Equivalence Transformations of Rational Matrices.

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Abstract.
The known theories of transformations between polynomial matrices are extended to the case of rational matrices. Specifically Ω-equivalence of rational matrices with possibly different dimensions is defined which has the property of preserving the zero structure of rational matrices in the region $\Omega \subseteq \mathbf{C} \cup \{\infty\}$.

Keywords: equivalence, transformation, rational matrices.

1. Introduction.

Some open questions surround the existence of transformations which preserve both the finite and infinite zero structure of polynomial matrices and more generally for rational matrices of different dimensions. Hayton et.al. (1988) proposed *full equivalence* (f.e.) for the special case of polynomial matrices, which includes both *extended unimodular equivalence* (e.u.e.) (Pugh & Shelton 1978) and *extended causal equivalence* (e.c.e.) (Walker 1988) within one single transformation. It thus preserves both the finite and infinite zero structure of polynomial matrices with different dimensions.

In this paper an extension in two separate senses of the above ideas is presented. The first extension is the development of a general equivalence transformation for rational matrices which are not necessarily of the same dimension. The second extension concerns the reference of the transformation to a specific region $\Omega \subseteq \mathbf{C} \cup \{\infty\}$ and the confinement of its invariants to the given region.

2. Structure of Rational Matrices in $\Omega \subseteq \mathbf{C}$.

Let $\Omega \subseteq \mathbf{C}$ be symmetrically located w.r.t. the real axis $\mathbf{R}$. Let $t(s) \in \mathbf{R}(s)$ be written as

$$t(s) = t_\Omega(s)\hat{t}(s) \tag{2.1}$$

where $t_\Omega(s) = n_\Omega(s)/d_\Omega(s)$ and $n_\Omega(s), d_\Omega(s) \in \mathbf{R}[s]$ are coprime with all their zeros within Ω and $\hat{t}(s) = \hat{n}(s)/\hat{d}(s)$ and $\hat{n}(s), \hat{d}(s) \in \mathbf{R}[s]$ are coprime with all their zeros outside Ω. Let $\delta_\Omega : \mathbf{R}(s) \to \mathbf{Z} \cup \{-\infty\}$ as

$$\delta_\Omega(t(s)) = \begin{cases} \deg n_\Omega(s) - \deg d_\Omega(s) & t_\Omega \neq 0 \\ -\infty & t_\Omega \equiv 0 \end{cases} \tag{2.2}$$

then $\delta_\Omega(\cdot)$ is a discrete valuation on $\mathbf{R}(s)$. The zeros of $n_\Omega(s)$ (resp. $d_\Omega(s)$) are termed the *zeros* (resp.*poles*) of $t(s)$ in Ω. Let $\mathbf{R}_\Omega(s)$ be the ring of rational functions with no poles in Ω. These will be called Ω *-polynomials* (Pernebo, 1981).

Lemma 1. Let $t_1(s)$, $t_2(s) \in \mathbf{R}(s)$, $t_2(s) \neq 0$ and $\delta_\Omega(t_1(s)) > \delta_\Omega(t_2(s))$. Then $\exists\ \hat{q}(s) \in \mathbf{R}_\Omega(s)$, $r(s) \in \mathbf{R}(s)$ such that:

$$t_1(s) = t_2(s)\hat{q}(s) + r(s) \tag{2.3}$$

where $\delta_\Omega(r(s)) < \delta_\Omega(t_2(s))$ or $r(s) = 0$. □

For $t(s)(\neq 0) \in \mathbf{R}_\Omega(s)$, $\delta_\Omega(t(s)) \geq 0$, and so $\delta_\Omega(t)$ serves as a *degree* on $\mathbf{R}_\Omega(s)$. Thus by Lemma 1 $\mathbf{R}_\Omega(s)$ is a Euclidean ring and hence a principal ideal domain.

