On a Fundamental Notion of Equivalence in Linear Systems Theory

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Abstract. A fundamental form of equivalence between two homogeneous systems of ordinary differential equations is defined in terms of isomorphisms between the "finite" and "infinite" solution sets. This equivalence is an alternative characterization of the transformation of full equivalence in the case of nonsingular matrices.

Keywords. Full equivalence, homogeneous systems, isomorphism, ordinary differential equations.

1. Introduction.

A number of authors (*e.g.*[4]) have considered the action of a linear mapping on the solution space of a given linear system of mixed algebraic and ordinary differential equations. This action has only partially been described, mainly as a result focusing attention solely on the finite frequency aspects of the system's behaviour. Even when the impulsive behaviour has been taken into account [7,11] a full account has not been given because certain basic questions concerning the mapping have been neglected.

The first part of this paper will therefore present a complete analysis of the action of such a mapping, both with regard to the interpretation of its injectiveness in terms of the finite and infinite frequency behaviour of the system and with the regard to the more fundamental question of whether the mapping is properly constituted in the formal sense, or merely a relation. The condition that the solution map be well constituted offers a simple explanation of why, for example, any map between the solution set of any general linear system of ordinary differential equations and the solution set of a system in state space form can contain no derivatives and is therefore constant [12].

A notion of equivalence can be assigned to two general linear systems by the requirement that the mapping relating their solution spaces is bijective. If only the finite frequency behaviour is of interest, the defined equivalence has a matrix representation which is well known to be an extended version of the usual equivalence by unimodular matrices [8]. When the infinite behaviour is additionally of interest then the equivalence so induced will be shown to be that defined in [2].

2. Preliminary Results.

Consider the set $P(p,m)$ of $(r+p)\times(r+m)$ polynomial matrices where the integer $r \geq \max(-p,-m)$. A matrix transformation with many system theory implications is the following [8]

Definition 1. $T_1(s)$, $T_2(s) \in P(p,m)$ are said to be ***extended unimodular equivalent*** (*e.u.e.*) in case $\exists$ polynomial matrices $M(s)$, $N(s)$ of appropriate dimensions such that :

$$M(s)T_1(s) = T_2(s)N(s) \qquad (1)$$

where $M(s)$, $T_2(s)$ (resp. $T_1(s)$, $N(s)$) are relative left (resp. right) prime. □

An important property of (*e.u.e.*) is that it is an equivalence relation on $P(p,m)$ which leave invariant the finite zero structure. An interesting interpretation of (*e.u.e.*) in terms of maps is the following [4]:

Lemma 1. Consider two linear homogeneous systems of ordinary differential equations:

$$T_i(\rho)\xi_i(t) = 0 \quad i = 1,2 \quad (\rho = d/dt) \qquad (2)$$

where $T_i(\rho)$ $i = 1,2$ are nonsingular polynomial matrices. Then $T_1(s)$, $T_2(s)$ are (*e.u.e.*) iff $\exists$ an isomorphism between the "finite" homogeneous solution spaces of (2) of the form :

$$\xi_2(t) = N(\rho)\xi_1(t) \qquad (3)\ \square$$

In fact (*e.u.e.*) represents a complete description of the relationship which holds between any two polynomial matrices from $P(p,m)$ whose finite zero structures are identical. If both the finite and infinite zero structure of a polynomial matrix are to be preserved, then (*e.u.e.*) needs to be further constrained. In this respect [2] proposed:

Definition 2. $T_1(s), T_2(s) \in P(p,m)$ are said to be ***fully equivalent*** (*f.e.*) in case $\exists$ polynomial matrices $M(s)$, $N(s)$ of appropriate dimensions such that :

$$[M(s)\ T_2(s)]\begin{bmatrix} T_1(s) \\ -N(s)\end{bmatrix} = 0 \qquad (4)$$

where the compound matrices

$$[M(s)\ T_2(s)]\ ;\ [T_1(s)^{\top} - N(s)^{\top}]^{\top} \qquad (5)$$

are such that

(a) they have full normal rank ;

