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A FUNDAMENTAL NOTION OF EQUIVALENCE FOR LINEAR MULTIVARIABLE SYSTEMS

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Abstract

A fundamental form of equivalence between Polynomial Matrix Descriptions of linear multivariable systems is defined. It is based on the existence of a bijective map between the finite and infinite solution sets of the differential equations describing the two systems. The connection with the system matrix relationship of full system equivalence is established.

1. Introduction

The transformation of *strict system equivalence* was proposed in [10] for the study of the finite frequency behaviour of linear systems. This transformation has the property of leaving invariant the *finite* structure of any Polynomial Matrix Description (PMD) of a linear multivariable system Σ to which it is applied. Further studies of this transformation have been conducted in [2], and notably [7] showed that it is equivalent to the existence of a certain bijective map between the *finite* solution sets of the differential equations describing the system.

A number of important properties are related to the infinite frequency behaviour of the system (due to certain of the natural modes of the system occurring at the point at infinity) and these are not left invariant by the previous transformation. For this reason [12] proposed the notion of *strong system equivalence* for systems in generalized state space form. [8] gave a closed - form description of this transformation termed *complete system equivalence*. Subsequently [3] gave interpretations of this form of equivalence in terms of bijective maps between the *finite and infinite* solution sets, as well as between the restricted initial conditions, of the differential equations which describe a system in generalized state space form.

More recently [1,5] have proposed extensions of these latter transformations to the general case of PMDs, which were termed respectively *strong system equivalence* and *full system equivalence*. The transformation of *full system equivalence* is a closed-form matrix description of the equivalence in this case.

This paper takes up the views of [3,7,12] and generalises these ideas to the case of general PMD's. It gives an alternative characterization of full system equivalence in terms of bijective maps between the finite and infinite solution sets as well as between the restricted initial conditions of the differential equations which describe the PMD of the system. Within this framework certain of the invariants appear naturally, as a consequence of the definition, and a complete explanation of the role and nature of the conditions of full system equivalence emerges.

2. Preliminary Results.

Consider a linear system Σ described by a (PMD)

$$A(\rho)\beta(t) = B(\rho)u(t) \tag{2.1a}$$

$$y(t) = C(\rho)\beta(t) + D(\rho)u(t) \tag{2.1b}$$

where $(\rho = d/dt)$, $A(\rho) \in \mathbb{R}[\rho]^{r\times r}$ with $|A(\rho)| \neq 0$, $B(\rho) \in \mathbb{R}[\rho]^{r\times m}$, $C(\rho) \in \mathbb{R}[\rho]^{p\times r}$, $D(\rho) \in \mathbb{R}[\rho]^{p\times m}$. Let

$$P(s) := \begin{bmatrix} A(s) & -B(s) \\ C(s) & D(s) \end{bmatrix} \in \mathbb{R}[s]^{(r+p)\times(r+m)} \tag{2.2}$$

be the Rosenbrock system matrix corresponding to Σ. The normalised form of $P(s)$ is [12]

$$\mathcal{P}(s) := \left[\begin{array}{ccc|c} A & -B & 0 & 0 \\ C & D & -I_p & 0 \\ 0 & I_m & 0 & -I_m \\ \hline 0 & 0 & I_p & 0 \end{array}\right] =: \begin{bmatrix} \mathcal{T}(s) & -\mathcal{U} \\ \mathcal{V} & 0 \end{bmatrix} \tag{2.3}$$

which has the advantage over (2.2) of permitting consistent definitions of finite and infinite frequency characteristics to be made.

Let p, m be fixed positive integers and r any positive integer. Define P_0 as the set of $(r+p)\times(r+m)$ Rosenbrock system matrices, and G_0 as the set of $(r+p)\times(r+m)$ sytem matrices in generalised state space form. Let $P(p,m)$ denote the set of $(r+p)\times(r+m)$ polynomial matrices where the integer $r \geq \max(-p,-m)$.