$T(s) \in \mathbf{R}(s)^{p\times m}$ is said to be Ω*-polynomial* if $lim_{s\to s_0}T(s)$ exists $\forall s_0 \in \Omega$. The set of such matrices is denoted $\mathbf{R}_\Omega(s)^{p\times m}$. $T(s) \in \mathbf{R}_\Omega(s)^{p\times p}$ is said to be Ω*- unimodular* in case $\exists\ \hat{T}(s) \in \mathbf{R}_\Omega(s)^{p\times p}$ such that: $T(s)\hat{T}(s) = I$ or equivalently iff $lim_{s\to s_0}T(s) = T_0$, $\forall s_0 \in \Omega$, with $\det|T_0| \neq 0$. Obvious row/column operations correspond to pre/post-multiplication by an appropriate Ω-unimodular matrix.

Definition 1. $T_1(s), T_2(s) \in \mathbf{R}(s)^{p\times m}$ are said to be Ω *-unimodular equivalent* ($\Omega - u.e.$) in case $\exists$ Ω-unimodular matrices $U_L(s) \in \mathbf{R}_\Omega(s)^{p\times p}$ and $U_R(s) \in \mathbf{R}_\Omega(s)^{m\times m}$, such that:

$$U_L(s)T_1(s)U_R(s) = T_2(s) \tag{2.4}$$ □

Theorem 1. $T(s) \in \mathbf{R}(s)^{p\times m}$ is ($\Omega - u.e.$) to (Verghese, 1978; Vardulakis, 1991)

$$S^\Omega_{T(s)} = \text{blockdiag}\left[\frac{\epsilon_1}{\psi_1}, \ldots, \frac{\epsilon_r}{\psi_r}, \mathrm{O}\right] \tag{2.5}$$

where $\epsilon_i(s), \psi_i(s) \in \mathbf{R}[s]$ are monic, have no zeros outside Ω, are pairwise coprime and $\epsilon_i(s)\backslash\epsilon_{i+1}(s), \psi_{i+1}(s)\backslash\psi_i(s)$ for each $i = 1, 2, .., r-1$. $S^\Omega_{T(s)}(s)$ is the *Smith-McMillan form of* $T(s)$ *in* Ω and r is the *rank* of T(s). In (2.5) and the sequel 0 denotes a zero matrix of the obvious appropriate dimensions. □

$\epsilon_i(s)/\psi_i(s) := f_i(s)$, called the Ω*-invariant rational functions* of $T(s)$, constitute a complete set of invariants for ($\Omega - u.e.$) of rational matrices. Further the *zeros* (resp.*poles*) in Ω of $T(s)$ are the zeros of $\epsilon_i(s)$ (resp. $\psi_i(s)$), $i \in \mathbf{R}$.

Definition 2. $A(s) \in \mathbf{R}_\Omega(s)^{p\times p}$ and $B(s) \in \mathbf{R}_\Omega(s)^{p\times m}$ are said to be Ω*-left coprime* iff

$$\text{rank}_\mathbf{R}[A(s_0)\ B(s_0)] = p \quad \forall s_0 \in \Omega \tag{2.6}$$

For $\Omega \subseteq \mathbf{C}$ the following hold

Proposition 1. Let $T_1(s) \in \mathbf{R}_\Omega(s)^{p\times \ell}$, $T_2(s) \in \mathbf{R}_\Omega(s)^{p\times t}$, $\ell + t := m \geq p = rank[T_1(s)\ T_2(s)]$. Then the following statements are equivalent:

(1) $T_1(s)$ and $T_2(s)$ are Ω-left coprime.
(2) $T(s) = [T_1(s)\ T_2(s)]$ has no zeros in Ω.
(3) $\exists\ X(s) \in \mathbf{R}_\Omega(s)^{\ell\times p}$, $Y(s) \in \mathbf{R}_\Omega(s)^{t\times p}$ s.t.

