On the Solution Space of Singular State–Space AR Representations.

by
N.P.Karampetakis and A.I.G.Vardulakis
Department of Mathematics
Aristotle University of Thessaloniki
Thessaloniki 54006, Greece.

Abstract.

The finite and infinite solution space of singular homogeneous state–space autoregressive representations is determined in relation to the structural invariants of the singular pencil which describe the autoregressive representation. A closed formula for the homogeneous solution of singular homogeneous state–space systems in Kronecker form is also proposed.

Keywords : singular pencils, structural invariants, homogeneous solutions.

1. Introduction.

Consider a system of linear homogeneous differential and algebraic equations described in matrix form by :

$$E\,\dot{x}(t) = A\,x(t) \tag{1.1}$$

where E, A $\in \mathbb{R}^{p\times m}$, $\mathrm{rank}_{\mathbb{R}(s)}(sE-A) = r \le \max(p,m)$ and $x(t) : [0-,\infty) \longrightarrow \mathbb{R}^m$ is an m–valued function called "*external variable*" of the system. Following the terminology of Willems (1986) we call the set of equations (1.1) an AR representation (*AutoRegressive representation*) of $\mathcal{B}$ (*behaviour*) where $\mathcal{B}$ is the solution set of equations (1.1). In system theory we needs sometimes descriptions where there is no distinction between inputs and outputs i.e. interconnection of systems. In such cases the model (1.1) is very useful (Blomberg 1983, Willems 1986,1991, Kuijper 1992). In case where sE–A is a *regular* matrix i.e. E, A $\in \mathbb{R}^{n\times n}$ and $\det|sE-A|\neq 0$, the above model has been studied by a number of authors (Rosenbrock 1970, Kailath 1980, Verghese 1978). It was shown that the model (1.1) exhibits smooth and impulsive behaviour which is due to the finite and infinite elementary divisors of sE–A. In case now where sE–A is *singular* i.e. 1) E, A $\in \mathbb{R}^{n\times n}$ and $\det|sE-A|=0$ or 2) E, A $\in \mathbb{R}^{p\times m}$, the system (1.1) the system may have an arbitrary number of solutions or may have no solution. This strange behaviour as we shall see in the sequel is due to the new structural invariants of the pencil sE–A, the left and right kernel of sE–A. More specifically we shall show that the finite and infinite elementary divisors and the right kernel of sE–A produce independent solutions of (1.1) whereas the left kernel produce conditions between the initial conditions Ex(0–) which must be satisfied so that the system (1.1) has a solution. A quite interesting result is that the solution vector space of (1.1) is consisted by equivalence classes in contrast to the solution vector space of regular systems which is consisted by specific vector valued functions. A closed formula for the homogeneous solution of the system (1.1) is also proposed in case where sE–A is in Kronecker canonical form.

2. Solution Vector Space of Singular State–Space AR Representations.

Consider the AR representation (1.1). We introduce a new m–vector valued function z(t) such that

$$x(t) = Q\,z(t) \qquad Q\in\mathbb{R}^{m\times m},\ |Q|\neq 0$$

Substituting Qz(t) for x(t) in (1.1) and multiplying on the left by $P\in\mathbb{R}^{p\times p}$ we obtain

$$\tilde{E}\,\dot{z}(t) = \tilde{A}\,z(t) \tag{2.1}$$

where

$$\tilde{E} = P\,E\,Q \quad \text{and} \quad \tilde{A} = P\,A\,Q$$

We choose the matrices P and Q such that the pencil sE–A has the Kronecker canonical form (Gantmacher 1971)

$$sE-A = \text{block diag}[0_{pg}\ \ L_\epsilon\ \ L_\eta\ \ sI_n-J\ \ I_\mu-sJ_\infty] \in \mathbb{R}[s]^{p\times m} \tag{2.2}$$

where

$$L_\epsilon = \text{block diag}[L_{\epsilon_{g+1}}\ \ L_{\epsilon_{g+2}}\ \cdots\ L_{\epsilon_p}]$$

