Polynomial Matrices and Equivalent Singular Pencils.

by

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Abstract

In this paper we investigate the problem of transforming a not necessary square and nonsingular polynomial matrix T(s) into a possibly singular pencil sE—A so that T(s) and sE—A have the same (i) finite zeros, (ii) infinite zeros, (iii) right minimal indices, (iv) left minimal indices.

Keywords : polynomial matrices, equivalent, singular pencil.

1. Introduction.

In this paper we examine the following problem : Given a not necessary square polynomial matrix T(s)

$$T(s) = T_0 + T_1 s + \cdots + T_q s^q \in \mathbb{R}[s]^{p\times m} \tag{1.1}$$

determine real matrices :

$$E \in \mathbb{R}^{\lambda_1 \times \lambda_2} \quad \text{and} \quad A \in \mathbb{R}^{\lambda_1 \times \lambda_2} \tag{1.2}$$

such that the "singular pencil" :

$$sE-A \in \mathbb{R}[s]^{\lambda_1 \times \lambda_2} \tag{1.3}$$

satisfies all the following requirements :

(i) T(s) and sE—A have the same *finite* and *infinite* zero structure [2],[7],[8].

(ii) T(s) and sE—A have the same left and right *minimal indices* [2].

We would like also to determine square unimodular polynomial matrices $M(s) \in \mathbb{R}[s]^{(\mu+p)\times(\mu+p)}$ and $N(s) \in \mathbb{R}[s]^{(\mu+m)\times(\mu+m)}$ such that :

$$s\tilde{E} - \tilde{A} = M(s) \begin{bmatrix} I_\mu & 0 \\ 0 & T(s) \end{bmatrix} N(s) \tag{1.4}$$

and where $s\tilde{E} - \tilde{A}$ is a singular pencil in the Kronecker canonical form [2] such that $s\tilde{E}-\tilde{A}$ and T(s) satisfy (i) and (ii) above.

A particularly interesting case of the above problem is the case when T(s) is a pencil i.e.

$$T(s) := T_0 + T_1 s \equiv sE - A \in \mathbb{R}[s]^{p\times m} \tag{1.5}$$

and M(s), N(s) are square constant and nonsingular matrices and this is a problem similar to the one was studied by Kronecker [2]. Another case of interest is when T(s) is a *square* and *nonsingular pencil* sE—A and M(s), N(s) are constant and nonsingular matrices. This is the problem studied by Weierstrass [2] and was generalized in the case of general *square* and *nonsingular polynomial matrices* by Vardulakis [6]. In this note we give a generalization of all the previous cases by assuming that T(s) is a not necessary square and nonsingular polynomial matrix.

2. Strongly Irreducible Systems and System Transformations.

Consider a linear time invariant multivariable system Σ described by a polynomial matrix model

$$A(\rho)\,\beta(t) = B(\rho)\,u(t) \tag{2.1a}$$

$$y(t) = C(\rho)\,\beta(t) + D(\rho)\,u(t) \tag{2.1b}$$

where (ρ=d/dt), $A(\rho) \in \mathbb{R}[\rho]^{r\times r}$ with $|A(\rho)| \neq 0$, $B(\rho) \in \mathbb{R}[\rho]^{r\times m}$, $C(\rho) \in \mathbb{R}[\rho]^{p\times r}$, $D(\rho) \in \mathbb{R}[\rho]^{p\times m}$, $\beta(t)$: $(0-,\infty) \longrightarrow \mathbb{R}^r$ the *pseudostate*, u(t): $(0-,\infty) \longrightarrow \mathbb{R}^m$ the *control input* and y(t) the *output* of Σ and

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let

$$P(s) := \begin{bmatrix} A(s) & -B(s) \\ C(s) & D(s) \end{bmatrix} \in \mathbb{R}[s]^{(r+p)\times(r+m)} \tag{2.2}$$

be the Rosenbrock *system matrix*. Σ may be also written in the form :

$$\begin{bmatrix} A(\rho) & -B(\rho) & 0 \\ C(\rho) & D(\rho) & -I_p \\ 0 & I_m & 0 \end{bmatrix} \begin{bmatrix} \beta(t) \\ u(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ I_m \end{bmatrix} u(t) \tag{2.3a}$$

$$y(t) = [\,0 \;\; 0 \;\; I_p\,] \begin{bmatrix} \beta(t) \\ u(t) \\ y(t) \end{bmatrix} \tag{2.3b}$$

The polynomial matrix

$$\mathcal{P}(s) := \left[\begin{array}{ccc|c} A(s) & -B(s) & 0 & 0 \\ C(s) & D(s) & -I_p & 0 \\ 0 & I_m & 0 & -I_m \\ \hline 0 & 0 & I_p & 0 \end{array}\right] := \begin{bmatrix} \mathcal{T}(s) & -\mathcal{U} \\ \mathcal{V} & 0 \end{bmatrix} \tag{2.4}$$

is defined by Verghese [8] as the *normalized form* of the system matrix P(s).

Definition 1 [7],[8] We define a Rosenbrock system matrix P(s) to be of *generalized least order* or *strongly irreducible* if and only if the compound matrices :

$$\begin{bmatrix} A(s) & -B(s) & 0 \\ C(s) & D(s) & I_p \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} A(s) & -B(s) \\ C(s) & D(s) \\ 0 & -I_m \end{bmatrix} \tag{2.5}$$

have no finite nor infinite zeros. □

Consider the set P(p,m) of $(r+p)\times(r+m)$ polynomial matrices where the integer $r \geq \max(-p,-m)$.

Definition 2 [3] Two matrices $T_1(s),T_2(s) \in P(p,m)\times P(p,m)$ are said to be *fully equivalent* (FE) if there exist polynomial matrices M(s),N(s) of appropriate dimensions such that :

$$\begin{bmatrix} M(s) & T_2(s) \end{bmatrix} \begin{bmatrix} T_1(s) \\ -N(s) \end{bmatrix} = 0 \tag{2.6}$$

where the compound matrices

$$[M(s)\;\; T_2(s)] \quad ; \quad \begin{bmatrix} T_1(s) \\ -N(s) \end{bmatrix} \tag{2.7}$$

satisfy

i) they have full normal rank (2.8a)

ii) they have no finite nor infinite zeros (2.8b)

iii) the following McMillan degree conditions hold

$$\delta_M(M(s)\; T_2(s)) = \delta_M(T_2(s)) \; ; \; \delta_M\begin{bmatrix} T_1(s) \\ N(s) \end{bmatrix} = \delta_M(T_1(s)) \tag{2.8c}$$

□

Proposition 1 [9] If $T_1(s)$, $T_2(s) \in P(p,m)$ are *fully equivalent* then they possess identical finite and infinite zero structures. □

Definition 3 [9] Let $(P_1(s), P_2(s)) \in \in P(p,m)\times P(p,m)$ be two Rosenbrock system matrices. $P_1(s)$ and $P_2(s)$ are said to be *full system equivalent* (FSE) if there exist polynomial matrices M(s),N(s),X(s),Y(s) such that

$$\begin{bmatrix} M(s) & 0 \\ X(s) & I \end{bmatrix} \begin{bmatrix} A_1(s) & -B_1(s) \\ C_1(s) & D_1(s) \end{bmatrix} = \begin{bmatrix} A_2(s) & -B_2(s) \\ C_2(s) & D_2(s) \end{bmatrix} \begin{bmatrix} N(s) & Y(s) \\ 0 & I \end{bmatrix} \tag{2.9}$$

where (2.9) is a transformation of full equivalence (FE). □

Proposition 2 [9] Under full system equivalence the following remain invariant :

(i) generalized order $f := \delta_M(\mathcal{T}(s))$, Rosenbrock degree d_r, (ii) finite and infinite *system zeros*, (iii) finite and infinite *system poles*, (iv) finite and infinite *decoupling zeros*, (v) finite and infiinite *transmission zeros and poles*. □

Proposition 3 Let P(s) be a Rosenbrock system matrix and let $\mathcal{P}(s)$ be its normalized form. Then P(s) and $\mathcal{P}(s)$ have the same finite and infinite zero structures and the same left and right (non—zero) minimal indices.