Definition 1. [4] $T_1(s), T_2(s) \in P(p,m)$ are *fully equivalent* (F.E.) in case $\exists$ polynomial matrices $M(s), N(s)$ of appropriate dimensions such that:

$$\begin{bmatrix} M & T_2 \end{bmatrix} \begin{bmatrix} T_1 \\ -N \end{bmatrix} = 0 \tag{2.4}$$

and where the compound matrices

$$\begin{bmatrix} M & T_2 \end{bmatrix}; \begin{bmatrix} T_1 \\ -N \end{bmatrix} \tag{2.5}$$

are such that

(i) they have full normal rank (2.6a)

(ii) they have no finite nor infinite zeros (2.6b)

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(iii) the following McMillan degree conditions hold

$$\delta_M([M\ T_2]) = \delta_M(T_2); \delta_M\left(\begin{bmatrix} T_1 \\ -N \end{bmatrix}\right) = \delta_M(T_1) \quad (2.6c)\ \square$$

One notable form of (F.E.) is obtained when $T_1(s)$, $T_2(s)$ are *regular* pencils and $M(s), N(s)$ are constant matrices. This is termed [8] *complete equivalence* (C.E.). Let

$$E_i\dot{x}_i(t) = A_i x_i(t) + B_i u(t) \quad (2.7a)$$

$$y_i(t) = C_i x_i(t) + D_i u(t) \quad i = 1, 2 \quad (2.7b)$$

be two generalized state space systems driven by the same input $u(t)$ and let $P_i(s), i = 1, 2$ be their system matrices. (C.E.) in the context of such systems yields

Definition 2. $(P_1(s), P_2(s)) \in G_0 \times G_0$ are *completely system equivalent* (C.S.E.) if $\exists$ constant matrices M, N, X, Y such that

$$\begin{bmatrix} M & 0 \\ X & I \end{bmatrix}\begin{bmatrix} sE_1 - A_1 & -B_1 \\ C_1 & D_1 \end{bmatrix} = \begin{bmatrix} sE_2 - A_2 & -B_2 \\ C_2 & D_2 \end{bmatrix}\begin{bmatrix} N & Y \\ 0 & I \end{bmatrix}$$

is a (C.E.) transformation. $\square$

Definition 3. [3] $(P_1(s), P_2(s)) \in G_0 \times G_0$ are *fundamentally equivalent* if $\exists$

(1) a constant, injective map

$$\begin{bmatrix} \hat{x}_2(s) \\ \hat{u}(s) \end{bmatrix} = \begin{bmatrix} N & Y \\ 0 & I \end{bmatrix}\begin{bmatrix} \hat{x}_1(s) \\ \hat{u}(s) \end{bmatrix} \quad (2.8)$$

and

(2) a constant, surjective map

$$\begin{bmatrix} E_2 x_2(0-) \\ \hat{y}(s) \end{bmatrix} = \begin{bmatrix} M & 0 \\ X & I \end{bmatrix}\begin{bmatrix} E_1 x_2(0-) \\ \hat{y}(s) \end{bmatrix}, XE_1 = 0. (2.9)\ \square$$

Some properties of this transformation are [3]

Lemma 1. (i). If $(P_1(s), P_2(s)) \in G_0 \times G_0$ which are fundamentally equivalent then the maps (2.8), (2.9) are bijective.
(ii) $(P_1(s), P_2(s)) \in G_0 \times G_0$ are fundamentally equivalent iff they are (C.S.E.) $\square$

The attraction of definition 3 lies in its reference to the solution-input pairs in (2.8) and initial condition/output pairs in (2.9), quantities which are fundamental to the system. Indeed the system equations define a relation between these pairs. It is the nature of these pairs which characterise the system from the point of view of its controllability and observability properties. For example the controllability properties of a given system are encoded within its set of solution, input pairs, while the initial condition/output pairs encapsulate the system's observability properties. Lemma 1(ii) thus guarantees that two generalised state space systems which are fundamental system equivalent automatically have identical controllability and observability properties. An extension of (C.S.E.) to general PMDs is [5]