$$T_1(s)X(s) + T_2(s)Y(s) = I_p \tag{2.7}$$ □

Lemma 2. Let $T(s) \in \mathbf{R}(s)^{p\times m}$ be of rank r. Then $\exists$ (non unique) Ω-left coprime matrices $A_1(s) \in \mathbf{R}_\Omega(s)^{p\times p}$, $B_1(s) \in \mathbf{R}_\Omega(s)^{p\times m}$ such that

$$T(s) = A_1^{-1}(s)B_1(s) \qquad (2.8)$$

Every other Ω-left coprime factorization $\tilde{A}_1^{-1}(s)\tilde{B}_1(s)$ is such that:

$$\tilde{A}_1(s) = U_L(s)A_1(s)\ ;\ \tilde{B}_1(s) = U_L(s)B_1(s) \qquad (2.9)$$

where $U_L(s) \in \mathbf{R}_\Omega(s)^{p\times p}$ is Ω-unimodular. □

(2.8) is called a *Ω-left coprime matrix fraction description* ($\Omega - MFD$) of $T(s)$. We call a $p\times m$ Ω-polynomial matrix such as $B_1(s)$ a *numerator* of $T(s)$ and a $p\times p$ Ω-polynomial matrix such as $A_1(s)$ a *denominator* of $T(s)$. In the usual way we have

$$\text{zeros in } \Omega \text{ of } T(s) \equiv \textit{zeros} \text{ in } \Omega \text{ of } B_1(s) \qquad (2.10)$$

$$\text{poles in } \Omega \text{ of } T(s) \equiv \textit{zeros} \text{ in } \Omega \text{ of } A_1(s) \qquad (2.11)$$

Definition 3. Let $A(s) \in \mathbf{R}(s)^{p\times m}$ and $B(s) \in \mathbf{R}(s)^{p\times \ell}$ and consider the left Ω-coprime MFD

$$[A(s)\ B(s)] = D_1^{-1}(s)[\bar{A}(s)\ \bar{B}(s)] \qquad (2.12)$$

$A(s)$, $B(s)$ are *Ω-left coprime* iff the Ω-polynomial matrices $\bar{A}(s), \bar{B}(s)$ are Ω-left coprime. □

Similar definitions to those above may be given in respect of *Ω-right coprimeness.*

Definition 4. If $\Omega \subseteq \mathbf{C}\cup\{\infty\}$ then the *Ω-least order* of $T(s)$, denoted $\nu_\Omega(T(s))$, is the number of poles of $T(s)$ occurring in Ω counted according to degree and multiplicity. □

3. Equivalence of Matrices in $\Omega \subseteq \mathbf{C}\cup\{\infty\}$.

It is of interest to know if $\exists$ transformations which preserve, in a given region, the zero structure of rational matrices with different dimensions. Initially we consider merely the case $\Omega \subseteq \mathbf{C}$ since there are some technical problems in treating the point at infinity and $\mathbf{C}$ together (Pernebo, 1981). In case we wish to consider $\Omega \subset \mathbf{C}\cup\{\infty\}$ a bilinear transformaion may be employed to reduce the problem to $\Omega \subseteq \mathbf{C}$. Let $P(p,m)$ and $P_\Omega(p,m)$ be respectively the sets of $(r+p)\times(r+m)$ of rational and Ω-polynomial matrices, where the integer $r \geq \max(-p,-m)$.

Definition 5. $P_1(s)$, $P_2(s) \in P(p,m)$ are said to be *Ω-equivalent* in case $\exists$ rational matrices $M(s)$, $N(s)$ such that

$$(M(s)\quad P_2(s))\begin{pmatrix} P_1(s) \\ -N(s)\end{pmatrix} = 0 \qquad (3.1)$$

where the elements of the compound matrices

$$(M(s)\ P_2(s));\ (P_1(s)^\top\ -N(s)^\top)^\top \qquad (3.2)$$

are Ω-left and Ω-right coprime respectively and sayisfy the following Ω-least order conditions

$$\nu_\Omega(M(s)\quad P_2(s)) = \nu_\Omega(P_2(s))$$
$$\nu_\Omega(P_1(s)^\top\ -N(s)^\top)^\top = \nu_\Omega(P_1(s)) \qquad \square\ (3.3)$$

The importance of Ω-equivalence is seen in

Theorem 2. Ω-equivalence on $P(p,m)$ preserves the zero structure in Ω.