Proof The proof is based on the fact that there exist a FSE transformation between P(s) and $\mathcal{P}(s)$ [9] as follows :

$$\left[\begin{array}{c|c} I_r & 0 \\ 0 & 0 \\ 0 & 0 \\ \hline 0 & I_p \end{array}\right]\begin{bmatrix} A(s) & -B(s) \\ C(s) & D(s) \end{bmatrix} = \left[\begin{array}{ccc|c} A(s) & -B(s) & 0 & 0 \\ C(s) & D(s) & -I_p & 0 \\ 0 & I_m & 0 & -I_m \\ \hline 0 & 0 & I_p & 0 \end{array}\right]\left[\begin{array}{c|c} I_r & 0 \\ 0 & I_m \\ C(s) & D(s) \\ \hline 0 & I_m \end{array}\right] \tag{2.10}$$

and so according proposition 2 the two system matrices have the same finite and infinite zero structures. As far as the left and right minimal indices are concerned we can see that P(s) and $\mathcal{P}(s)$ are related by the following ***strict equivalence*** transformation :

$$\left[\begin{array}{ccc|c} A(s) & -B(s) & 0 & 0 \\ C(s) & D(s) & 0 & 0 \\ 0 & 0 & I_p & 0 \\ \hline 0 & 0 & 0 & I_m \end{array}\right] = \begin{bmatrix} I_r & 0 & 0 & 0 \\ 0 & I_p & I_p & 0 \\ 0 & 0 & 0 & I_m \\ 0 & 0 & I_p & 0 \end{bmatrix}\left[\begin{array}{ccc|c} A(s) & -B(s) & 0 & 0 \\ C(s) & D(s) & -I_p & 0 \\ 0 & I_m & 0 & -I_m \\ \hline 0 & 0 & I_p & 0 \end{array}\right]\begin{bmatrix} I_r & 0 & 0 & 0 \\ 0 & I_m & 0 & 0 \\ 0 & I_m & 0 & I_m \\ 0 & 0 & I_p & 0 \end{bmatrix} \tag{2.11}$$

and so the two system matrices have the same left and right (non—zero) minimal indices because strict equivalence has the property to preserve the minimal indices. □

Theorem 1 [5] Let

$$P_i(s) = \begin{bmatrix} A_i(s) & -B_i(s) \\ C_i(s) & D_i(s) \end{bmatrix} \in P(p,m) \quad i=1,2 \tag{2.12}$$

be two strongly irreducible Rosenbrock system matrices and let

$$\mathcal{P}_i(s) = := \begin{bmatrix} \mathcal{T}_i(s) & -\mathcal{U}_i \\ \mathcal{V}_i & 0 \end{bmatrix} \tag{2.13}$$

their corresponding normalized forms. Then :

$$P_1(s) \overset{FSE}{\sim} P_2(s) \iff G_1(s) = G_2(s) \tag{2.14}$$

where

$$G_i(s) = C_i(s)A_i(s)^{-1}B_i(s)+D_i(s) = \mathcal{V}_i\mathcal{T}_i(s)^{-1}\mathcal{U}_i \quad i=1,2 \tag{2.15}$$

the transfer function matrix of $P_i(s)$. □

Theorem 2 [8] Consider a strongly irreducible Rosenbrock system matrix P(s) as in (2.2) and its normalized form $\mathcal{P}(s)$ as in (2.4). Then :

(i) The polar structure of the transfer function matrix $G(s) = C(s)A(s)^{-1}B(s)+D(s) = \mathcal{V}\mathcal{T}(s)^{-1}\mathcal{U}$ in $s\in\mathbb{C}\cup\{\infty\}$ is isomorphic with the zero structure of $\mathcal{T}(s)$ in $s\in\mathbb{C}\cup\{\infty\}$.

(ii) The zero structure of the transfer function matrix $G(s) = C(s)A(s)^{-1}B(s)+D(s) = \mathcal{V}\mathcal{T}(s)^{-1}\mathcal{U}$ in $s\in\mathbb{C}\cup\{\infty\}$ is isomorphic with the zero structure of $\mathcal{P}(s)$ in $s\in\mathbb{C}\cup\{\infty\}$ or equivalently from proposition 3 of the Rosenbrock system matrix P(s).

(iii) Let R(s) be a minimal polynomial basis for the *right* null—space of $\mathcal{P}(s)$ so that

$$\mathcal{P}(s)R(s)=0 \quad \text{or} \quad \begin{bmatrix} \mathcal{T}(s) & -\mathcal{U} \\ \mathcal{V} & 0 \end{bmatrix}\begin{bmatrix} R_1(s) \\ R_2(s) \end{bmatrix} = 0 \tag{2.16}$$

Then $R_2(s)$ is a minimal polynomial basis for the *right* null—space of $G(s)=\mathcal{V}\mathcal{T}(s)^{-1}\mathcal{U}$ and has the same indices as R(s). According proposition 3 the transfer function matrix G(s) and the Rosenbrock system matrix P(s) must have also the same right minimal indices.

(iv) Conversely let $R_2(s)$ be a minimal basis for the *right* null—space of G(s). Then

$$R(s) = \begin{bmatrix} \mathcal{T}(s)^{-1}\mathcal{U}R_2(s) \\ R_2(s) \end{bmatrix} \tag{2.17}$$

is a minimal basis for the *right*—null space of $\mathcal{P}(s)$, and has the same indices as $R_2(s)$.

(v) and (vi) are the respective duals of the above, relating *left* null—spaces. □

3. An Equivalent Singular Pencil of a Polynomial Matrix.

In this section we prove that the well known linearizations of Verghese [8] as well as this of Bosgra and Van Der Weiden [1] preserve the algebraic structure of polynomial matrices. We shall find also transformations which relate a polynomial matrix with this linearizations.