$$L_{\epsilon_{g+i}} = \begin{bmatrix} s & 1 & \cdots & 0 & 0\\ 0 & s & \cdots & 0 & 0\\ \vdots & & \ddots & \ddots & \\ \vdots & & & & \\ 0 & 0 & \cdots & s & 1\end{bmatrix} \in \mathbb{R}^{\epsilon_{g+i}\times(\epsilon_{g+i}+1)}$$

$$L_\eta = \text{block diag}[L_{\eta_{h+1}}\ L_{\eta_{h+2}} \cdots L_{\eta_g}]$$

$$L_{\eta_{h+i}} = \begin{bmatrix} s & 0 & \cdots & 0 \\ 1 & s & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdot & s \\ 0 & 0 & \cdots & 1 \end{bmatrix} \in \mathbb{R}^{(\eta_{h+i}+1)\times\eta_{h+i}}$$

$$J = \text{block diag}[J_1\ J_2 \cdots J_k]$$

$$J_i = \text{block diag}[J_{ik}\ J_{ik+1} \cdots J_{ir}]$$

$$J_{ij} = \begin{bmatrix} \lambda_i & 1 & \cdots & 0 \\ 0 & \lambda_i & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \cdot & \cdot & \cdot & 1 \\ 0 & 0 & \cdots & \lambda_i \end{bmatrix} \in \mathbb{R}^{\sigma_{ij}\times\sigma_{ij}}$$

$$J_\infty = \text{block diag}[J_{\infty 1}\ J_{\infty 2} \cdots J_{\infty k'}]$$

$$J_{\infty i} = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \cdot & \cdot & \cdot & 1 \\ 0 & 0 & \cdots & 0 \end{bmatrix} \in \mathbb{R}^{(\hat{q}_i+1)\times(\hat{q}_i+1)}$$

where ϵ_i, η_i, σ_{ij} and $\hat{q}_i$ are respectively the orders of the right minimal indices, the left minimal indices, the multiplicities of the eigenvalues λ_i and the orders of the zeros at $s=\infty$.

In accordance with the form (2.2) the singular homogeneous system of differential equations in (2.1) splits into $r=p-g+q-h+s+n$ separate systems of the forms

$$0_{pg}\,\overset{1}{z}(t) = 0_{p1} \qquad (2.3)$$

$$L_{\epsilon_{g+i}}\left[\frac{d}{dt}\right]\overset{1+i}{z} = 0 \quad (i=1,2,..,p-g) \qquad (2.4)$$

$$L_{\eta_{h+j}}\left[\frac{d}{dt}\right]\overset{p-g+1+j}{z} = 0 \quad (j=1,2,..,q-h) \qquad (2.5)$$

$$J_{\infty k}\left[\frac{d}{dt}\right]\overset{p-g+q-h+1+k}{z} = \overset{p-g+q-h+1+k}{z} \quad (k=1,2,..,s) \qquad (2.6)$$

$$\left[\frac{d}{dt}\right]\overset{v}{z} = J\,\overset{v}{z} \qquad (2.7)$$

where

$$z = (\overset{1}{z}\ \overset{2}{z} \ldots\ \overset{v}{z})^T$$

and

$$\overset{1}{z} = [z_1, z_2, .., z_g]^T,\ \overset{2}{z} = [z_{g+1}, z_{g+2}, .., z_{g+\epsilon_{g+1}+1}]^T, \ldots$$

1) The system (2.3) is consistent for any $\overset{1}{z}$.