Proof. Let $P_1(s)$, $P_2(s) \in P(p,m)$ and let

$$[M(s)\quad P_2(s)] = D_2^{-1}(s)[\tilde{M}(s)\quad \tilde{P}_2(s)]$$

$$\begin{bmatrix} P_1(s) \\ -N(s)\end{bmatrix} = \begin{bmatrix} \tilde{P}_1(s) \\ -\tilde{N}(s)\end{bmatrix} D_1^{-1}(s) \qquad (3.4)$$

be Ω-coprime MFDs. Then

$$\delta_\Omega(|D_2|) = \nu_\Omega[M\quad P_2] \overset{(3.3)}{=} \nu_\Omega(P_2) \qquad (3.5)$$

which means that the Ω-MFD

$$P_2(s) = D_2^{-1}(s)\tilde{P}_2(s) \qquad (3.6)$$

is Ω-left coprime and so the zero structure in Ω and the rank defect of $P_2(s)$ and $\tilde{P}_2(s)$ coincide. Similarly the zero structure in Ω and the rank defect of $P_1(s)$ and $\tilde{P}_1(s)$ coincide. Thus from (3.1) and (3.4)

$$[\tilde{M}(s)\quad \tilde{P}_2(s)]\begin{bmatrix} \tilde{P}_1(s) \\ -\tilde{N}(s)\end{bmatrix} = 0 \qquad (3.7)$$

where $[\tilde{M}(s)\ \tilde{P}_2(s)]$ and $[\tilde{P}_1(s)^\top - \tilde{N}(s)^\top]^\top$ have full rank $\forall s_0 \in \Omega$. From (2.7) it follows that $\exists$ Ω-polynomial matrices $A_1(s), A_2(s)$, $B_1(s)$, $B_2(s)$ such that

$$\tilde{M}(s)A_1(s) + \tilde{P}_2(s)B_1(s) = I_1 \qquad (3.8)$$
$$A_2(s)\tilde{N}(s) + B_2(s)\tilde{P}_1(s) = I_2 \qquad (3.9)$$

From (3.7), (3.8) and (3.9) we obtain

$$\begin{bmatrix} \tilde{P}_2 & \tilde{M} \\ -A_3 & B_3\end{bmatrix}\begin{bmatrix} B_1 & -\tilde{N} \\ A_1 & \tilde{P}_1\end{bmatrix} = \begin{bmatrix} I_1 & 0 \\ 0 & I_2\end{bmatrix} \qquad (3.10)$$

where

$$A_3 = (A_2B_1 - B_2A_1)\tilde{P}_2 - A_2$$
$$\text{and}\quad B_3 = (A_2B_1 - B_2A_1)\tilde{M} + B_2 \qquad (3.11)$$

From (3.10) the two Ω-polynomial matrices on the left hand side are Ω-unimodular. Thus

$$\begin{bmatrix} B_1 & -\tilde{N} \\ A_1 & \tilde{P}_1\end{bmatrix}\begin{bmatrix} \tilde{P}_2 & \tilde{M} \\ -A_3 & B_3\end{bmatrix} = \begin{bmatrix} I_3 & 0 \\ 0 & I_4\end{bmatrix} \qquad (3.12)$$

and it follows immediately that

$$\begin{bmatrix} B_1 & -\tilde{N} \\ A_1 & \tilde{P}_1\end{bmatrix}\begin{bmatrix} \tilde{P}_2 & 0 \\ -A_3 & I_2\end{bmatrix} = \begin{bmatrix} I_3 & -\tilde{N} \\ 0 & \tilde{P}_1\end{bmatrix} \qquad (3.13)$$

Thus

$$\begin{bmatrix} I_2 & 0 \\ 0 & \tilde{P}_2(s)\end{bmatrix} \quad\text{and}\quad \begin{bmatrix} I_3 & 0 \\ 0 & \tilde{P}_1(s)\end{bmatrix} \qquad (3.14)$$

are Ω-u.e. and so $\tilde{P}_1(s)$, $\tilde{P}_2(s)$ or equivalently $P_1(s)$, $P_2(s)$ have the same zero structure and rank defect in Ω. □

Ω-equivalence has the property of preserving the zero structure in Ω. The proof also indicates that the Ω-polynomial matrices $D_1(s)$ and $D_2(s)$ which define the pole structure in Ω of $P_1(s)$, $P_2(s)$ may be cancelled. Thus the pole structure in Ω is not invariant under Ω-equivalence.