Lemma 1 [9] Let $A(s)\in\mathbb{R}[s]^{p\times m}$,$B(s)\in\mathbb{R}[s]^{p\times k}$, $C(s)\in\mathbb{R}[s]^{\sigma\times m}$ be polynomial matrices and let $D\in\mathbb{R}^{\sigma\times k}$ i.e. a constant matrix. Suppose

$$\delta_M\left[\begin{bmatrix} A(s) & B(s) \\ C(s) & D \end{bmatrix}\right] = \delta_M(B(s)) \tag{3.1}$$

Then $C(s)\in\mathbb{R}^{\sigma\times m}$. □

Lemma 2 [6],[9] Let $sE-A\in\mathbb{R}[s]^{p\times m}$ and $C(s)\in\mathbb{R}[s]^{q\times m}$. Suppose that

$$\delta_M\begin{bmatrix} sE-A \\ C(s) \end{bmatrix} = \delta_M(sE-A) \tag{3.2}$$

Then $C(s)=C_0+HEs$ for some constant matrix H. □

Definition 4 [7] Let $T(s)\in\mathbb{R}[s]^{p\times m}$. Then a triple of matrices $C_\infty\in\mathbb{R}^{p\times m}$, $J_\infty\in\mathbb{R}^{\mu\times\mu}$, $B_\infty\in\mathbb{R}^{\mu\times m}$ $\mu\in\mathbb{Z}^+$ such that

$$T(s) = C_\infty(I_\mu-sJ_\infty)^{-1}B_\infty \tag{3.3}$$

is called a *realization* of T(s). □

A realization of T(s) can be obtained always from a realization of the strictly proper rational matrix $\bar{T}(w) := \frac{1}{w}T(\frac{1}{w})\in\mathbb{R}_{pr}(w)^{p\times m}$ because if $\{C_\infty\in\mathbb{R}^{p\times\mu}, J_\infty\in\mathbb{R}^{\mu\times\mu}, B_\infty\in\mathbb{R}^{\mu\times m}\}$ is a realization of $\bar{T}(w)$ i.e. if

$$\bar{T}(w) := \frac{1}{w}T(\frac{1}{w}) = C_\infty(wI_\mu-J_\infty)^{-1}B_\infty \tag{3.4}$$

then (3.4) by substitution $\frac{1}{w}=s$ gives (3.3).

Definition 5 [7] Let $T(s)\in\mathbb{R}[s]^{p\times m}$,$\text{rank}_{\mathbb{R}(s)}T(s)=r$ and let

$$S^\infty_{T(s)} = \text{diag}[s^{q_1},s^{q_2},\ldots,s^{q_v},I_{k-v},\frac{1}{s^{\hat{q}_{k+1}}},\frac{1}{s^{\hat{q}_{k+2}}},\ldots,\frac{1}{s^{\hat{q}_r}},0_{p-r,m-r}] \tag{3.5}$$

be the Smith—McMillan form at $s=\infty$ of T(s), where

$$q_1\geq q_2\geq\cdots\geq q_v>0=q_{v+1}=\cdots=q_k \quad\text{and}\quad \hat{q}_r\geq\hat{q}_{r-1}\geq\cdots\geq\hat{q}_{k+1}>0 \tag{3.6}$$

are respectively the orders of the poles and zeros at $s=\infty$ of T(s).

Let $C_\infty\in\mathbb{R}^{p\times\mu}$, $J_\infty\in\mathbb{R}^{\mu\times\mu}$, $B_\infty\in\mathbb{R}^{\mu\times m}$ be a realization of $T(s)\in\mathbb{R}[s]^{p\times m}$ and consider the Rosenbrock system matrix

$$P_1(s) = \begin{bmatrix} I_\mu-sJ_\infty & -B_\infty \\ C_\infty & 0 \end{bmatrix} \tag{3.7}$$

Then we have :

Proposition 4 [7] If one of the following equivalent conditions is satisfied

(i) $P_1(s)$ is strongly irreducible

(ii)
$$\begin{bmatrix} I_\mu-sJ_\infty & -B_\infty & 0 \\ C_\infty & 0 & I_p \end{bmatrix} \quad\text{and}\quad \begin{bmatrix} I_\mu-sJ_\infty & -B_\infty \\ C_\infty & 0 \\ 0 & I_m \end{bmatrix} \tag{3.8}$$

have no zeros in $\mathbb{C}\cup\{\infty\}$

(iii)
$$[I_\mu-sJ_\infty \;\; -B_\infty] \quad\text{and}\quad \begin{bmatrix} I_\mu-sJ_\infty \\ -C_\infty \end{bmatrix} \tag{3.9}$$

have no zeros in $\mathbb{C}\cup\{\infty\}$

Then

$$\mu=\sum_{i=1}^{k}(q_i+1) \tag{3.10}$$

If J_∞ is in Jordan normal form then

$$J_\infty = \text{block diag}[J_{\infty 1},J_{\infty 2},\ldots,J_{\infty k}]\in\mathbb{R}^{\mu\times\mu} \tag{3.11}$$

where

$$J_{\infty i} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & \ddots & & \vdots \\ \vdots & & & \ddots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \in \mathbb{R}^{(q_i+1)\times(q_i+1)} \quad i=1,2,\ldots,k \tag{3.12}$$

□

Now using lemmata 1,2 and Theorems 1 and 2 we prove

Theorem 3 Let $T(s)\in\mathbb{R}[s]^{p\times m}$ and let $\{C_\infty\in\mathbb{R}^{p\times\mu}, J_\infty\in\mathbb{R}^{\mu\times\mu}, B_\infty\in\mathbb{R}^{\mu\times m}\}$ be a strongly irreducible realization of $T(s)$ with J_∞ in Jordan normal form. Then the polynomial matrix $T(s)\in\mathbb{R}^{p\times m}$ and the singular pencil

$$sE-A = \begin{bmatrix} I_\mu - sJ_\infty & -B_\infty \\ C_\infty & 0 \end{bmatrix} = s\begin{bmatrix} -J_\infty & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} -I_\mu & B_\infty \\ -C_\infty & 0 \end{bmatrix} \tag{3.13}$$

have the same :

(i) finite, and infinite zero structure,

(ii) left and right minimal indices.

and there exist the following transformation between them :

$$\left[\begin{array}{c|c} I_\mu - sJ_\infty & -B_\infty \\ \hline C_\infty & 0 \end{array}\right] = \left[\begin{array}{c|c} I_\mu & 0 \\ \hline C_\infty(I_\mu - sJ_\infty)^{-1} & I_p \end{array}\right]\left[\begin{array}{c|c} I_\mu & 0 \\ \hline 0 & T(s) \end{array}\right]\left[\begin{array}{c|c} I_\mu - sJ_\infty & -B_\infty \\ \hline 0 & I_m \end{array}\right] \tag{3.14a}$$

$$\left[\begin{array}{c|c} I_\mu - sJ_\infty & -B_\infty \\ \hline C_\infty & 0 \end{array}\right] = \left[\begin{array}{c|c} I_\mu - sJ_\infty & 0 \\ \hline C_\infty & I_p \end{array}\right]\left[\begin{array}{c|c} I_\mu & 0 \\ \hline 0 & T(s) \end{array}\right]\left[\begin{array}{c|c} I_\mu & -(I_\mu - sJ_\infty)^{-1}B_\infty \\ \hline 0 & I_m \end{array}\right] \tag{3.14b}$$

Proof Consider the singular pencil sE—A in (3.13) as a Rosenbrock system matrix i.e. let $A(s)=I_\mu - sJ_\infty$ $C(s)=C_\infty$, $B(s)=B_\infty$, $D(s)=0_{p,m}$ where $C_\infty, J_\infty, B_\infty$ a strongly irreducible realization of $T(s)$, or equivalently let

$$P_1(s) = \left[\begin{array}{c|c} I_\mu - sJ_\infty & -B_\infty \\ \hline C_\infty & 0 \end{array}\right] \tag{3.15}$$