(b) they have no finite nor infinite zeros ;

(c) the following McMillan degree conditions holds

$$\delta_M([M(s)\ T_2(s)]) = \delta_M(T_2(s))$$
$$\delta_M[T_1(s)^{\top} - N(s)^{\top}]^{\top} = \delta_M(T_1(s)) \qquad (6)\ \square$$

One notable particular form of (*f.e.*) is obtained when $T_1(s)$ and $T_2(s)$ are pencils and $M(s), N(s)$

are constant matrices. This special form of $(f.e.)$ is termed [6] *complete equivalence* $(c.e.)$. Some properties of $(f.e.)$ are that it keeps invariant the finite and infinite zero structure and that it is an equivalence relation on $P^r(m,m)$, *i.e.* the class of square matrices with full normal rank. An interpretation of $(c.e.)$ in terms of maps follows from [1] and is:

Lemma 2. Consider two linear homogeneous systems of algebraic and differential equations:

$$(\rho E_i - A_i)\xi_i(t) = 0 \quad i = 1,2 \qquad (7)$$

where $\rho E_i - A_i$ $i = 1,2$ are nonsingular. Then $\rho E_1 - A_1$ and $\rho E_2 - A_2$ are $(c.e.)$ iff $\exists$ an isomorphism between the "finite" and the "infinite" homogeneous solution spaces of (7) of the form:

$$\xi_2(t) = N\,\xi_1(t) \qquad (8)\ \square$$

An extension of this form of equivalence (and so of Lemma 1) to the case of general systems of the form (2) will be given here.

3. Mappings of Solution Sets.

Consider a system of the form (2):

$$T(\rho)\xi(t) = 0 \qquad (9)$$

where $T(\rho) = T_q\rho^q + \cdot + T_1\rho + T_0$ is nonsingular. Define $\mathcal{X}$ as the set of all solutions of (9) corresponding to all possible initial conditions on $\xi(t)$ and its first $q-1$ derivatives. Suppose $\mathcal{X}$ is mapped onto $\mathcal{X}_1$ which is defined as the set of all functions $\xi_1(t)$ forming the range of the relation :

$$\xi_1(t) = M(\rho)\xi(t) \qquad (10)$$

where $M(\rho) = M_q\rho^q + \cdots + M_1\rho + M_0$ and one of T_q, M_q is nonzero. The following example shows that relation (10) is not always a map in the formal sense.

Example 1. Consider the homogeneous system

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \rho \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \xi_1(t) \\ \xi_2(t) \\ \xi_3(t) \end{bmatrix} = 0 \qquad (E.1)$$

with solution [9], $\xi_1(t) = \xi_3(t) = 0$ and $\xi_2(t) = \xi_3(0-)\delta(t)$ and consider the relation

$$y(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} L_1\rho + L_0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \xi_1(t) \\ \xi_2(t) \\ \xi_3(t) \end{bmatrix} \qquad (E.2)$$

from which we obtain

$$y(t) = \begin{bmatrix} -L_1\xi_1(0-)\delta(t) \\ \xi_3(0-)\delta(t) \end{bmatrix} \qquad (E.3)$$

(E.2) is not formally a map because for any one solution $\xi(t) = [\xi_1(t)\ \xi_2(t)\ \xi_3(t)]^\top$ of (E.1) we obtain many images $y(t)$ which will depend on the initial condition $\xi_1(0-)$. $\square$

The first question regarding (10) is therefore if it represents a mapping of $\mathcal{X}$ to $\mathcal{X}_1$ in the formal sense, or whether it is merely a relation.