Definition 4. $(P_1(s), P_2(s)) \in P_0 \times P_0$ are *full system equivalent* (F.S.E.) if $\exists$ polynomial matrices $M(s), N(s), X(s), Y(s)$ such that

$$\begin{bmatrix} M & 0 \\ X & I \end{bmatrix}\begin{bmatrix} A_1 & -B_1 \\ C_1 & D_1 \end{bmatrix} = \begin{bmatrix} A_2 & -B_2 \\ C_2 & D_2 \end{bmatrix}\begin{bmatrix} N & Y \\ 0 & I \end{bmatrix} \quad (2.10)$$

is a transformation of (F.E.). $\square$

Definition 5. The normalized forms $\mathcal{P}_1(s)$, $\mathcal{P}_2(s)$ of $(P_1(s), P_2(s)) \in P_0 \times P_0$ are *Normal Full System Equivalent* (N.F.S.E.) if $\exists$ polynomial matrices $M(s), N(s), X(s), Y(s)$, such that $\mathcal{P}_1(s)$ and $\mathcal{P}_2(s)$ are related by:

$$\begin{bmatrix} M & 0 \\ X & I \end{bmatrix}\begin{bmatrix} T_1(s) & -\mathcal{U}_1 \\ \mathcal{V}_1 & 0 \end{bmatrix} = \begin{bmatrix} T_2(s) & -\mathcal{U}_2 \\ \mathcal{V}_2 & 0 \end{bmatrix}\begin{bmatrix} N & Y \\ 0 & I \end{bmatrix} \quad (2.11)$$

where (2.11) is a transformation of (F.E.)

(F.S.E.) and (N.F.S.E.) define equivalence relations on P_0 which give rise to identical equivalence classes. Further important properties of this equivalence are [5,6]

Lemma 2. Under (F.S.E.), (N.F.S.E.) the following are invariant:

(i) the *generalized order* $f := \delta_M(T(s))$ and the *Rosenbrock degree* d_R.

(ii) the transfer function matrix $G(s)$ and so the *finite* and *infinite transmission poles and zeros* of $G(s)$

(iii) the *finite* and *infinite system poles and zeros*

(iv) the *finite* and *infinite input (output) decoupling zeros*. $\square$

It is seen that fundamental equivalence (definition 3) is a natural form of equivalence from which the invariance of many of the system's properties follows automatically from the definition. It also has a matrix representation in (C.S.E.). Unfortunately however this notion of equivalence appears to be confined to the generalised state space context. We therefore seek an extension of this equivalence to general PMDs.

3. Mappings of Solution Sets.

Define X_u as the set of all solutions of (2.1a) corresponding to the input $u(t)$ for all possible initial conditions on $\beta(t)$ and its $q-1$ derivatives where q is the highest power of s occurring in $P(s)$. Let X_u be related to the set X_u^1, the range of the relation

$$\begin{bmatrix} \beta_1(t) \\ u(t) \end{bmatrix} = \begin{bmatrix} \tilde{N}(p) & \tilde{Y}(p) \\ 0 & I_m \end{bmatrix}\begin{bmatrix} \beta(t) \\ u(t) \end{bmatrix} \quad (3.1)$$

where $\tilde{N}(p) \in \mathbb{R}[p]^{r_1 \times r}$ and $\tilde{Y}(p) \in \mathbb{R}[p]^{r_1 \times m}$. It is easily seen that (3.1) may be written as

$$\begin{bmatrix} \beta_1(t) \\ u(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} \tilde{N}(p) & \tilde{Y}(p) & 0 \\ 0 & I_m & 0 \\ 0 & 0 & I_p \end{bmatrix}\begin{bmatrix} \beta(t) \\ u(t) \\ y(t) \end{bmatrix} \Longrightarrow$$