Then $P_1(s)$ is by construction strongly irreducible and gives rise to the transfer function matrix $G(s):=C(s)A(s)^{-1}B(s)+D(s)=T(s)$. Hence (i),(ii) follows from theorem 2. If we now consider the Rosenbrock system matrix

$$P_2(s) = \left[\begin{array}{c|c} I_\mu & 0 \\ \hline 0 & T(s) \end{array}\right] \tag{3.16}$$

then both $P_1(s), P_2(s)$ are strongly irreducible and give rise to the same transfer function matrix $T(s)$ and so according theorem 1, $P_1(s)$ and $P_2(s)$ are FSE or equivalently there exist polynomial matrices $M(s)\in\mathbb{R}[s]^{\mu\times\mu}$, $X(s)\in\mathbb{R}[s]^{p\times\mu}$, $N(s)\in\mathbb{R}[s]^{\mu\times\mu}$ and $Y(s)\in\mathbb{R}[s]^{\mu\times m}$ such that

$$\left[\begin{array}{c|c} M(s) & 0 \\ \hline X(s) & I_p \end{array}\right]\left[\begin{array}{c|c} I_\mu - sJ_\infty & -B_\infty \\ \hline C_\infty & 0 \end{array}\right] = \left[\begin{array}{c|c} I_\mu & 0 \\ \hline 0 & T(s) \end{array}\right]\left[\begin{array}{c|c} N(s) & Y(s) \\ \hline 0 & I_m \end{array}\right] \tag{3.17}$$

where (3.17) is a full equivalence relation. From the McMillan degree conditions of the f.e. relation (3.17) we obtain that

$$\delta_M\left[\begin{array}{cc} I_\mu - sJ_\infty & -B_\infty \\ C_\infty & 0 \\ \hline N(s) & Y(s) \\ 0 & I_m \end{array}\right] = \delta_M\left[\begin{array}{c|c} I_\mu - sJ_\infty & -B_\infty \\ \hline C_\infty & 0 \end{array}\right] \tag{3.18}$$

and

$$\delta_M\left[\begin{array}{cc|cc} M(s) & 0 & I_\mu & 0 \\ X(s) & I_p & 0 & T(s) \end{array}\right] = \delta_M\left[\begin{array}{c|c} I_\mu & 0 \\ \hline 0 & T(s) \end{array}\right] \tag{3.19}$$

which according lemmatta 1 and 2 give that $Y(s)=Y\in\mathbb{R}^{\mu\times m}$, $M(s)=M\in\mathbb{R}^{\mu\times\mu}$ (constant matrices) and $N(s) = N_0+HJ_\infty s \in\mathbb{R}[s]^{\mu\times\mu}$ for some constant matrix H. So (3.17) may be rewritten as

$$\left[\begin{array}{c|c} M & 0 \\ \hline X(s) & I_p \end{array}\right]\left[\begin{array}{c|c} I_\mu - sJ_\infty & -B_\infty \\ \hline C_\infty & 0\end{array}\right] = \left[\begin{array}{c|c} I_\mu & 0 \\ \hline 0 & T(s)\end{array}\right]\left[\begin{array}{c|c} N_0+HJ_\infty s & Y \\ \hline 0 & I_m\end{array}\right] \tag{3.20}$$

From (3.20) we obtain that

$$M(I_\mu - sJ_\infty) = N_0+HJ_\infty s \Leftrightarrow M=N_0 \text{ and } -MJ_\infty=HJ_\infty \tag{3.21a}$$

$$-MB_\infty = Y \tag{3.21b}$$

$$X(s)\,[I_\mu - sJ_\infty] + C_\infty = 0 \Leftrightarrow X(s) = -C_\infty(I_\mu - sJ_\infty)^{-1} \tag{3.21c}$$

We can also see that the compound matrices in (3.20)

$$Q(s)=\left[\begin{array}{cc} I_\mu - sJ_\infty & -B_\infty \\ C_\infty & 0 \\ \hline N_0+HJ_\infty s & Y \\ 0 & I_m \end{array}\right] \quad\text{and}\quad L(s)=\left[\begin{array}{cc|cc} M & 0 & I_\mu & 0 \\ X(s) & I_p & 0 & T(s)\end{array}\right] \tag{3.22}$$

have no finite nor infinite zeros because L(s) has a unit submatrix I_{p+m} and constant and thus biproper and unimodular elementary operation [7] reduce Q(s) to :

$$\left[\begin{array}{cccc} I & 0 & 0 & B_\infty \\ 0 & I & 0 & 0 \\ H & 0 & I & 0 \\ 0 & 0 & 0 & I \end{array}\right]\left[\begin{array}{cc} I_\mu - sJ_\infty & -B_\infty \\ C_\infty & 0 \\ \hline N_0+HJ_\infty s & Y \\ 0 & I_m \end{array}\right] = \left[\begin{array}{cc} I_\mu - sJ_\infty & 0 \\ C_\infty & 0 \\ \hline 0 & 0 \\ 0 & I_m \end{array}\right] \tag{3.23}$$

which has no finite nor infinite zeros since $\{C_\infty\in\mathbb{R}^{p\times\mu}, J_\infty\in\mathbb{R}^{\mu\times\mu}, B_\infty\in\mathbb{R}^{\mu\times m}\}$ is a strongly irreducible realization of T(s). As far as the constant matrix $M \in \mathbb{R}^{\mu\times\mu}$ is concerned this plays no role and we may assume that $M=I_\mu$. If we take into account relations (3.21) then relation (3.20) may be rewritten as

$$\left[\begin{array}{c|c} I_\mu & 0 \\ \hline -C_\infty(I_\mu - sJ_\infty)^{-1} & I_p \end{array}\right]\left[\begin{array}{c|c} I_\mu - sJ_\infty & -B_\infty \\ \hline C_\infty & 0\end{array}\right] = \left[\begin{array}{c|c} I_\mu & 0 \\ \hline 0 & T(s)\end{array}\right]\left[\begin{array}{c|c} I_\mu - sJ_\infty & -B_\infty \\ \hline 0 & I_m\end{array}\right] \Leftrightarrow$$

$$\left[\begin{array}{c|c} I_\mu - sJ_\infty & -B_\infty \\ \hline C_\infty & 0\end{array}\right] = \left[\begin{array}{c|c} I_\mu & 0 \\ \hline C_\infty(I_\mu - sJ_\infty)^{-1} & I_p \end{array}\right]\left[\begin{array}{c|c} I_\mu & 0 \\ \hline 0 & T(s)\end{array}\right]\left[\begin{array}{c|c} I_\mu - sJ_\infty & -B_\infty \\ \hline 0 & I_m\end{array}\right] \tag{3.24}$$

Using the symmetry property of the full system equivalence relation and the previous methodology we also obtain that

$$\left[\begin{array}{c|c} I_\mu - sJ_\infty & -B_\infty \\ \hline C_\infty & 0\end{array}\right] = \left[\begin{array}{c|c} I_\mu - sJ_\infty & 0 \\ \hline C_\infty & I_p \end{array}\right]\left[\begin{array}{c|c} I_\mu & 0 \\ \hline 0 & T(s)\end{array}\right]\left[\begin{array}{cc} I_\mu & -(I_\mu - sJ_\infty)^{-1}B_\infty \\ 0 & I_m\end{array}\right] \tag{3.25}$$

and so the theorem is proved. □

Corollary 1 That the polynomial matrix $T(s)\in\mathbb{R}[s]^{p\times m}$ and the singular pencil of (3.13) have the same finite and infinite zero structure may be explained by the property of full equivalence (Proposition 1) which preserve the finite and infinite zero structure of polynomial matrices. Unfortunately we are not in a position to state similar results for the left and right null–space structure because these are not preserved always under the full equivalence transformation. □

