{R}(X_U[\bar{A}_1, X_{A_1}]) = \mathscr{R}(X_U\bar{A}_1) + \mathscr{R}(X_U X_{A_1}) = \mathscr{R}(X_U\bar{A}_1)$. Thus, we obtain

$$\mathscr{R}(X_U) \subseteq \mathscr{R}(X_{A_2})$$

which implies that

$$\operatorname{rank}[X_U, X_{A_2}] = \operatorname{rank} X_{A_2}$$

i.e.

$$\delta_M[U(s), A_2(s)] = \delta_M(A_2(s)) \tag{31}$$

which proves that (i) implies (ii).

Conversely, if (ii) holds it is easy to check that Equations (31)–(27) are equivalent. Then if $\xi_1 \in \mathscr{B}_1$, Equation (23) is satisfied and thus (26) holds, which in view of (22) implies successively (25) and (24) which verifies that $\xi_2 \in \mathscr{B}_2$. □

An interesting map connects the initial condition spaces of the two systems as described by the following.

Corollary 11

If $\mathscr{B}_1 \subseteq \mathscr{B}_2$ then an induced injective mapping between the initial conditions of the two systems is given by

$$X_{A_2}\xi_0^2 = \bar{U}(X_{A_1}\xi_0^1) \tag{32}$$

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and thus

$$\bar{U}\mathscr{R}(X_{A_1}) \subseteq \mathscr{R}(X_{A_2}) \tag{33}$$

Proof
Equation (32) is simply the second equation of (28). Obviously (33) must hold since (32) is solvable with respect to $\bar{\xi}_0^2$. □

In case $\mathscr{B}_1 = \mathscr{B}_2$ then $U(s)$ is unimodular and thus $\det(U_0) = \det(U(0)) \neq 0$ or equivalently the matrix $\bar{U}$ is invertible and thus (32) is an injective map (not necessarily surjective) between the initial condition spaces and (33) reduces to $\mathscr{R}(X_{A_1}) \subseteq \mathscr{R}(X_{A_2})$.

Example 12
Consider the Example 1. We have the following injective map between the initial condition spaces:

$$\underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{X_{A_1}} \underbrace{\begin{bmatrix} \xi_{11} \\ \xi_{21} \\ \xi_{31} \\ \xi_{41} \end{bmatrix}}_{\xi_0^1} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{U_0} \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{X_{A_2}} \underbrace{\begin{bmatrix} \xi_{12} \\ \xi_{22} \\ \xi_{32} \\ \xi_{42} \end{bmatrix}}_{\xi_0^2}$$

since the matrix $U(s)$ is unimodular. However, note that the dimension of the initial condition space $X_{A_1}\xi_1$ is 2, which is different from the dimension of the initial condition space $X_{A_2}\xi_2$ which is 1, and thus the map U_0 although is injective it is not surjective.

According to Theorem 10 if additionally $\mathscr{B}_2 \subseteq \mathscr{B}_1$ then $\exists V(s) \in \mathbb{R}^{r\times r}[s] : V(s)A_2(s) = A_1(s)$ and $\delta_M[V(s), A_1(s)] = \delta_M(A_1(s))$ and thus, $V(s)A_2(s) = A_1(s) \Leftrightarrow V(s)U(s)A_1(s) = A_1(s) \Leftrightarrow (V(s)U(s) - I)A_1(s) = 0$. Since $A_1(s)$ has full row rank then $U(s) \in \mathbb{R}^{r\times r}[s]$ defined in the above Theorem is unimodular (see also [26]). According to Corollary 11 we have from the above relation that $\mathscr{R}(X_{A_2}) \subseteq \mathscr{R}(X_{A_1})$ and thus $\mathscr{R}(X_{A_1}) = \mathscr{R}(X_{A_2})$ or otherwise $\delta_M(A_1(s)) = \delta_M(A_2(s))$. Therefore, an isomorphism exists between the initial condition spaces of the two systems. The McMillan degree condition of full unimodular equivalence thus ensures that the dimension of the initial condition space remains invariant under the particular unimodular equivalence transformation, $U(s)$, which acts between the matrices describing the behaviour.

The inclusion (33) ensures that the image of every initial condition of Σ_1 is mapped through $\bar{U}$, to an initial condition of Σ_2. The following corollary guarantees that every solution of Σ_1 starting from the zero initial condition will be mapped to a zero initial condition solution of Σ_2.