Example 1. Let $P_1(s) = \frac{s+1}{s+2}$ and $P_2(s) = s+1$.

With $\Omega = C$ consider the transformation:

$$(s+2 \quad s+1)\begin{pmatrix} \frac{s+1}{s+2} \\ -1 \end{pmatrix} = 0 \qquad (3.15)$$

(3.15) is an Ω-equivalence transformation and so $P_1(s)$, $P_2(s)$ have the same zero structure in **C**. Note however that $P_1(s)$ has one pole at $s = -2$ while $P_2(s)$ has no poles in **C**. □

Theorem 3. If $P_1(s)$, $P_2(s) \in P(p,m)$ have the same zero structure in Ω and the same rank defect then they are Ω-equivalent.

Proof. Let $S^{\Omega}_{P_1(s)}$, $S^{\Omega}_{P_2(s)}$ be the Smith-McMillan forms in Ω of $P_1(s)$, $P_2(s)$. Then $\exists$ Ω-unimodular matrices $U_1(s)$, $U_2(s)$, $U_3(s)$, $U_4(s)$ such that:

$$U_1(s)P_1(s)U_2(s) = S^{\Omega}_{P_1(s)} = \text{blockdiag}\left[\frac{\epsilon_1(s)}{\psi_1(s)}, \ldots, \frac{\epsilon_r(s)}{\psi_r(s)}, 0\right] \qquad (3.16a)$$

$$U_3(s)P_2(s)U_4(s) = S^{\Omega}_{P_2(s)}(s) = \text{blockdiag}\left[\frac{1}{\hat{\psi}_1(s)}, \ldots, \frac{1}{\hat{\psi}_{k-1}(s)}, \frac{\epsilon_1(s)}{\hat{\psi}_k(s)}, \ldots, \frac{\epsilon_r(s)}{\hat{\psi}_{r+k-1}(s)}, 0\right] \qquad (3.16b)$$

where $S^{\Omega}_{P_1(s)}$, $S^{\Omega}_{P_2(s)}$ have the same zero structure and rank defect.

$$\left[\begin{array}{c|c} \Psi' & 0 \\ \hline 0 & I \end{array}\right] S^{\Omega}_{P_2(s)} = S^{\Omega}_{P_1(s)} \left[\begin{array}{c|c} \Psi & 0 \\ \hline 0 & I \end{array}\right] \qquad (3.17)$$

$$\Psi' = (0,\ \text{diag}[\hat{\psi}_k(s), \ldots, \hat{\psi}_{r+k-1}(s)])$$
$$\Psi = (0,\ \text{diag}[\psi_1(s), \ldots, \psi_r(s)])$$

is an Ω-equivalence transformation since the compound matrices (3.2) arising from (3.17) satisfy all the conditions of Definition 5. From (3.17)

$$\left[U_1^{-1}(s)\left[\begin{array}{c|c} \Psi' & 0 \\ \hline 0 & I \end{array}\right] U_3(s)\right] P_2(s) = P_1(s)\left[U_2(s)\left[\begin{array}{c|c} \Psi & 0 \\ \hline 0 & I \end{array}\right] U_4^{-1}(s)\right] \qquad (3.18)$$

(3.18) is still an Ω-equivalence transformation since multiplication by Ω-unimodular matrices does not alter the conditions on (3.17). □

From Theorems 2 and 3 we see that Ω-equivalence is a necessary and sufficient condition for two rational matrices to have the same zero structure and rank defect in Ω. Also note from Theorem 3 that

Corollary 1 The transforming matrices under Ω-equivalence may be taken to be Ω-polynomial. □

Theorem 4. Ω-equivalence is an equivalence relation on $P(p,m)$.