Corollary 2 We may see according definition 5 or equivalently from theorem 2 (i) that the infinite pole structure of T(s) is isomorphic to the zero structure of $I_\mu - sJ_\infty$ and so the equivalent pencil of (3.10) give us information also as concerns the infinite pole structure of T(s). □

Remark 1 Using lemma 2 we can easily see that the dimension of the equivalent singular pencil of (3.10) is

$$\lambda_1\times\lambda_2 = (p+\mu)\times(\mu+m) = \Big(p+\sum_{i=1}^{k}(q_i+1)\Big)\times\Big(\sum_{i=1}^{k}(q_i+1)+m\Big) \tag{3.26}$$

Due to the fact that $q_i=0$ $i=v+1,..,k$ there will be $k-v$ 1×1 zero blocks in the Jordan normal form J_∞ which are associated with the "non–dynamic" variables in the set of equations :

$$J_\infty \dot{x}(t) = I_\mu x(t) + B_\infty u(t) \tag{3.27a}$$

$$y(t) = C_\infty x(t) \tag{3.27b}$$

We can always absorb these "non–dynamic" variables in the output expression via a constant feedthrough matrix D_∞ and so we obtain a set of equations

$$\bar{J}_\infty \dot{x}_r(t) = I_{\mu'}\, x_r(t) + \bar{B}_\infty u(t) \tag{3.28a}$$

$$y(t) = \bar{C}_\infty x_r(t) + \bar{D}_\infty u(t) \tag{3.28b}$$

and thus an equivalent pencil

$$s\bar{E}-\bar{A} = \begin{bmatrix} I_{\mu'}-s\bar{J}_\infty & -\bar{B}_\infty \\ \bar{C}_\infty & \bar{D}_\infty \end{bmatrix} = s\begin{bmatrix} -\bar{J}_\infty & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} -I_{\mu'} & \bar{B}_\infty \\ -\bar{C}_\infty & -\bar{D}_\infty \end{bmatrix} \tag{3.29}$$

with dimension less than the one in (3.26)

$$\lambda_1'\times\lambda_2' = (p+\mu')\times(\mu+m') = (p+\sum_{i=1}^{v}(q_i+1))\times(\sum_{i=1}^{v}(q_i+1)+m) \tag{3.30}$$

where now $q_i>0$ $i=1,2,..,v$ are the positive orders of the infinite poles of T(s). We can easily see that the new singular pencil will be related with the polynomial matrix T(s) via the transformations :

$$\left[\begin{array}{c|c} I_{\mu'}-s\bar{J}_\infty & -B_\infty \\ \hline \bar{C}_\infty & \bar{D}_\infty \end{array}\right] = \left[\begin{array}{c|c} I_{\mu'} & 0 \\ \hline \bar{C}_\infty(I_{\mu'}-s\bar{J}_\infty)^{-1} & I_p \end{array}\right]\left[\begin{array}{c|c} I_{\mu'} & 0 \\ \hline 0 & T(s) \end{array}\right]\left[\begin{array}{c|c} I_{\mu'}-s\bar{J}_\infty & -\bar{B}_\infty \\ \hline 0 & I_m \end{array}\right] \tag{3.31}$$

and

$$\left[\begin{array}{c|c} I_{\mu'}-s\bar{J}_\infty & -\bar{B}_\infty \\ \hline \bar{C}_\infty & \bar{D}_\infty \end{array}\right] = \left[\begin{array}{c|c} I_{\mu'}-s\bar{J}_\infty & 0 \\ \hline \bar{C}_\infty & I_p \end{array}\right]\left[\begin{array}{c|c} I_{\mu'} & 0 \\ \hline 0 & T(s) \end{array}\right]\left[\begin{array}{c|c} I_{\mu'} & -(I_{\mu'}-s\bar{J}_\infty)^{-1}\bar{B}_\infty \\ \hline 0 & I_m \end{array}\right] \tag{3.32}$$

□

Example 1 Consider the polynomial matrix

$$T(s) = \begin{bmatrix} s^3+s^2 & s^3+s^2-1 & s^3-s \\ -s^2-s & -s^2-s & -s^2+1 \end{bmatrix} \tag{E.1.1}$$

The algebraic structure of $T(s)\in\mathbb{R}[s]^{2\times3}$ can be easily seen from the Smith Form of T(s) at $s\in\mathbb{C}$, the McMillan Form of T(s) at $s=\infty$ and the left and right minimal indices of T(s).

We can see that

$$S^{\infty}_{T(s)} = \begin{bmatrix} s^3 & 0 & 0 \\ 0 & 1/s & 0 \end{bmatrix} \;;\; S^{\mathbb{C}}_{T(s)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & s+1 & 0 \end{bmatrix} \;;\; T(s)\begin{bmatrix} 1-s \\ 0 \\ s \end{bmatrix} = 0_{2\times1} \tag{E.1.2}$$

In this case $v=1$, $q_1=3$ and $\mu=q_1+1=4$. Let $C_\infty\in\mathbb{R}^{2\times4}$, $J_\infty\in\mathbb{R}^{4\times4}$, $B_\infty\in\mathbb{R}^{4\times3}$ be a minimal realization of T(s) such that

$$T(s) = C_\infty(I_\mu-sJ_\infty)^{-1}B_\infty = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & -s & 0 & 0 \\ 0 & 1 & -s & 0 \\ 0 & 0 & 1 & -s \\ 0 & 0 & 0 & 1 \end{bmatrix}^{-1}\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \tag{E.1.3}$$

Then the pencil

$$
sE-A=\left[\begin{array}{c|c} I_{\mu}-sJ_{\infty} & -B_{\infty} \\ \hline C_{\infty} & 0 \end{array}\right]=\left[\begin{array}{cccc|ccc} 1 & -s & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & -s & 0 & -1 & -1 & 0 \\ 0 & 0 & 1 & -s & -1 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 & -1 & -1 \\ \hline 1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 & 0 \end{array}\right] \tag{E.1.4}
$$

has the same algebraic structure with the polynomial matrix T(s). In particular

$$
S^{\infty}_{sE-A}=[\mathrm{diag}[s,s,s,1,1,\tfrac{1}{s}] \,\vdots\, 0_{6\times1}] \quad ; \quad S^{\mathbb{C}}_{sE-A}=[\mathrm{diag}[1,1,1,1,1,(s+1)] \,\vdots\, 0_{6\times1}]
$$

$$
\text{and} \quad (sE-A)\,[1,1,1,1,1-s,0,s]^{T}=0 \tag{E.1.5}
$$

The dimension of the equivalent pencil sE—A is $\lambda_1\times\lambda_2=(p+q_1+1)\times(q_1+1+m)=6\times7$. We have also that the singular pencil sE—A and the polynomial matrix T(s) are related via the following transformation

$$
\left[\begin{array}{cccc|ccc} 1 & -s & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & -s & 0 & -1 & -1 & 0 \\ 0 & 0 & 1 & -s & -1 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 & -1 & -1 \\ \hline 1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 & 0 \end{array}\right]=\left[\begin{array}{cccc|c} & I_4 & & & 0_{4\times2} \\ \hline 1 & s & s^2-1 & s^3-s & I_2 \\ 0 & -1 & 0 & -s^2+1 & \end{array}\right]
$$

$$
\left[\begin{array}{c|ccc} I_4 & & 0_{4\times3} & \\ \hline 0_{2\times4} & s^3+s^2 & s^3+s^2-1 & s^3-s \\ & -s^2-s & -s^2-s & -s^2+1 \end{array}\right]\left[\begin{array}{cccc|ccc} 1 & -s & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & -s & 0 & -1 & -1 & 0 \\ 0 & 0 & 1 & -s & -1 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 & -1 & -1 \\ \hline & 0_{3\times4} & & & & I_3 & \end{array}\right] \tag{E.1.6}
$$