$$\xi_1(t) = N_1(p)\xi(t) \quad (3.2)$$

where $\xi_1(t) = [\beta_1(t)^T, u(t)^T, y(t)^T]^T$, $N_1(p) \in \mathbb{R}[p]^{\hat{r}_1 \times \hat{r}}$, $\hat{r}_1 := r_1 + p + m$, $\hat{r} := r + p + m$ and $\xi(t) = [\beta(t)^T, u(t)^T, y(t)^T]^T$ is the pseudostate of the normalized form of (2.1) (see (2.3)):

$$T(p)\xi(t) = \mathcal{U}u(t) \quad (3.3.a)$$

$$y(t) = \mathcal{V}\xi(t) \quad (3.3.b)$$

Define X_u as the set of all solutions $\xi(t)$ (3.3.a) corresponding to a given input $u(t)$ for all possible initial conditions on $\xi(t)$ and its $q-1$ derivatives. Suppose X_u is mapped onto the set X_u^1 defined as the range of the relation

$$\begin{bmatrix} \xi_1(t) \\ u(t) \end{bmatrix} = \begin{bmatrix} M(p) & N_1(p) \\ 0 & I_m \end{bmatrix}\begin{bmatrix} \xi(t) \\ u(t) \end{bmatrix} \quad (3.4)$$

where $M(p) \in \mathbb{R}[p]^{r_1 \times r}$ and $N_1(p) \in \mathbb{R}[p]^{r_1 \times m}$. Now writing

$$M(p) = [M_1(p)\quad M_2(p)\quad M_3(p)] \quad (3.5)$$

it follows from (3.4) that

$$
\begin{aligned}
\xi_1(t) &= [M_1(\rho), M_2(\rho), M_3(\rho)]\xi(t) + N_1(\rho)u(t) \\
&= [M_1(\rho) \quad M_2(\rho) + N_1(\rho) \quad M_3(\rho)]\xi(t) \qquad (3.6) \\
&= M'(\rho)\xi(t)
\end{aligned}
$$

For the above reasons we shall be interested only in relations of the form (3.2). Many authors [1,7,11] have considered the action of such a linear mapping on the solution space of a given PMD. The effects have only been partially quantified, generally as a result of focusing attention simply on the finite frequency behaviour. However even when the impulsive behaviour has been taken into account [7] full consideration has not been given, specifically in regard to the issue of the map being well constituted. A complete analysis will therefore be given here of the action of a linear mapping on the solution space of a PMD of a linear system, both in regard to its effect on the finite and infinite frequency behaviour of the system and in regard to whether (3.2) can be considered a mapping in the formal sense or merely a relation. This latter issue is considered first, and the following example indicates the nature of the difficulty.

Example 1. Consider the homogeneous system of differential equations

$$
\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \rho \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \xi_1(t) \\ \xi_2(t) \\ \xi_3(t) \end{bmatrix} = 0 \qquad \text{(E.1.1)}
$$

with solution

$$
\begin{bmatrix} \xi_1(t) \\ \xi_2(t) \\ \xi_3(t) \end{bmatrix} = \begin{bmatrix} 0 \\ \xi_3(0-)\delta(t) \\ 0 \end{bmatrix} \qquad \text{(E.1.2)}
$$

and consider the relation

$$
y(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} L_1\rho - L_0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \xi_1(t) \\ \xi_2(t) \\ \xi_3(t) \end{bmatrix} \qquad \text{(E.1.3)}
$$

from which we obtain

$$
y(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} -L_1\xi_1(0-)\delta(t) \\ \xi_3(0-)\delta(t) \end{bmatrix} \qquad \text{(E1.4)}
$$

It is easily seen from the above that the previous relation is not a map in the strict sense because for any one solution $\xi(t) = [\xi_1(t) \quad \xi_2(t) \quad \xi_3(t)]^T$ of (E.1.1), which is determined solely by $\xi_3(0-)$, we obtain many images $y(t)$ which will depend on the initial value $\xi_1(0-)$. □