Laplace transforming (9) and (10) with initial conditions $\xi(0-), \xi^{(1)}(0-), \ldots, \xi^{(q-1)}(0-)$ gives [5]

$$T(s)\hat{\xi}(s) = \hat{a}_T(s) \qquad (11)$$

$$\hat{\xi}_1(s) = M(s)\hat{\xi}(s) - \hat{a}_M(s) \qquad (12)$$

where $\hat{a}_T(s)$, $\hat{a}_M(s)$ are given by

$$\hat{a}_F(s) = S_{q-1}X_F\bar{\xi}(0-) \quad F = T, M \qquad (13)$$

where

$$S_{q-1} = \begin{bmatrix} s^{q-1}I \\ s^{q-2}I \\ \vdots \\ I \end{bmatrix}^T ; \quad X_F = \begin{bmatrix} F_q & 0 & \ldots & 0 \\ F_{q-1} & F_q & \ldots & 0 \\ \vdots & & \ddots & \vdots \\ F_1 & F_2 & \ldots & F_q \end{bmatrix}$$

$$\bar{\xi}(0-) = [\xi(0-)^\top, .., \xi^{(q-1)}(0-)^\top]^\top \qquad (14)$$

Theorem 1. The relation (10) is a mapping in the formal sense iff

$$\delta_M[T(s)^\top\ M(s)^\top]^\top = \delta_M(T(s)) \qquad (15)$$

Proof. The relation (10) is a mapping in the formal sense iff it uniquely specifies an image, $\xi_1(t)$, for each solution $\xi(t)$ of (9). Accordingly the relation (10) is a mapping iff for each $\hat{\xi}(s)$ determined by (11), the relation (12) determines $\hat{\xi}_1(s)$ uniquely. It is apparent that if a given $\hat{\xi}(s)$ has two images $\hat{\xi}_1(s)$ and $\hat{\xi}_1'(s)$ under (12) then this is entirely due to the $\hat{a}_M(s)$ vector or more particularly the associated initial condition vector $\bar{\xi}(0-)$ of (13). Let therefore $\bar{\xi}(0-) \neq \bar{\xi}'(0-)$ be two initial condition vectors. It then follows that these determine the same solution $\xi(t)$ provided that $\bar{\xi}(0-) - \bar{\xi}'(0-) \in ker\ X_T$. Given that this condition is satisfied, it then follows that $\hat{\xi}_1(s)$ is determined via (12) as the unique image of $\hat{\xi}(s)$ provided $\bar{\xi}(0-) - \bar{\xi}'(0-) \in ker\ X_M$. Hence (10) is a map in the formal sense iff

$$kerX_T \subseteq kerX_M \Leftrightarrow ker[X_T^\top\ X_M^\top]^\top = kerX_T \qquad (16)$$

From the dimension theorem of linear mappings, (16) holds iff

$$\text{Rank}_{\mathbb{R}}[X_T^\top\ X_M^\top]^\top = \text{Rank}_{\mathbb{R}}X_T^\top$$

$$\overset{[5]}{\Leftrightarrow} \delta_M[T(s)^\top\ M(s)^\top]^\top = \delta_M(T(s)) \qquad (17)$$

Remark 1. It can be seen directly from (E.3) that (E.2) is a map iff $L_1 = 0$. It is also clear that $\delta_M([T(s)^\top M(s)^\top]^\top) \geq \delta_M(T(s))$ with equality holding iff $L_1 = 0$, which verifies the theorem. $\square$

The existence condition for the map (10) to be properly constituted is thus revealed as the McMillan degree condition (6). Specifically this condition indicates that the polynomial part of $\hat{\xi}_1(s)$ is uniquely specified by $\xi(s)$ and so the map is well formed. The polynomial part of $\hat{\xi}_1(s)$ in (12), and thus the "infinite" part of $\xi_1(t)$ in (10), is not considered in [4] and so the conditions (6) play no role in $(e.u.e.)$.

Theorem 1 explains why any map between the solution set of a general system (9) and the solution set of a homogeneous system in state space form can contain no derivatives [12]:

Theorem 2. Consider the following systems:

$$S_1 : T(\rho)\xi(t) = 0 \; ; \; S_2 : E\dot{x}(t) = Ax(t) \qquad (18)$$

Suppose $\exists$ a map

$$\xi(t) = M(\rho)x(t) \qquad (19)$$

where $M(\rho) = M_q\rho^q + \cdots + M_1\rho + M_0$. Then $M(\rho)$ is constant.