Proof.

(i) *Reflexivity property.* Clearly

$$I.P(s) = P(s).I \qquad (3.19)$$

is an Ω-equivalence transformation.

(ii) *Symmetry property.* Let $P_1(s)$, $P_2(s) \in P(p,m)$ be Ω-equivalent

i.e. $$M(s)P_1(s) = P_2(s)N(s) \qquad (3.20)$$

Then $P_1(s)$, $P_2(s)$ will have the same zero structure and rank defect in Ω by Theorem 2. Thus $\exists$ a transformation of the form (3.18) between $P_2(s)$ and $P_1(s)$, which proves the symmetry property.

(iii) *Transitivity property.* Let $P_1(s)$, $P_2(s) \in P(p,m)$ and $P_2(s)$, $P_3(s) \in P(p,m)$ be Ω-equivalent respectively. Then $P_1(s)$, $P_2(s)$ and $P_2(s)$, $P_3(s)$ have the same zero structure and rank defect in Ω. Thus $P_1(s)$, $P_3(s)$ have the same zero structure and rank defect in Ω and so by Theorem 3, $P_1(s)$, $P_3(s)$ are Ω-equivalent. □

Example 2 In the special case $\Omega \equiv C$ and $P_1(s)$, $P_2(s)$ are polynomial matrices then Ω-equivalence coincides with (e.u.e.), (Pugh & Shelton, 19[illegible]). This is because the Ω-least order conditions require the transforming matrices to be polynomial, while the conditions that the compound matrices of (3.2) possess no zeros in Ω reduce to the usual relative primeness conditions of (e.u.e.). □

Example 3 In special case that $\Omega \equiv s_0$ where $s_0 \in C$, and $P_1(s)$, $P_2(s)$ are polynomial matrices then Ω-equivalence reduces to *local equivalence* (Cullen, 1987). If $\Omega = \{\infty\}$, then from Corollary 1 the transforming matrices of Ω-equivalence may be taken to be proper rational matrices. If in addition $P_1(s), P_2(s)$ are polynomial then the Ω-least order conditions are redundant and Ω-equivalence reduces immediately to bicausal equivalence (Vardulakis, 1991) if $P_1(s), P_2(s)$ are of the same dimensions, and to (e.c.e.) (Walker, 1988) otherwise. □

An interesting question still remains as to what transformations will preserve the zero structure of a rational matrix $P(s)$ in the region $\Omega = C \cup \{\infty\}$. The following result shows that Ω-equivalence of Definition 5 with $\Omega = C \cup \{\infty\}$ under the same conditions still provides an answer.

Theorem 5. If $P_1(s), P_2(s) \in P(p,m)$ are $C\cup\{\infty\}$-equivalent then they have the same finite and infinite zero structure.

Proof. Let $\mathcal{A}$ denote the set of all locations of poles and zeros of $P_1(s)$, $P_2(s)$ in $C \cup \{\infty\}$. Since $P_1(s)$, $P_2(s)$ are rational matrices over **R** it follows that $\mathcal{A}$ is finite and symmetric w.r.t. the real axis, and hence that $\mathcal{A} \subset C \cup \{\infty\}$. Thus $\exists$ a real number $\alpha \notin \mathcal{A}$. In particular $P_1(s)$, $P_2(s)$ are $C \cup \{\infty\} \setminus \{\alpha\}$-equivalent since they are $C\cup\{\infty\}$-equivalent. Thus by Theorem 2 $P_1(s)$, $P_2(s)$ have identical zero structure in $C \cup \{\infty\} \setminus \{\alpha\}$. By definition of α $P_1(s)$, $P_2(s)$ have no zeros at α which completes the proof. □

Corollary 2 In the case that $P_1(s)$, $P_2(s)$ are polynomial matrices, then $C \cup \{\infty\}$-equivalence coincides with (f.e.) (Hayton et.al. 1988).