2) The system (2.4) has $\hat{\epsilon} = \epsilon_{g+1} + \epsilon_{g+2} + \cdots + \epsilon_p$ independent solutions of the form

$$\overset{1+i}{z}_w = \begin{bmatrix} \vdots \\ 0 \\ (-1)^w \end{bmatrix}\delta^{(w)}(t) + \begin{bmatrix} \vdots \\ (-1)^{w-1} \\ 0 \end{bmatrix}\delta^{(w-1)}(t) + \cdots + \begin{bmatrix} \vdots \\ 1 \\ 0_{w,1} \end{bmatrix}\delta(t) \quad \begin{matrix} w=0,1,..,\epsilon_{g+i}-1 \\ i=1,2,\ldots,p-g \end{matrix}$$

for initial conditions

$$\overset{1+i}{z}_{(w)}(0-) = \begin{bmatrix} \vdots \\ 1 \\ 0_{w,1} \end{bmatrix}$$

$\omega+1$

However the system (2.4) has an arbitrary number of solutions. More analytically define the vectors

$$\underline{\overset{1+i}{z}} = \begin{bmatrix} \phi_i(t) \\ -\phi_i^{(1)}(t) \\ \\ (-1)^{(\epsilon_{g+i}-1)}\phi_i^{(\epsilon_{g+i})}(t) \end{bmatrix} \qquad (2.8)$$

where $\phi_i(t)$ are arbitrary real functions. Then any solution of (2.4) is of the form

$$z + \lambda_1 \begin{bmatrix} \underline{\overset{2}{z}} \\ 0 \\ \vdots \end{bmatrix} + \lambda_2 \begin{bmatrix} 0 \\ \underline{\overset{3}{z}} \\ \vdots \end{bmatrix} + \cdots + \lambda_{p-g}\begin{bmatrix} \vdots \\ 0 \\ \underline{\overset{p-g}{z}} \end{bmatrix}$$

where z is any solution of (2.4).

3) From the system (2.5) we obtain that the following $\hat{\eta} = \eta_{h+1} + \eta_{h+2} + \cdots + \eta_q$ conditions between the initials conditions must be satisfied :

$$\overset{p-g+1+j}{z}(0-) = 0 \quad (j=1,2,..,q-h)$$

so that the system (2.1) or equivalently the system (1.1) has a solution.

4) The basis of the solution vector space X^∞ of the system (2.6) which is due to the infinite zero structure of $sE-A$ is given by the $\hat{q}=\hat{q}_{r-s}+\hat{q}_{r-s+1}+\cdots\hat{q}_r$ i.e. where $\hat{q}_r \geq \hat{q}_{r-1} \geq \cdots \geq \hat{q}_{r-s}$ are the orders of the zeros at $s=\infty$ of $sE-A$, columns

of the following matrix

$$[\,\overset{p-g+q-h+2}{z}\,,\ \overset{p-g+q-h+3}{z}\,,\ \ldots,\ \overset{p-g+q-h+s+1}{z}\,]$$

with

$$\overset{p-g+q-h+k+1}{z} = [\,\overset{p-g+q-h+k+1}{z}_0\,,\ \overset{p-g+q-h+k+1}{z}_1\,,\ \ldots,\ \overset{p-g+q-h+k+1}{z}_{\hat{q}_{r-s-1+k}-1}\,] \quad (k=1,2,..,s)$$

and

$$\overset{p-g+q-h+k+1}{z}_{\mu} = \begin{bmatrix} 1 \\ 0 \\ \vdots \end{bmatrix} \delta^{(\mu)}(t) + \begin{bmatrix} 0 \\ 1 \\ \vdots \end{bmatrix} \delta^{(\mu-1)}(t) + \cdots + \begin{bmatrix} 0_{\mu,1} \\ 1 \\ \vdots \end{bmatrix} \delta(t)$$

$$\mu = 0, 1, \ldots, \hat{q}_{r-s-1+i}-1$$

Note that $\overset{p-g+q-h+k+1}{z}_{\mu}$ is a solution of the system (2.6) for initial conditions of the form

$$\overset{p-g+q-h+k+1}{z}_{(\mu)}(0-) = \begin{bmatrix} 0_{\mu+1,1} \\ -1 \\ \vdots \end{bmatrix} \quad \begin{matrix} \mu=0,1,..,\hat{q}_{r-s-1+k}-1 \\ k=1,2,..,s \end{matrix}$$

Thus the solution vector space X^{∞} of the homogeneous system (2.6) has dimension equal to

$$\dim X^{\infty} = \hat{q} := \textbf{total number of infinite zeros of } sE-A \in \mathbb{R}[s]^{p \times m}$$