Proof. A necessary and sufficient condition for relation (19) to be a map in the formal sense is according to Theorem 1 that :

$$\delta_M\,[(\rho E - A)^\top \;\; M(\rho)^\top]^\top = \delta_M(\rho E - A) \overset{[5]}{\Leftrightarrow}$$

$$\mathrm{Rank}_{\mathbb{R}}\,[E^\top \quad X_M^\top]^\top = \mathrm{Rank}_{\mathbb{R}}[E] \Leftrightarrow$$

$$M_1 = HE \; ; \; M_i = 0 \quad i = 2, \ldots, q \qquad (20)$$

for some constant matrix H, and so

$$\xi(t) = (M_0 + HE\rho)x(t) \overset{(18)}{=} (M_0 + HA)x(t) \quad \square \; (21)$$

By definition of $\mathcal{X}_1$ it follows that (10) is an onto mapping of $\mathcal{X}$ onto $\mathcal{X}_1$ whenever it is properly constituted *i.e.* whenever it satisfies (15). It is now appropriate to consider when (10) might be injective and consequently invertible.

Theorem 3. The relation (10) is an injective mapping iff (15) holds and the polynomial matrices $M(s)$, $T(s)$ have no finite nor infinite zeros in common.

Proof. Note that (15) is a necessary and sufficient condition for the relation (10) to be a mapping. Note also from Lemma 1 that $\exists$ an injective mapping between the "finite" parts of $\xi(t)$ and $\xi_1(t)$ iff $M(s)$, $T(s)$ have no finite zeros in common. It remains to show that $\exists$ an injective mapping between the polynomial parts of $\hat{\xi}(s)$ and $\hat{\xi}_1(s)$ in (12) iff $T(s)$, $M(s)$ have no infinite zeros in common.

Assume $\exists$ two solutions $\hat{\xi}(t), \hat{\xi}'(t) \in \mathcal{X}$ with corresponding initial condition vectors $\bar{\xi}(0-)$, $\bar{\xi}'(0-)$ and polynomial vectors $\hat{a}_T(s), \hat{a}'_T(s)$, $\hat{a}_M(s)$, $\hat{a}'_M(s)$, which map onto $\hat{\xi}_1(t) \in \mathcal{X}_1$. Then from (11),(12)

$$T(s)[\hat{\xi}(s) - \hat{\xi}'(s)] = \hat{a}_T(s) - \hat{a}'_T(s) \qquad (22)$$

$$0 = M(s)[\hat{\xi}(s) - \hat{\xi}'(s)] - [\hat{a}_M(s) - \hat{a}'_M(s)] \qquad (23)$$

A necessary and sufficient condition for $T(s)$, $M(s)$ to have no common infinite zeros [10] is that $\exists$ proper rational matrices $R_i(s) = R_{0i} + R_{1i}(1/s) + \ldots$, $i = 1, 2$ such that

$$R_1(s)T(s) + R_2(s)M(s) = I \qquad (24)$$

Substituting $T(s)$ and $M(s)$ into (24) and equating positive powers of s we obtain :

$$\mathcal{R}_1 X_T + \mathcal{R}_2 X_M = 0$$

$$\mathcal{R}_i = \begin{bmatrix} R_{0i} & 0 & \ldots & 0 \\ R_{1i} & R_{0i} & \ldots & 0 \\ \vdots & & \ddots & \vdots \\ R_{q-1,i} & R_{q-2,i} & \ldots & R_{0i} \end{bmatrix} \qquad (25)$$

If $pol(\cdot)$ denotes the polynomial part of the indicated matrix then

$$pol(R_i(s)S_{q-1}) = S_{q-1}\mathcal{R}_i \quad i = 1, 2 \qquad (26)$$

Premultiplying (25) by S_{q-1} and postmultiplying by $(\bar{\xi}(0-) - \bar{\xi}'(0-))$ yields, by virtue of (26), that

$$pol\,(R_1(s)\{\hat{a}_T(s) - \hat{a}'_T(s)\} + R_2(s)\{\hat{a}_M(s) - \hat{a}'_M(s)\}) = 0 \qquad (27)$$