□

We shall give now the Bosgra and Van Der Weiden equivalent singular pencil which has less dimension than this of (3.13). Let $T(s)\in\mathbb{R}[s]^{p\times m}$ as in (1.1). Define the block Hankel matrices (Bosgra and Van Der Weiden 1981) :

$$
\Pi_E=\begin{bmatrix} T_2 & T_3 & \cdots & T_q \\ T_3 & T_4 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ T_q & 0 & \cdots & 0 \end{bmatrix},\ \Pi_A=\begin{bmatrix} T_3 & T_4 & \cdots & T_q & 0 \\ T_4 & T_5 & \cdots & 0 & 0 \\ \vdots & & \ddots & & \vdots \\ T_q & 0 & & & \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix},\ \Pi_B=\begin{bmatrix} T_2 \\ T_3 \\ \vdots \\ T_q \end{bmatrix}
$$

$$
\Pi_C=[T_2,T_3,\ldots,T_q] \tag{3.33}
$$

and let $r=\mathrm{rank}_{\mathbb{R}}\Pi_E$. Let I,J denote sets of r row and column indices such that the rows I and columns J of Π_E are linearly independent respectively. Let T_E,T_A,T_B,T_C be submatrices of Π_E,Π_A,Π_B,Π_C, respectively such that T_E,T_A,T_B are formed by the rows I and T_E,T_A,T_C are formed by the columns J (see [1]). Define the system matrix :

$$
T_F(s)=\begin{bmatrix} T_E-sT_A & -T_Bs \\ T_Cs & T_1s+T_0 \end{bmatrix} \tag{3.34}
$$

Lemma 3 [1],[4]

(i) $T(s)=(T_Cs)(T_E-sT_A)^{-1}(T_Bs)+T_1s+T_0$

(ii) The compound matrix pencils

$$\begin{bmatrix} T_F(s) \\ [0 \ I_p] \end{bmatrix} \quad \text{and} \quad [T_F(s) \begin{bmatrix} 0 \\ -I_m \end{bmatrix}] \tag{3.35}$$

have no finite nor infinite zero in common. □

Theorem 4 The polynomial matrix T(s) of (1.1) and the singular pencil $T_F(s)$ have the same

(i) finite and infinite zero structure,
(ii) left and right minimal indices.

and there exist the following transformations between them

$$\begin{bmatrix} T_E-sT_A & -T_B s \\ T_C s & T_1 s+T_0 \end{bmatrix} = \begin{bmatrix} I_r & 0 \\ T_C s(T_E-sT_A)^{-1} & I_p \end{bmatrix}\begin{bmatrix} I_r & 0 \\ 0 & T(s) \end{bmatrix}\begin{bmatrix} T_E-sT_A & -T_B s \\ 0 & I_m \end{bmatrix} \tag{3.36a}$$

$$\begin{bmatrix} T_E-sT_A & -T_B s \\ T_C s & T_1 s+T_0 \end{bmatrix} = \begin{bmatrix} T_E-sT_A & 0 \\ T_C s & I_p \end{bmatrix}\begin{bmatrix} I_r & 0 \\ 0 & T(s) \end{bmatrix}\begin{bmatrix} I_r & -(T_E-sT_A)^{-1}T_B s \\ 0 & I_m \end{bmatrix} \tag{3.36b}$$

Proof That the singular pencil $T_F(s)$ has the same finite, infinite zero structure and the same left and right minimal indices with the polynomial matrix $T(s) \in \mathbb{R}[s]^{p \times m}$ is based in theorem 2 if we assume that $G(s) := C(s)A^{-1}(s)B(s)+D(s) = T(s)$ with $C(s)=T_C s$, $A(s)=T_E-sT_A$, $B(s)=T_B s$ and $D(s)=T_1 s+T_0$.

If we consider the Rosenbrock system matrices

$$P_1(s) = \begin{bmatrix} T_E-sT_A & -T_B s \\ T_C s & T_1 s+T_0 \end{bmatrix} \quad \text{and} \quad P_2(s) = \begin{bmatrix} I_r & 0 \\ 0 & T(s) \end{bmatrix} \tag{3.37}$$

then $P_1(s)$ and $P_2(s)$ are strongly irreducible and give rise to the same transfer function T(s) and so according theorem 1, $P_1(s)$ and $P_2(s)$ are full equivalent or equivalently there exist polynomial matrices $M(s)\in\mathbb{R}[s]^{r\times r}$, $X(s)\in\mathbb{R}[s]^{p\times r}$, $N(s)\in\mathbb{R}[s]^{r\times r}$ and $Y(s)\in\mathbb{R}[s]^{r\times m}$ such that

$$\begin{bmatrix} M(s) & 0 \\ X(s) & I_p \end{bmatrix}\begin{bmatrix} T_E-sT_A & -T_B s \\ T_C s & T_1 s+T_0 \end{bmatrix} = \begin{bmatrix} I_r & 0 \\ 0 & T(s) \end{bmatrix}\begin{bmatrix} N(s) & Y(s) \\ 0 & I_m \end{bmatrix} \tag{3.38}$$

where (3.38) is a full equivalence transformation. According the McMillan degree conditions of the f.e. transformation (3.38)

$$\delta_M\begin{bmatrix} M(s) & 0 & I_r & 0 \\ X(s) & I_p & 0 & T(s) \end{bmatrix} = \delta_M\begin{bmatrix} M(s) & 0 \\ X(s) & I_p \end{bmatrix} \tag{3.39a}$$

and

$$\delta_M\begin{bmatrix} T_E-sT_A & -T_B s \\ T_C s & T_1 s+T_0 \\ -N(s) & -Y(s) \\ 0 & -I_m \end{bmatrix} = \delta_M\begin{bmatrix} T_E-sT_A & -T_B s \\ T_C s & T_1 s+T_0 \end{bmatrix} \tag{3.39b}$$

and from lemma 1,2 we obtain that $M(s)=M \in\mathbb{R}^{r\times r}$, $N(s)=N_0+N_1 s \in\mathbb{R}[s]^{r\times r}$ and $Y(s)=Y_0+Y_1 s \in\mathbb{R}[s]^{r\times m}$. We have also that

$$M\,(T_E-sT_A) = N_0+N_1 s \Leftrightarrow N_0=MT_E \text{ and } N_1=-MT_A \tag{3.40a}$$

$$-M\,T_B s = Y_0+Y_1 s \Leftrightarrow Y_0=0 \text{ and } Y_1 = -MT_B \tag{3.40b}$$

$$X(s)[T_E-sT_A] + T_C s = 0 \Leftrightarrow X(s) = -T_C s(T_E-sT_A)^{-1} = -[T_2 s+T_3 s^2+\cdots+T_q s^q] \tag{3.40c}$$