Corollary 13
If $\mathscr{B}_1 \subseteq \mathscr{B}_2$ then

$$\mathscr{Z}_1 \subseteq \mathscr{Z}_2 \tag{34}$$

where

$$\mathscr{Z}_i = \{\xi_i \in \mathscr{B}_i : X_{A_i}\bar{\xi}_0^i = 0\}, \quad i = 1, 2$$

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4. INDUCED MAPS OF FUNDAMENTAL EQUIVALENCE

Suppose that $\mathscr{B}_1 \subseteq \mathscr{B}_2$ and thus from Corollary 13 $\mathscr{Z}_1 \subseteq \mathscr{Z}_2$. Then there exists a unique mapping between the quotient solution spaces

$$I^* : [\xi_1] \in \mathscr{B}_1/\mathscr{Z}_1 \mapsto [\xi_2] \in \mathscr{B}_2/\mathscr{Z}_2 \tag{35}$$

such that the diagram

$$\begin{array}{ccc} \mathscr{B}_1 & \xrightarrow{I} & \mathscr{B}_2 \\ I_{\mathscr{B}_1} \downarrow & & \downarrow I_{\mathscr{B}_2} \\ \mathscr{B}_1/\mathscr{Z}_1 & \xrightarrow[I^*]{} & \mathscr{B}_2/\mathscr{Z}_2 \end{array}$$

is commutative, where I is the unit map and I^* is its restriction to $\mathscr{B}_i/\mathscr{Z}_i$. $I_{\mathscr{B}_1}, I_{\mathscr{B}_2}$ are the natural projections. Note that I^* is injective iff $\mathscr{Z}_1 = I^{-1}(\mathscr{Z}_2)$ and surjective iff $\mathscr{Z}_2 + \operatorname{im} I = \mathscr{B}_2$. Furthermore $\mathscr{B}_1 \subseteq \mathscr{B}_2$ implies that restriction of the I to $\mathscr{Z}_1$, denoted I_*, maps $\mathscr{Z}_1$ into $\mathscr{Z}_2$. We need the conditions for I_* to be a bijection. Note that $\mathscr{B}_1, \mathscr{B}_2$ are infinite dimensional and it turns out to be easier to study the properties of I through the structures of I_* and I^*. The map I is injective by definition. We note the following.

Theorem 14

Let I^*, I_* be the maps defined above, then I^*, I_* are surjective $\Rightarrow$ I is bijective

Furthermore we have

Theorem 15

Let I^*, I_* be the maps defined above, then

(i) If I is surjective then I^* is surjective.
(ii) If I_* is surjective then I^* is injective.
(iii) The map I_* is injective.
(iv) If I is surjective and I^* is injective then I_* is surjective.

The above theorems give a complete picture of the conditions for I^*, I_* to be bijections. It is clear that the injectiveness of I^* (surjectiveness of I_*) is not a direct consequence of the injectiveness (surjectiveness) of the unit map I. Nevertheless the following is true.

Corollary 16

Let $I : \mathscr{B}_1 \to \mathscr{B}_2$ and I^*, I_* be the maps defined above, then

$$I \text{ is bijection } \Leftrightarrow I_* \text{ and } I^* \text{ are bijective}$$

We establish the properties of I, I^* and I_*, in terms of the matrices of the Σ_1 and Σ_2.

Theorem 17

For $\bar{U}$ of (32) the following hold

(i) I^* is injective iff $\ker \bar{U} \cap \mathscr{R}(X_{A_1}) = \{0\}$.
(ii) I^* is surjective iff $\mathscr{R}(X_{A_2}) = \bar{U}\mathscr{R}(X_{A_1})$.

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Proof
(i) Injectiveness of I^* means that $I^*[\xi_1] = \mathscr{X}_2 \Rightarrow [\xi_1] = \mathscr{X}_1$ or equivalently $\xi_1 \in \mathscr{X}_2 \Rightarrow \xi_1 \in \mathscr{X}_1$. If $X_{A_2}\bar{\xi}_0^2$ are the initial conditions corresponding to $\xi_2 = \xi_1$ then the fact that $\xi_2 \in \mathscr{X}_2$ means that $X_{A_2}\bar{\xi}_0^2 = 0$, which in view of (32) implies

$$\bar{U}X_{A_1}\bar{\xi}_0^1 = 0 \tag{36}$$

Now obviously the requirement $\xi_1 \in \mathscr{X}_1$ gives $X_{A_1}\bar{\xi}_0^1 = 0$. Thus, I^* is injective iff Equation (36) implies $X_{A_1}\bar{\xi}_0^1 = 0$ for every ξ_1, which establishes (i).