Postmultiplying (24) by $\hat{\xi}(s) - \hat{\xi}'(s)$ yields

$$\hat{\xi}(s) - \hat{\xi}'(s) = R_1(s)\{\hat{a}_T(s) - \hat{a}'_T(s)\} + R_2(s)\{\hat{a}_M(s) - \hat{a}'_M(s)\} \qquad (28)$$

Now (27) indicates that the polynomial part of the right half side of (28) is zero and so

$$\mathrm{pol}\,(\hat{\xi}(s) - \hat{\xi}'(s)) = 0 \qquad (29)$$

Thus (29) is the necessary condition which any two solutions $\hat{\xi}(s), \hat{\xi}'(s)$ of (11) must satisfy if they are to map onto the same element $\hat{\xi}_1(s)$ under (12) when $T(s)$, $M(s)$ have no common infinite zeros.

For the converse of the theorem suppose that $[T(\rho)^\top, \; M(\rho)^\top]^\top$ has infinite zeros then according to [10] $\exists$ a unimodular matrix $Y(\rho)$ such that

$$[T(\rho)^\top \; M(\rho)^\top]^\top = [T_1(\rho)^\top \; M_1(\rho)^\top]^\top Y(\rho) \quad (30)$$

where $[T_1(\rho)^\top, M_1(\rho)^\top]^\top$ has no infinite zeros and $Y(\rho)$ is the *g.c.d.* of $[T(\rho)^\top, M(\rho)^\top]^\top$ as concerns the infinite zero structure. If $z_2(t)$ is an impulsive solution [9], of $Y(\rho)z(t) = 0$ and $z_1(t)$ is an impulsive solution of (9), then both $z_1(t)$ and $\xi(t) = z_1(t) + z_2(t)$ are mapped onto the same solution $\xi_1(t)$. Thus (10) is not injective, which is a contradiction. $\square$

Corollary 1. The absence of finite zeros in the polynomial matrix $[T(\rho)^\top, M(\rho)^\top]^\top$ is a necessary and sufficient condition for the "finite" part of the inverse of (10) to be uniquely determined. The absence of infinite zeros in $[T(\rho)^\top, M(\rho)^\top]^\top$ together with the condition (15) is a necessary and sufficient condition for the "impulsive" part of the inverse of (10) to be uniquely determined. $\square$

4. Fundamental Equivalence.

If $T(\rho)$, $M(\rho)$ satisfy the conditions of (*f.e.*) then (10) is a particularly interesting map. In this case the set of solutions $\mathcal{X}$ of (9) ([10], [9]) forms a vector space with dimension equal to the generalized order $f := \delta_M(T(s))$ of the system, which is preserved by the vector space isomorphism (10).

Suppose that $\mathcal{X}_1$ is the solution set of a second system of the form (9). It is then reasonable to call these systems "equivalent". It thus follows:

Definition 3. Consider two systems S_1 and S_2 in the form (2). S_1, S_2 are said to be *fundamentally equivalent* in case $\exists$ an isomorphism (3) of their "finite" and "infinite" homogeneous solution vector spaces. $\square$

It will be shown that Definition 3 is the extension of the definition of equivalence given in Lemma 1. Note that definition 3 additionally refers to the impulsive solution set [9], of the systems (2).

Theorem 4. Let S_1, S_2 be as in (2). $T_1(\rho)$, $T_2(\rho)$ are $(f.e.)$ iff S_1 and S_2 are fundamentally equivalent.
Proof. Let $T_1(\rho)$, $T_2(\rho)$ be $(f.e.)$ then by Definition 2, $\exists$ polynomial matrices $M(\rho)$, $N(\rho)$ of appropriate dimensions such that