The relation (3.38) may be rewritten now as

$$\begin{bmatrix} M & 0 \\ -T_C s(T_E-sT_A)^{-1} & I_p \end{bmatrix}\begin{bmatrix} T_E-sT_A & -T_B s \\ T_C s & T_1 s+T_0 \end{bmatrix} = \begin{bmatrix} I_r & 0 \\ 0 & T(s) \end{bmatrix}\begin{bmatrix} M(T_E-sT_A) & -MT_B s \\ 0 & I_m \end{bmatrix} \tag{3.41}$$

which can be easily seen as in theorem 3 that satisfy all the f.e. conditions. As far as concern the constant matrix $M \in \mathbb{R}^{r\times r}$ is concerned this plays no role and we may assume that $M=I_r$ and so (3.38) may be rewritten as

$$\begin{bmatrix} T_E - sT_A & -T_B s \\ T_C s & T_1 s + T_0 \end{bmatrix} = \begin{bmatrix} I_r & 0 \\ T_C s(T_E - sT_A)^{-1} & I_p \end{bmatrix} \begin{bmatrix} I_r & 0 \\ 0 & T(s) \end{bmatrix} \begin{bmatrix} T_E - sT_A & -T_B s \\ 0 & I_m \end{bmatrix} \quad (3.42)$$

Using the symmetry property of full system equivalence and the same methodology we obtain that

$$\begin{bmatrix} T_E - sT_A & -T_B s \\ T_C s & T_1 s + T_0 \end{bmatrix} = \begin{bmatrix} T_E - sT_A & 0 \\ T_C s & I_p \end{bmatrix} \begin{bmatrix} I_r & 0 \\ 0 & T(s) \end{bmatrix} \begin{bmatrix} I_r & -(T_E - sT_A)^{-1} T_B s \\ 0 & I_m \end{bmatrix} \quad (3.43)$$

and so the theorem is proved. □

Corollary 2 The dimension of the singular pencil $T_F(s)$ is equal to

$$\lambda_1 \times \lambda_2 = (p+r)\times(r+m) = (p+\mathrm{rank}_{\mathbb{R}}\Pi_E)\times(\mathrm{rank}_{\mathbb{R}}\Pi_E+m)$$

$$= (p+\delta_M(\frac{T_1 s + T_2 s^2 + \cdots + T_q s^q}{s}))\times(\delta_M(\frac{T_1 s + T_2 s^2 + \cdots + T_q s^q}{s})+m)$$

$$= (p+\sum_{i=1}^{v}(q_i-1))\times(\sum_{i=1}^{v}(q_i-1)+m) \quad (3.44)$$

where q_i $i\in k$ are the positive orders of the infinite poles of $T(s)$ (see 3.5). We can see so that the dimension of the singular pencil $T_F(s)$ is less than these of the singular pencils of (3.13) and (3.34). □

Theorem 5 [2] Let $sE-A \in \mathbb{R}[s]^{p\times m}$, $\mathrm{rank}_{\mathbb{R}(s)} sE-A = r$ with

$$S^{\mathbb{C}}_{sE-A} = \mathrm{diag}[\underbrace{1, 1, \ldots, 1}_{a}, (s-\lambda_1)^{\sigma_{11}} \cdots (s-\lambda_f)^{\sigma_{1f}}, \ldots, (s-\lambda_1)^{\sigma_{d1}} \cdots (s-\lambda_f)^{\sigma_{df}}, 0_{p-d-a, m-d-a}] \quad (3.45)$$

with $0<\sigma_{11}\leq\sigma_{21}\leq\cdots\leq\sigma_{d1}, \ldots, 0<\sigma_{1f}\leq\sigma_{2f}\leq\cdots\leq\sigma_{df}$ the *Smith form* of sE—A at $\mathbb{C}$ [2] and

$$S^{\infty}_{sE-A} = \mathrm{diag}\left\{sI_k, \frac{1}{s^{\hat{q}_{k+1}}}, \frac{1}{s^{\hat{q}_{k+2}}}, \ldots, \frac{1}{s^{\hat{q}_r}}, 0_{p-r, m-r}\right\} \quad (3.46)$$

with $q_1=q_2=\cdots=q_k=1$ και $\hat{q}_{k+1}\leq\hat{q}_{k+2}\leq\cdots\leq\hat{q}_r\leq 0$, the *McMillan form* at $s=\infty$ of sE—A [7]. Then there exist constant and nonsingular matrices $M\in\mathbb{R}^{p\times p}$ and $N\in\mathbb{R}^{m\times m}$ such that

$$s\tilde{E} - \tilde{A} = M\,(sE-A)\,N = \text{block diag}\,\{0_{h,g}, L_\varepsilon(s), L_\eta(s), sI_n+J, I_\mu+sJ_\infty\} \quad (3.47)$$

where

1)

$$J = \text{block diag}[J_{11}, J_{12}, \ldots, J_{df}] \quad (3.48)$$

$$J_{ij} = \underbrace{\begin{bmatrix} \lambda_j & 1 & 0 \cdots & 0 \\ 0 & \lambda_j & 1 \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 \cdots & \lambda_j \end{bmatrix}}_{\sigma_{ij}} \Bigg\updownarrow \sigma_{ij} \quad i \in f \quad (3.49)$$

2)

$$J_\infty = \text{block diag}[J_{\infty_1}, J_{\infty_2}, \ldots, J_{\infty_{r-k}}] \quad (3.50)$$

$$J_{\infty_i} = \underbrace{\begin{bmatrix} 0 & 1 & 0 \cdots & 0 \\ 0 & 0 & 1 \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 \cdots & 1 \\ 0 & 0 & 0 \cdots & 0 \end{bmatrix}}_{\hat{q}_i+1} \Bigg\updownarrow \hat{q}_i+1 \quad i \in r-k \quad (3.51)$$

3) $L_\varepsilon(s) = \text{block diag}[L_{\varepsilon_{g+1}}(s), L_{\varepsilon_{g+2}}(s), \ldots, L_{\varepsilon_l}(s)]$ (3.52)

$$L_{\varepsilon_i}(s) = \begin{bmatrix} s & 1 & 0 \cdots & 0 & 0 \\ 0 & s & 1 \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \ddots & 1 & 0 \\ 0 & 0 & 0 \cdots & s & 1 \end{bmatrix} \in \mathbb{R}^{\varepsilon_i \times (\varepsilon_i+1)}[s] \tag{3.53}$$

4) $$L_{\eta}(s) = \text{block diag}[L_{\eta_{h+1}}(s), L_{\eta_{h+2}}(s), \ldots, L_{\eta_t}(s)] \tag{3.54}$$

$$L_{\eta_i}(s) = \begin{bmatrix} s & 0 & 0 \cdots & 0 \\ 1 & s & 0 \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 \cdots & s \\ 0 & 0 & 0 & & 1 \end{bmatrix} \in \mathbb{R}^{(\eta_i+1) \times \eta_i}[s] \tag{3.55}$$

where $\varepsilon_1=\varepsilon_2=\cdots=\varepsilon_g=0<\varepsilon_{g+1}\leq\varepsilon_{g+2}\leq\cdots\leq\varepsilon_l$ are the *right minimal indices* of sE—A , $\eta_1=\eta_2=\cdots=\eta_h=0<\eta_{h+1}\leq\eta_{h+2}\leq\cdots\leq\eta_t$ are the *left minimal indices* of sE—A [2] □

Proposition 5 Let $T(s) \in \mathbb{R}[s]^{p\times m}$, $\text{rank}_{\mathbb{R}(s)}T(s)=r$. There always exist unimodular polynomial matrices $M(s)\in\mathbb{R}[s]^{(r+p)\times(r+p)}$ and $N(s)\in\mathbb{R}^{(r+m)\times(r+m)}$ such that

$$s\tilde{E} - \tilde{A} = M(s) \begin{bmatrix} I_r & 0 \\ 0 & T(s) \end{bmatrix} N(s) \tag{3.56}$$

where $s\tilde{E} - \tilde{A}$ is a singular pencil in the Kronecker canonical form with the same algebraic structure with this of the polynomial matrix $T(s)\in\mathbb{R}[s]^{p\times m}$.