(ii) We first note that $\bar{U}\mathscr{R}(X_{A_1}) \subseteq \mathscr{R}(X_{A_2})$ holds from (33). We thus need the reverse inclusion. Now I^* is surjective iff for every $\xi_2 \in \mathscr{B}_2$ there exists $\xi_1 \in \mathscr{B}_1$ such that $[\xi_2] = I^*[\xi_1]$ or equivalently $\xi_2 - \xi_1 \in \mathscr{X}_2$. Following the proof of Theorem 10 we conclude that (32) should be solvable for every initial condition $X_{A_2}\bar{\xi}_0^2$, hence the condition in (ii). □

If $U(s)$ is unimodular then $\ker \bar{U} = \{0\}$, since $\det U(0) = \det(U_0) \neq 0$. Thus, (i) in Theorem 17 holds and so I^* is injective. For the surjectiveness of I_* (and thus bijectiveness, since it is an injective map), note that $\mathscr{X}_1, \mathscr{X}_2$ are isomorphic to $\ker A_1(p)$, $\ker A_2(p)$, respectively, which are finite dimensional vector spaces over ℓ_f having the same dimension i.e. $m - r$. Thus, I_* is always surjective. Now I is always injective according to its definition. We can now establish conditions under which two AR-representations are fundamentally equivalent. The following is required.

Lemma 18
Let $T(s) \in \mathbb{R}^{k\times(k+r)}[s]$, $V(s) \in \mathbb{R}^{(k+r)\times r}$ be polynomial matrices with $\operatorname{rank}_{\mathbb{R}(s)} T(s) = k$ and $\operatorname{rank}_{\mathbb{R}(s)} V(s) = r$, such that $T(s)V(s) = 0$. Furthermore, let $V(s)$ have no zeros in $\mathbb{C} \cup \{\infty\}$ and denote by $\varepsilon = \sum_{i=1}^{r} \varepsilon_i$ the sum of right minimal indices of $T(s)$. Then

$$\varepsilon = \delta_M(V(s)) \tag{37}$$

Proof
If $V(s)$ is column proper then it forms a minimal basis of $\ker T(s)$ and its column degrees are the orders of the poles at $s = \infty$ of $V(s)$ (see [23]). The lemma then holds since the only poles of $V(s)$ are those at $s = \infty$.

Assume that $V(s)$ is not column proper and let $S_{V(s)}^{\infty}(s) = \operatorname{diag}\{s^{q_1}, s^{q_2}, \ldots, s^{q_r}\}$ be its Smith–McMillan form at $s = \infty$. It is known [23] that the orders of the poles at $s = \infty$ can be obtained from the formula $q_i = m_i - m_{i-1} \geqslant 0$, $i = 1, 2, \ldots, r$ where $m_0 = 0$ and $m_i = \max\deg$ {minors of order i of $V(s)$}, $i = 1, 2, \ldots, r$. It is easy to see that $q = \sum_{i=1}^{r} q_i = m_r$ and since $V(s)$ is polynomial

$$\delta_M(V(s)) = q = m_r \tag{38}$$

Consider now the unimodular matrix $W(s)$ which reduces $V(s)$ to $\bar{V}(s)$, which is column proper, i.e. $\bar{V}(s) = V(s)W(s)$. Then $\bar{V}(s)$ is a minimal basis of $\ker T(s)$, with column degrees equal to the right minimal indices of $T(s)$. Moreover, the column degrees of $\bar{V}(s)$ will be the orders of its poles at $s = \infty$. Thus, if $\bar{q}$ is the total number of poles at $s = \infty$ then

$$\bar{q} = \varepsilon = \bar{m}_r \tag{39}$$

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where $\bar{m}_r = \max\deg\{\text{minors of order } r \text{ of } \bar{V}(s)\}$. Now the minors of order r of $V(s)$ remain invariant (up to multiplication by a non-zero constant) in $\bar{V}(s)$. Thus, $\bar{m}_r = m_r$ which in view of (38), (39) proves the lemma. □

The main result is thus

Theorem 19
The systems in (2) are fundamentally equivalent iff there exists a unimodular matrix $U(s)$ satisfying

(i) $U(s)A_1(s) = A_2(s)$.
(ii) $\delta_M[U(s)\ \ A_2(s)] = \delta_M(A_2(s))$.
(iii) $[U(s)\ \ A_2(s)]$ have no zeros at $s = \{\infty\}$.