Consider

$$
\xi_1(t) = M(\rho)\xi(t) \qquad (3.7)
$$

The first question regarding (3.7) is therefore whether it formally represents a mapping of χ_u to χ_u^1, or whether it is merely a relation. Without loss of generality it will be assumed that $u(t) = 0$. Thus consider the solution space χ_0 of the unforced system

$$
T(\rho)\xi(t) = 0 \qquad (3.8)
$$

under the relation (3.7). Equivalently consider the equations

$$
\begin{bmatrix} 0 \\ \xi_1(t) \end{bmatrix} = \begin{bmatrix} T(\rho) \\ M(\rho) \end{bmatrix} \xi(t) \qquad (3.9)
$$

Without loss of generality it may be assumed that

$$
\begin{aligned}
T(\rho) &:= T_k\rho^k + \ldots + T_1\rho + T_0 \\
M(\rho) &:= M_k\rho^k + \ldots + M_1\rho + M_0
\end{aligned} \qquad (3.10)
$$

where at least one of $T_k, M_k \neq 0$. Taking Laplace transforms of (3.9) with assumed values $\xi^{(i)}(0-), i = 1, \ldots, k-1$ gives

$$
\begin{bmatrix} 0 \\ \hat{\xi}_1(s) \end{bmatrix} = \begin{bmatrix} T(s) \\ M(s) \end{bmatrix} \hat{\xi}(s) - \begin{bmatrix} \hat{a}_T(s) \\ \hat{a}_M(s) \end{bmatrix} \qquad (3.11)
$$

where the polynomial vectors $\hat{a}_T(s), \hat{a}_M(s)$ are given by

$$
\hat{a}_Q(s) = [s^{k-1}I, \ldots, sI, I] \begin{bmatrix} Q_k & 0 & \ldots & 0 \\ \vdots & \vdots & \ldots & \vdots \\ Q_2 & Q_3 & & 0 \\ Q_1 & Q_2 & \ldots & Q_k \end{bmatrix} \begin{bmatrix} \xi(0-) \\ \vdots \\ \xi^{(k-2)}(0-) \\ \xi^{(k-1)}(0-) \end{bmatrix}
$$

$$
:= S_{k-1}\mathcal{X}_Q\bar{\xi}(0-) \quad Q = T, M \qquad (3.12)
$$

Theorem 1. (3.7) is a mapping in the formal sense iff

$$
\delta_M \begin{bmatrix} T(s) \\ M(s) \end{bmatrix} = \delta_M(T(s)) \qquad (3.13)
$$

Proof The relation (3.7) is a mapping in the formal sense iff it uniquely specifies an image, $\xi_1(t)$, for each solution $\xi(t)$ of (3.8). Accordingly, in respect of (3.11), the relation (3.7) is a mapping iff for each $\hat{\xi}(s)$ determined by

$$
T(s)\hat{\xi}(s) = \hat{a}_T(s) \qquad (3.14)
$$

the relation

$$
\hat{\xi}_1(s) = M(s)\hat{\xi}(s) - \hat{a}_M(s) \qquad (3.15)
$$

determines $\hat{\xi}_1(s)$ uniquely. It is apparent that if a given $\hat{\xi}(s)$ has two images $\hat{\xi}_1(s)$ and $\hat{\xi}_1'(s)$ under (3.15) then this is due to $\hat{a}_M(s)$ or more particularly the associated initial condition vector $\bar{\xi}(0-)$ of (3.12). Let $\bar{\xi}(0-) \neq \bar{\xi}'(0-)$ be two initial condition vectors. It then follows that these determine the same solution $\xi(t)$ provided that $\bar{\xi}(0-) - \bar{\xi}'(0-) \in \ker\mathcal{X}_T$. Given that this condition is satisfied, it follows that $\hat{\xi}_1(s)$ is determined via (3.15) as the unique image of $\hat{\xi}(s)$ provided $\bar{\xi}(0-) - \bar{\xi}'(0-) \in \ker\mathcal{X}_M$. Hence (3.7) is a map in the formal sense iff