Proof We have from (3.42) that there exist unimodular polynomial matrices $M_1(s) \in\mathbb{R}[s]^{(r+p)\times(r+p)}$ and $N_1(s) \in\mathbb{R}[s]^{(r+m)\times(r+m)}$ such that

$$T_F(s) = M_1(s) \begin{bmatrix} I_r & 0 \\ 0 & T(s) \end{bmatrix} N_1(s) \tag{3.57}$$

We know also from proposition 5 that there always exist square constant and nonsingular matrices $M_2 \in\mathbb{R}^{(p+r)\times(p+r)}$ and $N_2 \in\mathbb{R}^{(r+m)\times(r+m)}$ such that

$$s\tilde{E} - \tilde{A} = M_2 T_F(s) N_2 \tag{3.58}$$

From (3.57) and (3.58) we obtain that there exist unimodular polynomial matrices $M(s)=M_2^{-1}M_1(s) \in\mathbb{R}[s]^{(p+r)\times(p+r)}$ and $N(s)=N_1(s)N_2^{-1} \in\mathbb{R}^{(r+m)\times(r+m)}$ such that relation (3.56) hold, true. □

Example 2 Consider the polynomial matrix T(s) of the example 1. Then

$$\Pi_E = \begin{bmatrix} P_2 & P_3 \\ P_3 & 0 \end{bmatrix} = \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 1 & 1 \\ -1 & -1 & -1 & 0 & 0 & 0 \\ \hline 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \text{ and } \Pi_A = \begin{bmatrix} P_3 & 0 \\ 0 & 0 \end{bmatrix} = \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\Pi_B = \begin{bmatrix} P_2 \\ P_3 \end{bmatrix} = \left[\begin{array}{ccc} 1 & 1 & 0 \\ -1 & -1 & -1 \\ \hline 1 & 1 & 1 \\ 0 & 0 & 0 \end{array}\right] \text{ and } \Pi_C = \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 1 & 1 \\ -1 & -1 & -1 & 0 & 0 & 0 \end{array}\right] \tag{E.2.1}$$

We can see that $r := \text{rank}_{\mathbb{R}}\Pi_E = 2$ and so if we select two independent rows {1,2} and columns {3,4} of Π_E we obtain

$$T_E = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} ; \ T_A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} ; \ T_B = \begin{bmatrix} 1 & 1 & 0 \\ -1 & -1 & -1 \end{bmatrix} ; \ T_C = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \tag{E.2.2}$$

and so the singular pencil

$$T_F(s) = \begin{bmatrix} T_E - sT_A & -T_B s \\ T_C s & T_1 s + T_0 \end{bmatrix} = \begin{bmatrix} -s & 1 & -s & -s & 0 \\ -1 & 0 & s & s & s \\ 0 & s & 0 & -1 & -s \\ -s & 0 & -s & -s & 1 \end{bmatrix} \tag{E.2.3}$$

has the same algebraic structure as the one of the polynomial matrix $T(s)\in\mathbb{R}[s]^{2\times 3}$.

$$S^{\mathbb{C}}_{T_F(s)} = [\mathrm{diag}[1,1,1,(s+1)] \mid 0_{4\times 1}] \ ; \ S^{\infty}_{T_F(s)} = [\mathrm{diag}[s,s,s,\tfrac{1}{s}] \mid 0_{4\times 1}]$$

$$T_F(s)\,[-s,-s,1-s,0,s]^T = 0_{4\times 1} \tag{E.2.4}$$

We can find also according (3.42) that the singular pencil $T_F(s)$ and the polynomial matrix T(s) are related by the following transformation :

$$\begin{bmatrix} -s & 1 & -s & -s & 0 \\ -1 & 0 & s & s & s \\ 0 & s & 0 & -1 & -s \\ -s & 0 & -s & -s & 1 \end{bmatrix} = \left[\begin{array}{cc|c} \multicolumn{2}{c|}{I_2} & 0_2 \\ \hline s & -s^2 & \\ 0 & s & I_2 \end{array}\right] \left[\begin{array}{c|ccc} I_2 & & 0_2 & \\ \hline 0_2 & s^3+s^2 & s^3+s^2-1 & s^3-s \\ & -s^2-s & -s^2-s & -s^2+1 \end{array}\right] \left[\begin{array}{cc|ccc} -s & 1 & -s & -s & 0 \\ -1 & 0 & s & s & s \\ \hline \multicolumn{2}{c|}{0_{3\times 2}} & & I_3 & \end{array}\right] \tag{E.2.5}$$

We know also that there exist square constant and nonsingular matrices $M \in \mathbb{R}^{4\times 4}$ and $N \in \mathbb{R}^{5\times 5}$ such that

$$s\tilde{E} - \tilde{A} = M\,T_F(s)\,N \Leftrightarrow$$

$$\left[\begin{array}{c|cc|cc} s+1 & & & 0 & \\ \hline & 1 & -s & & \\ 0 & 0 & 1 & & \\ \hline & & & 1 & -s \end{array}\right] = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -s & 1 & -s & -s & 0 \\ -1 & 0 & s & s & s \\ 0 & s & 0 & -1 & -s \\ -s & 0 & -s & -s & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 1 & 1 & -1 & 1 & -1 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{bmatrix} \tag{E.2.6}$$

and so if we relate (E.2.5) and (E.2.6) we obtain that

$$\left[\begin{array}{c|cc|cc} s+1 & & & 0 & \\ \hline & 1 & -s & & \\ 0 & 0 & 1 & & \\ \hline & & & 1 & -s \end{array}\right] = \begin{bmatrix} 0 & s+1 & 0 & 1 \\ s & -s^2 & 1 & 0 \\ -1 & s & 0 & 1 \\ -1 & 0 & 0 & 0 \end{bmatrix} \left[\begin{array}{c|ccc} I_2 & & 0_2 & \\ \hline 0_2 & s^3+s^2 & s^3+s^2-1 & s^3-s \\ & -s^2-s & -s^2-s & -s^2+1 \end{array}\right] \begin{bmatrix} 0 & 0 & 0 & -1 & s \\ s+1 & 0 & -1 & 1 & -s \\ 1 & 1 & -1 & 1 & -1 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{bmatrix} \tag{E.2.7}$$

□

4. Conclusions.

Two equivalent singular pencils of a not necessary square polynomial matrix have been proposed as well as the transformations which relate these two singular pencils with the specific polynomial matrix has also been determined. These two singular pencils have been also presented by Verghese [8] and Bosgra and Van Der Weiden [9] but without mention of anything regarding the null space structure of these pencils as well as the transformations which relate these two singular pencils with the equivalent polynomial matrix. Based in these results an extension of the Kronecker canonical form has also be given for singular polynomial matrices. An extension of the present theory to polynomial system matrices of *singular* dynamical systems will also be given in the future by the authors.

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