$$M(\rho)T_1(\rho) = T_2(\rho)N(\rho) \tag{31}$$

Postmultiplying both sides of (31) with $\xi_1(t)$ gives

$$T_2(\rho)[N(\rho)\xi_1(t)] = 0 \tag{32}$$

and so the relation

$$\xi_2(t) = N(\rho)\xi_1(t) \tag{33}$$

gives a solution of the homogeneous system S_2. From the $(f.e.)$ conditions governing (31)

$$\delta_M[T_1(\rho)^\top N(\rho)^\top]^\top = \delta_M(T_1(\rho)) \tag{34}$$

which by Theorem 1 implies that (33) is a map. That $[T_1(\rho)^\top N(\rho)^\top]^\top$ has no finite nor infinite zeros implies by Theorem 3 that (33) is a monomorphism between the solution spaces $\mathcal{X}_1$, $\mathcal{X}_2$ of S_1 and S_2 respectively. In the same way under the symmetry property of $(f.e.)$ $\exists$ polynomial matrices $M'(\rho)$, $N'(\rho)$ such that

$$M'(\rho)T_2(\rho) = T_1(\rho)N'(\rho) \tag{35}$$

Hence the relation

$$\xi_1(t) = N'(\rho)\xi_2(t) \tag{36}$$

is also a monomorphism between $\mathcal{X}_2$, $\mathcal{X}_1$ and so the relations (33) and (36) are bijective maps.

Conversely assume that $\exists$ a bijection

$$\xi_2(t) = N(\rho)\xi_1(t) \tag{37}$$

between the pseudostates $\xi_i(t)$ $i = 1, 2$ of S_1 and S_2 of (2). Let $\rho E_1 - A_1$ and $\rho E_2 - A_2$ be two $(f.e.)$ regular pencils of $T_1(\rho)$ and $T_2(\rho)$ respectively [3]. $\exists$ matrices $M(\rho), N$ and $M', N'(\rho)$ such that

$$M(\rho)(\rho E_1 - A_1) = T_1(\rho)N \tag{38}$$

$$M'\ T_2(\rho) = (\rho E_2 - A_2)N'(\rho) \tag{39}$$

and so $\exists$ bijective maps

$$x_2(t) = N'(\rho)\xi_2(t)\ ;\ \xi_1(t) = Nx_1(t) \tag{40}$$

where $x_i(t)$ are the pseudostates of the systems corresponding to the pencils $\rho E_i - A_i$ $i = 1, 2$. Hence from (37) and (40) $\exists$ a bijective map

$$x_2(t) = N'(\rho)N(\rho)Nx_1(t) \tag{41}$$

where the matrix $N'(\rho)N(\rho)N$ must be constant by Theorem 2, and so from Lemma 2 $\rho E_i - A_i$ $i = 1, 2$ are $(c.e.)$ or equivalently $(f.e.)$. The following relations hold

$$T_1(\rho) \overset{f.e.}{\sim} \rho E_1 - A_1 \overset{f.e.}{\sim} \rho E_2 - A_2 \overset{f.e.}{\sim} T_2(\rho) \tag{42}$$

Hence from the transitivity property of $(f.e.)$ $T_1(s)$ and $T_2(s)$ are $(f.e.)$. □

Fundamental equivalence and $(f.e.)$ thus define identical equivalence classes on $P'(m, m)$. Consequently fundamental equivalence leaves invariant the finite and infinite zero structures of $T_1(s)$ and $T_2(s)$, which determine the "finite" and "infinite" solution spaces of S_1, S_2 [9].

5. Conclusions.

A neat characterization of the transformation of full equivalence has been given in terms of the existence of a bijective map between the finite and infinite solution sets of the homogeneous systems constructed by the $(f.e.)$ polynomial matrices. This characterization has enabled the true nature and role of the conditions of full equivalence in case where square and nonsingular polynomial matrices are involved. Thus for example the McMillan degree conditions (6), which have previously appeared somewhat arbitrarily attached to the transformation, are seen to be vital in a quite fundamental way. It is concluded therefore that the transformation of full equivalence, with its various characterizations, is the basic transformational tool for the simultaneous study of the finite and infinite frequency behaviour of general autonomous systems.

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