Proof
(if) Assume that there exists the unit map $I : \mathscr{B}_1 \to \mathscr{B}_2$ satisfying (i)–(iii). Conditions (i)–(ii) guarantee (see Theorem 10) that I is a well-defined map $I : \mathscr{B}_1 \to \mathscr{B}_2$. The restriction of I to $\mathscr{L}_1$, i.e. I_*, will be bijective since $\ker A_1(p)$, $\ker A_2(p)$ have the same dimension over ℓ_f. Using Theorem 15 shows that I^* is injective. Now write (i) as

$$[U(s)\quad A_2(s)]\begin{bmatrix} A_1(s) \\ -I \end{bmatrix} = 0 \tag{40}$$

$[U(s)\ A_2(s)]$ has full row rank and so the dimension of its kernel is $\dim\ker[U(s)\ A_2(s)] = r + m - r = m$. The matrices in (40) satisfy Lemma 18, so

$$\delta_M\begin{bmatrix} A_1(s) \\ -I \end{bmatrix} = \varepsilon$$

where ε is the sum of the right minimal indices of $[U(s)\ A_2(s)]$. Now since $[U(s)\ A_2(s)]$ has no zeros in $\mathbb{C} \cup \{\infty\}$,

$$\delta_M([U(s)\quad A_2(s)]) = \varepsilon$$

Using the McMillan degree conditions in (ii) we conclude $\delta_M(A_1(s)) = \delta_M(A_2(s))$ or equivalently $\operatorname{rank} X_{A_1} = \operatorname{rank} X_{A_2}$. Combining this with the fact that I^* is injective and using statement (i) of Theorem 17 and Equation (32) we obtain

$$\operatorname{rank} X_{A_2} = \operatorname{rank}(\bar{U}X_{A_1})$$

From (33) we thus conclude that $\bar{U}\mathscr{R}(X_{A_1}) = \mathscr{R}(X_{A_2})$, which is the necessary and sufficient condition (Theorem 17, (ii)) for surjectiveness of I^*. The surjectiveness of I follows from Theorem 14, which proves the sufficiency of (i)–(iii).

(only if) Assume now that the systems in (2) are fundamentally equivalent. Then conditions (i)–(ii) holds since I is a well-defined map from $\mathscr{B}_1$ to $\mathscr{B}_2$. Since I is surjective, I^* will be surjective, thus using (ii) of Theorem 17 we have

$$\mathscr{R}(X_{A_2}) = \bar{U}\mathscr{R}(X_{A_1})$$

or equivalently

$$\begin{aligned}\operatorname{rank} X_{A_2} &= \operatorname{rank}(\bar{U} X_{A_1}) \\ &= \operatorname{rank} X_{A_1} - \dim(\ker \bar{U} \cap \mathscr{R}(X_{A_1}))\end{aligned}$$

Hence

$$\begin{aligned}\dim(\ker \bar{U} \cap \mathscr{R}(X_{A_1})) &= \delta_M A_1(s) - \delta_M A_2(s) \\ &= \delta_M \begin{bmatrix} A_1(s) \\ -I \end{bmatrix} - \delta_M [U(s) \quad A_2(s)]\end{aligned} \tag{41}$$

Following similar lines as the (if) part it is easy to see that $\delta_M \left[\begin{smallmatrix} A_1(s) \\ -I \end{smallmatrix}\right] = \varepsilon$ and $\delta_M[U(s)\ A_2(s)] = n + \hat{q} + \varepsilon$, where $n, \hat{q}$ is the total number of finite and infinite zeros of $[U(s)\ A_2(s)]$, respectively, (note that $[U(s)\ A_2(s)]$ has full row rank since $A_2(s)$ has full row rank). Now (41) becomes

$$\dim(\ker \bar{U} \cap \mathscr{R}(X_{A_1})) = -n - \hat{q} \geqslant 0 \tag{42}$$

which implies $n + \hat{q} = 0$, i.e. that $[U(s)\ A_2(s)]$ has no zeros in $\mathbb{C} \cup \{\infty\}$ which proves condition (iii). Moreover, from (42) we get $\dim(\ker \bar{U} \cap \mathscr{R}(X_{A_1})) = 0$ or equivalently $\ker \bar{U} \cap \mathscr{R}(X_{A_1}) = \{0\}$, which by (i) of Theorem 17 implies that I^* is injective. Now it easy to see that I_* is injective since I is injective and surjective because Theorem 17 is satisfied. □

Remark 20

Clearly, the conditions of the above theorem coincide with those in Definition 7. The structural invariants of the matrices preserved by full unimodular equivalence i.e. finite and infinite zero structure and right minimal indices will also therefore be preserved by fundamental equivalence.