5) The basis of the solution vector space $X^{\mathbb{C}}$ of the system (2.7) which is due to the finite zero structure of sE–A is given by the columns of the following matrix

$$[\check{z}_1, \check{z}_2, \ldots, \check{z}_{\hat{n}}] = [e^{Jt}]$$

where

$$\dim X^{\mathbb{C}} = \hat{n} := \textbf{total number of finite zeros of } sE-A \in \mathbb{R}[s]^{p \times m}$$

Consider now the space

$$Z = \{ z \mid z = \lambda_1 \begin{bmatrix} 0 \\ \overset{2}{\underline{z}} \\ \vdots \\ 0 \end{bmatrix} + \lambda_2 \begin{bmatrix} 0 \\ 0 \\ \overset{3}{\underline{z}} \\ \vdots \\ 0 \end{bmatrix} + \cdots + \lambda_{p-g} \begin{bmatrix} 0 \\ \vdots \\ \overset{p-g}{\underline{z}} \\ \vdots \\ 0 \end{bmatrix} + \lambda \begin{bmatrix} \overset{1}{z} \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

where $\overset{2}{\underline{z}}, \overset{3}{\underline{z}}, .., \overset{p-g}{\underline{z}}$ have been defined in (2.8) and $\overset{1}{z}$ is an arbitrary real function which comes from the solution of the system (2.3). Define now the following relation between the set of solutions $\mathcal{B}$, of the system (2.1)

$$R(z_1, z_2) = \{ (z_1, z_2) \mid z_1, z_2 \in \mathcal{B} \text{ and } z_1 - z_2 \in Z \} \tag{2.9}$$

It can be easily seen that (2.9) defines an equivalence relation. We call an *equivalence class* of the element $z \in \mathcal{B}$ and we denote this with $[z]$ the set of all solutions of (2.1) which are equivalent to z, or equivalently

$$[z] = \{ z_1 \in \mathcal{B} \mid (z, z_1) \in R \} = z \oplus Z \tag{2.10}$$

We can see that any equivalence class of an element $z \in \mathcal{B}$ gives the solution of (2.1) under some specific initial conditions. In case where sE–A has not a right kernel then every equivalence class is composed by a unique element contrary to the singular case where to each equivalence class corresponds an arbitrary number of elements of $\mathcal{B}$. We conclude therefore that the whole solution space $\mathcal{B}$ of the singular state–space representation (2.1) is divided into equivalence classes which are defined by (2.10). Define now the following "*sum*" between equivalence classes of the form (2.10)

$$[z_1] + [z_2] := (z_1 + z_2) \oplus Z = \{ z_1 + z_2 + z \mid z_1, z_2 \in \mathcal{B} \text{ and } z \in Z \}$$

and the "*product*"

$$\lambda [z] := \lambda z \oplus Z = \{ \lambda z + \tilde{z} \mid z \in \mathcal{B} \text{ and } \tilde{z} \in Z \}$$

It is now easily to prove the following

Theorem 1. The space which is spanned by the equivalence classes which are defined in (2.10) is a vector space $\hat{\mathcal{B}}$ and this is the solution vector space of the homogeneous singular state–space representation (2.1). □

Consider now the following spaces

$$\hat{X}^{\mathbb{C}} = \langle [\check{z}_1], [\check{z}_2], \ldots, [\check{z}_{\hat{n}}] \rangle = X^{\mathbb{C}}/R$$

$$\hat{X}^{\infty} = < \overset{p-g+q-h+2}{[z]}, \overset{p-g+q-h+3}{[z]}, \ldots, \overset{p-g+q-h+s+1}{[z]} > = X^{\infty}/R$$

with

$$\overset{p-g+q-h+k+1}{[z]} = < \overset{p-g+q-h+k+1}{[z_0]}, \overset{p-g+q-h+k+1}{[z_1]}, \ldots, \overset{p-g+q-h+k+1}{[z_{\hat{q}_{r-s-1+k}-1}]} >$$