Proof. Note that if $P_1(s)$, $P_2(s)$ are polynomial then the $C \cup \{\infty\}$ conditions (3.3) imply that the transforming matrices can have no finite poles and are therefore polynomial. These particular conditions then reduce immediately to the McMillan degree conditions of (f.e.). The other conditions of $C \cup \{\infty\}$-equivalence coincide directly with those of (f.e.), as required. □

Example 4. Let $G(s) \in R(s)^{p \times m}$ be the transfer function matrix of an open-loop system and let $G_F(s)$ be the transfer function matrix of the closed loop system under constant output feedback of the form $u(t) = -Fy(t) + v(t)$

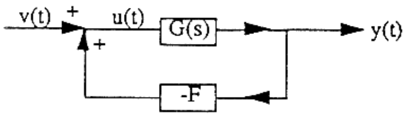

Diagram 1

It is well known that the finite and infinite zero structure of $G(s)$ and $G_F(s)$ is the same, but what is interesting here is the explanation which can be given for this by the notion of Ω-equivalence. Recall that the transfer function matrix of the closed loop system is

$$G_F(s) = G(s)(I + FG(s))^{-1} = (I + G(s)F)^{-1}G(s) \tag{3.21}$$

Consider therefore the transformation

$$I \quad G(s) = G_F(s) \quad (I + FG(s)) \tag{3.22}$$

and the compound matrices

$$[I \quad G_F(s)] \quad ; \quad \begin{pmatrix} G(s) \\ -I - FG(s) \end{pmatrix} \tag{3.23}$$

We will show that (3.22) is a $C \cup \{\infty\}$-equivalence transformation. Consider therefore the left coprime MFD

$$G_F(s) = D_F^{-1}(s)N_F(s) \tag{3.24}$$

Then

$$[I \quad G_F(s)] = D_F^{-1}(s)[D_F(s) \quad N_F(s)] \tag{3.25}$$

is clearly a left coprime MFD. Thus the finite zero structure of [I $G_F(s)$] coincides with that of $[D_F(s)\ N_F(s)]$ and so it has no finite zeros. In a similar way [I $G_F(1/w)$] can be seen to have no zeros at $w = 0$ and so [I $G_F(s)$] has no infinite zeros. Obviously

$$\nu_{C\cup\{\infty\}}(I \quad G_F(s)) = \nu_{C\cup\{\infty\}}(G_F(s)) \tag{3.26}$$

Thus [I $G_F(s)$] satisfies the conditions of $C \cup \{\infty\}$-equivalence. For the other compound matrix in (3.23) note that

$$\begin{bmatrix} G(s) \\ -I \end{bmatrix} = \begin{bmatrix} I & 0 \\ F & I \end{bmatrix} \begin{bmatrix} G(s) \\ -I - FG(s) \end{bmatrix} \tag{3.27}$$

is a strict equivalence transformation which preserves the pole and zero structure in $C \cup \{\infty\}$. Thus the second compound matrix in (3.23) satisfies the conditions of $C \cup \{\infty\}$-equivalence, since $[G(s)^T \quad -I]^T$ satisfies these conditions. Therefore (3.22) is a transformation of $C \cup \{\infty\}$-equivalence and so $G(s)$ and $G_F(s)$ have the same finite and infinite zero structure. □

4. Conclusions.

A transformation, called Ω-equivalence, between rational matrices of different dimensions has been defined. It is shown that Ω-equivalence preserves the zero structure of the given matrices, within the region $\Omega \subseteq C \cup \{\infty\}$, and defines an equivalence relation on the set $P(p,m)$ in case where $\Omega \subset C \cup \{\infty\}$. It is observed that Ω-equivalence is an generalisation of many known transformations, for example (e.u.e.), (e.c.e.), local equivalence and (f.e.). The notion of Ω-equivalence provides an interesting explanation of the well known invariance of the finite and infinite zero structure of a given rational transfer function matrix under constant output feedback.

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