The following result is a direct consequence of Theorem 19.

Corollary 21

If Σ_1, Σ_2 are fundamentally equivalent then $I : \mathscr{B}_1 \to \mathscr{B}_2$ induces a bijective map I^* between the quotient solution spaces $\mathscr{B}_1/\mathscr{Z}_1, \mathscr{B}_2/\mathscr{Z}_2$ of the systems. Further the restriction of I to $\mathscr{Z}_1$, $I_* : \mathscr{Z}_1 \to \mathscr{Z}_2$, is bijective.

We have the following commutative diagram:

$$\begin{array}{ccccc} \mathscr{B}_1 & \xrightarrow{\pi_1} & \mathscr{B}_1/\mathscr{Z}_1 & \xrightarrow{\phi_1} & \mathscr{X}_{A_1} \\ \downarrow_I & & \downarrow_{I^*} & & \downarrow_{\bar{U}} \\ \mathscr{B}_2 & \xrightarrow{\pi_2} & \mathscr{B}_2/\mathscr{Z}_2 & \xrightarrow{\phi_2} & \mathscr{X}_{A_2} \end{array}$$

where I, I^* and $\bar{U}$ are the maps between the behaviours, the quotient solution spaces and the initial conditions spaces, respectively. π_i, $i = 1, 2$ are the natural projections, ϕ_i, $i = 1, 2$ are the isomorphisms between the quotient solution spaces and the initial condition spaces, that is the map that takes equivalence classes to their corresponding initial conditions.

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It is obvious that if I is a bijection, so that $\mathscr{B}_1 = \mathscr{B}_2$, then due to Theorem 21, I^* will also be an isomorphism and as a consequence of the commutativity of the diagram, $\bar{U}$ will be bijective. Thus, two fundamentally equivalent systems will also have isomorphic initial condition spaces.

Example 22
Note that the following unimodular transformation is valid:

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -s \\ 0 & 0 & 1 \end{bmatrix}}_{U_1(s)} \underbrace{\begin{bmatrix} -1 & s+1 & 0 & 0 \\ 0 & 0 & s & -1 \\ 0 & 1 & -1 & 0 \end{bmatrix}}_{A_1(s)} = \underbrace{\begin{bmatrix} -1 & s+1 & 0 & 0 \\ 0 & -s & 2s & -1 \\ 0 & 1 & -1 & 0 \end{bmatrix}}_{A_2(s)}$$

since $\delta_M(U_1(s)\ A_2(s)) = 2 = \delta_M(A_2(s))$ and $[U_1(s)\ A_2(s)]$ has no infinite zeros. However, the unimodular transformation of Example 1 is not valid since $\delta_M(U(s)\ A_2(s)) = 2 \neq 1 = \delta_M(A_2(s))$. Note that the bijective map between the initial conditions $X_{A_i}\bar{\xi}_0^i$, $i = 1, 2$ in the first case is given by

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\bar{U}} \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{X_{A_1}} \underbrace{\begin{bmatrix} \xi_{11} \\ \xi_{21} \\ \xi_{31} \\ \xi_{41} \end{bmatrix}}_{\bar{\xi}_0^1} = \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{X_{A_2}} \underbrace{\begin{bmatrix} \xi_{12} \\ \xi_{22} \\ \xi_{32} \\ \xi_{42} \end{bmatrix}}_{\bar{\xi}_0^2}$$

which is now seen to be a bijection.

5. CONCLUSIONS

A characterization of C-solvability of a non-regular AR-representation in terms of the left minimal structure of the polynomial matrix that describes the AR-representation is given and is a generalization of the corresponding conditions appearing in [17, 18]. The definition of equivalence between AR-representations that has been presented in the behavioural context by [10, 26] has been extended to the case where the smooth-impulsive solution sets are of interest. An alternative characterization of this equivalence, has been given in terms of a transformation between polynomial matrices, named full unimodular equivalence. Full unimodular equivalence is a special case of the known full matrix equivalence, appearing in [25], and has the nice property of preserving both the finite and infinite zero structure and the right minimal indices of the associated polynomial matrices. These invariants are playing a key role in the description of the smooth-impulsive behaviour of AR-representations [24]. In this sense, fundamental equivalence of non-regular systems provides a dynamical interpretation of known algebraic results.

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