and

$$\overset{p-g+q-h+k+1}{[z_\mu]} = \left[\begin{bmatrix}1\\0\\\vdots\end{bmatrix}\delta^{(\mu)}(t) + \begin{bmatrix}0\\1\\\vdots\end{bmatrix}\delta^{(\mu-1)}(t) + \cdots + \begin{bmatrix}0_{\mu,1}\\1\\\vdots\end{bmatrix}\delta(t)\right]$$

$$\mu=0,1,\ldots,\hat{q}_{r-s-1+k}-1$$

$$\hat{X}^{\epsilon} = < \overset{2}{[z]}, \overset{3}{[z]}, \ldots, \overset{p-g}{[z]} >$$

where

$$\overset{1+i}{[z]} = < \overset{1+i}{[z_0]}, \overset{1+i}{[z_1]}, \ldots, \overset{1+i}{[z_{\epsilon_{g+i}-1}]} >$$

with

$$\overset{1+i}{[z_w]} = \left[\begin{bmatrix}\vdots\\0\\(-1)^{w}\end{bmatrix}\delta^{(w)}(t) + \begin{bmatrix}\vdots\\(-1)^{w-1}\\0\end{bmatrix}\delta^{(w-1)}(t) + \cdots + \begin{bmatrix}\vdots\\1\\0_{w,1}\end{bmatrix}\delta(t)\right] \quad \begin{matrix}w=0,1,\ldots,\epsilon_{g+i}-1\\ i=1,2,\ldots,p-g\end{matrix}$$

It is obvious that the spaces $\hat{X}^{C}$, $\hat{X}^{\infty}$ and $\hat{X}^{\epsilon}$ are vector spaces which are due to the finite, infinite zero structure and the right minimal indices of sE–A. We can also prove the following

Theorem 2. The vector space $\hat{\mathcal{B}} = \{ [z] \mid z \in \mathcal{B} \}$ has dimension

$$\boxed{\dim \hat{\mathcal{B}} = f := \hat{n} + \hat{q} + \hat{\epsilon}}$$

$f = \dim\hat{\mathcal{B}}$ is called *generalized order* of the singular state–space representation (1.1). □

Remark 3. Note that if sE–A is regular then $\hat{\epsilon}=0$ and f coincides with what Verghese defined as generalized order of (1.1). □

The above results may also applied to the singular state–space AR representation (1.1) if we take into account that transformations of the form $x(t) = Q\, z(t)$ with $|Q| \neq 0$ does not alter the algebraic structure of the pencil sE–A and so the behaviour of the system (1.1).

Example 4. Consider the homogeneous singular state–space representation :

$$\begin{bmatrix}\rho & 1 & 0 & 0 & 0 & 0\\ 0 & 0 & \rho & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1 & -\rho & 0\\ 0 & 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 0 & \rho+1\end{bmatrix}\begin{bmatrix}z_1(t)\\z_2(t)\\z_3(t)\\z_4(t)\\z_5(t)\\z_6(t)\end{bmatrix} = 0_{61} \tag{E.1}$$

In this case $\hat{n}=1$, $\hat{q}=1$, $\hat{\epsilon}=1$ and $\hat{\eta}=1$. The minimal left indices gives rise to the conditions between the initial conditions which must be satisfied so that (E.1) has a solution and are the following

$$z_3(0-) = 0 \tag{E.2}$$

Thus under the assumption that condition (E.2) is satisfied the generalized order of (E.1) is $f = \dim\hat{\mathcal{B}} = \hat{n} + \hat{q} + \hat{\epsilon} = 1 + 1 + 1 = 3$ and $\hat{\mathcal{B}}$ is given by

$$\hat{\mathcal{B}} = \{\lambda_1[\begin{bmatrix}0\\1\\0\\0\\0\\0\end{bmatrix}\delta(t)] + \lambda_2[\begin{bmatrix}0\\0\\0\\1\\0\\0\end{bmatrix}\delta(t)] + \lambda_3[\begin{bmatrix}0\\0\\0\\0\\0\\e^{-t}\end{bmatrix}]\}$$

$$\{\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}\}$$

$$=\{\lambda_1\begin{bmatrix}0\\1\\0\\0\\0\\0\end{bmatrix}\delta(t)+\lambda_2\begin{bmatrix}0\\0\\0\\1\\0\\0\end{bmatrix}\delta(t)+\lambda_3\begin{bmatrix}0\\0\\0\\0\\0\\e^{-t}\end{bmatrix}+\begin{bmatrix}z(t)\\-z^{(1)}(t)\\0\\0\\0\\0\end{bmatrix}\}$$

$$\{\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}\} \tag{E.3}$$

where $z(t)$ is an arbitrary real function. □

3. Solutions of Homogeneous Singular AR Representations.

Consider the singular system (2.1) or equivalently the singular systems (2.3)–(2.7). In what follows we derive the solution of these systems.

1) The solution of the system (2.3) is of the form :

$$\overset{1}{z} = z$$

for any arbitrary real function $z(t)$ i.e. the system is consistent for any z.

2) The solution of the system (2.4) is of the form

$$\overset{1+i}{z} = [\overset{1+i}{z}_0\ \overset{1+i}{z}_1 \cdots \overset{1+i}{z}_{\epsilon_{g+i}-1}\ \overset{1+i}{\underline{z}}] \begin{bmatrix} \overset{1+i}{z}_{\epsilon_{g+i}-1}(0-) \\ \vdots \\ \overset{1+i}{z}_0(0-) \\ 1 \end{bmatrix}$$

(i=1,2,..,p−g)

3) The solution of the system (2.5) is

$$\overset{p-g+1+j}{z} = 0 \quad (j=1,2,...,q-h)$$

under the assumption that

$$\overset{p-g+1+j}{z}(0-) = 0$$

4) The solution of the system (2.6) is

$$\overset{p-g+q-h+k+1}{z} = -[\overset{p-g+q-h+k+1}{z}_0,\ \overset{p-g+q-h+k+1}{z}_1,\ \ldots,\ \overset{p-g+q-h+k+1}{z}_{\hat{q}_{r-s-1+k}-1}] \begin{bmatrix} \overset{p-g+q-h+k+1}{z}_2(0-) \\ \overset{p-g+q-h+k+1}{z}_3(0-) \\ \vdots \\ \overset{p-g+q-h+k+1}{z}_{\hat{q}_{r-s-1+k}+1}(0-) \end{bmatrix}$$

(k=1,2,..,s)

5) The solution of the system (2.7) is

$$\overset{v}{z} = e^{Jt}\overset{v}{z}(0-)$$

Thus if we denote with

$$\Psi(t) = \text{block diag}[\underbrace{\overset{1}{z}}_{1}\ \underbrace{\{\overset{2}{z}_0\ \overset{2}{z}_1 \cdots \overset{2}{z}_{\epsilon_{g+1}-1}\ \overset{2}{\underline{z}}\} \ldots \overset{p-g+1}{z}_0 \ldots}_{2}\ \underbrace{\{0..0\}}_{3}\ \underbrace{\{\overset{p-g+q-h+2}{z}_0,\ \overset{p-g+q-h+2}{z}_1,\ \ldots,\ \overset{p-g+q-h+2}{z}_{\hat{q}_{r-s}-1}\},\ldots,\ \overset{p-g+q-h+s+1}{z}_0}_{4},\ \ldots\underbrace{\{e^{Jt}\}}_{5}]$$

and

$$\tilde{z}(0-) = \begin{bmatrix} 1 \\ \hline \overset{1+i}{z}_{\epsilon_{g+i}-1}(0-) \\ \vdots \\ \overset{1+i}{z}_0(0-) \\ 1 \\ \hline \vdots \end{bmatrix} \ldots$$

then the solution of the system (2.1) will be

$$z(t) = \Psi(t)\,\tilde{z}(0-)$$

if the initial conditions $\overset{p-g+1+j}{z}(0-) = 0$ are satisfied.

Example 5. Consider the AR representation of Example 4. We can see that

$$\Psi(t)=\text{block diag}[\begin{bmatrix} 0 & z(t) \\ \delta(t) & -z^{(1)}(t) \end{bmatrix}\ 0\ \begin{bmatrix} \delta(t) \\ 0 \end{bmatrix}\ e^{-t}] \quad \text{(E.1)}$$

and

$$\tilde{z}(0-) = \begin{bmatrix} z_1(0-) \\ 1 \\ 0 \\ z_5(0-) \\ z_6(0-) \end{bmatrix} \quad \text{(E.2)}$$

Thus the solution of the AR representation in Example 4 will be

$$z(t) = \Psi(t)\,\tilde{z}(0-) =$$

$$= \text{block diag}[\begin{bmatrix} 0 & z(t) \\ \delta(t) & -z^{(1)}(t) \end{bmatrix}\ 0\ \begin{bmatrix} \delta(t) \\ 0 \end{bmatrix}\ e^{-t}]$$

$$\begin{bmatrix} z_1(0-) \\ 1 \\ 0 \\ z_5(0-) \\ z_6(0-) \end{bmatrix} = \begin{bmatrix} z(t) \\ z_1(0-)\delta(t)-z^{(1)}(t) \\ 0 \\ z_5(0-)\delta(t) \\ 0 \\ z_6(0-)\delta(t) \end{bmatrix}$$

(E.3) □

4. Conclusions.

In this paper we extend known results concerning the homogeneous solution vector spaces of regular state–space AR representations to the case of singular systems. More specifically it was shown that the right kernel of the pencil which describe the singular state–space AR representation is responsible for the arbitrary number of solutions while the left kernel give us certain conditions between the initial conditions which must be satisfied so that the homogeneous system has a solution. It was proved that the whole solution vector space $\hat{B}$, of the singular state–space AR representations is strongly related with the finite and infinite zero structure and the right minimal indices of the pencil which describe the system, its elements are equivalence classes and its dimension is equal to the total number of its finite zeros and infinite zeros and right minimal indices (orders accounted for). Thus the meaning of the algebraic structure of a singular pencil in relation to the solution vector spaces of singular AR representations has been elucidated. A closed formula for the solution of a homogeneous singular system in Kronecker form has also been provided.

REFERENCES

[1] BLOMBERG H. and YLINEN R., 1983, *Algebraic Theory for Multivariable Linear Systems.* , Mathematics in Science and Engineering, Vol.166, Academic Press, London.

[2] COPPEL W.A. and CULLEN D.J., 1985,Strong System Equivalence (II)., *J.Austral.Math.Soc., Ser. B* 27, pp.223–237.

[3] GOHBERG I., LANGASTER P and RODMAN I., 1982, *Matrix Polynomials.*, Academic Press, New York.

[4] GANTMACHER F.R., 1971, *The Theory of Matrices.*, Chelsea, New York.

[5] KAILATH T., 1980, *Linear Systems.*, Prentice Hall.

[6] KUIJPER M., 1992, First Order Representations of Linear Systems., Ph.D. Thesis, Center for Mathematics and Computer Science, Amsterdam, The Netherlands.

[7] ROSENBROCK H.H., 1970, *State–Space and Multivariable Theory.*, Nelson.

[8] VARDULAKIS A.I.G, 1991, *Linear Multivariable Control, Algebraic Analysis and Synthesis Methods*, John Wiley and Sons

[9] VERGHESE G., 1978, *Infinite–Frequency Behaviour in Dynamical Systems*, Ph.D.Dissertation, Department of Electronic Engineering, Stanford University.

[10] WILLEMS J.C., 1986, From Time Series to Linear System – Part I. Finite Dimensional Time Invariant Systems., *Automatica*, 22, pp.561–580.

[11] WILLEMS J.C., 1991, Paradigms and Puzzles in the Theory of Dynamical Systems., *IEEE Trans.Auto.Control*, 36, pp.259–294.