$$
\ker\mathcal{X}_T \subseteq \ker\mathcal{X}_M \qquad (3.16)
$$

or equivalently

$$
\ker \begin{bmatrix} \mathcal{X}_T \\ \mathcal{X}_M \end{bmatrix} = \ker\mathcal{X}_T \qquad (3.17)
$$

From the dimension theorem of linear maps, (3.17) holds iff

$$
\mathrm{Rank}_{\mathbb{R}} \begin{bmatrix} \mathcal{X}_T \\ \mathcal{X}_M \end{bmatrix} = \mathrm{Rank}_{\mathbb{R}}\mathcal{X}_T \qquad (3.18)
$$

Hence from the characterisation of the McMillan degree

$$
\delta_M \left(\begin{bmatrix} T(s) \\ M(s) \end{bmatrix} \right) = \delta_M(T(s)) \qquad (3.19) \ \square
$$

Remark In the example 1 it can be seen directly from (E.1.4) that the relation (E.1.3) is a map iff $L_1 = 0$. Further

$$\delta_M \begin{bmatrix} \mathcal{T}(s) \\ M(s) \end{bmatrix} \geq \delta_M(\mathcal{T}(s)) \tag{3.20}$$

with equality holding iff $L_1 = 0$, which verifies the theorem. □

The existence condition for the map (3.7) to be properly constituted is thus revealed as the McMillan degree condition appearing in the definition of (F.E.). Thus far from being a condition somewhat arbitrarily attached to the transformation, it is in fact a vital condition if the map (3.7) is to be considered in relation to its effect on the complete solution space χ_a.

Now by definition of χ_a^1 above, it follows that (3.7) is an onto mapping of χ_a onto χ_a^1 whenever it is properly constituted. It is now appropriate to consider under which circumstances it might be injective and consequently invertible.

Theorem 2. For any fixed $\alpha(t)$, (3.7) is an injective mapping of χ_a to χ_a^1 iff the existence condition (3.13) holds and $M(s)$ and $T(s)$ have no common finite (or infinite) zeros.

Proof. Again there is no loss of generality in considering the unforced system (3.8). Note that (3.13) is a necessary and sufficient condition for the relation (3.7) to be a mapping.

Assume that two solutions $\xi(t), \xi'(t)$ from χ_a map onto $\xi_1(t) \in \chi_a^1$. Then from (3.14) and (3.15)

$$\mathcal{T}(s)[\hat{\xi}(s) - \hat{\xi}'(s)] = \hat{\alpha}_T(s) - \hat{\alpha}'_T(s) \tag{3.21}$$

$$0 = M(s)[\hat{\xi}(s) - \hat{\xi}'(s)] - \hat{\alpha}_M(s) - \hat{\alpha}'_M(s) \tag{3.22}$$

A necessary and sufficient condition for $\mathcal{T}(s)$, $M(s)$ to have no common finite zeros is that $\exists$ polynomial matrices $\bar{R}_1(s)$ and $\bar{R}_2(s)$ of the appropriate dimensions such that

$$\bar{R}_1(s)\mathcal{T}(s) - \bar{R}_2(s)M(s) = I, \tag{3.23}$$

Postmultiplying (3.23) by $\hat{\xi}(s) - \hat{\xi}'(s)$ and substituting from (3.21) and (3.22) gives

$$\hat{\xi}(s) - \hat{\xi}'(s) = \bar{R}_1(s)\{\hat{\alpha}_T(s) - \hat{\alpha}'_T(s)\} - \bar{R}_2(s)\{\hat{\alpha}_M(s) - \hat{\alpha}'_M(s)\} \tag{3.24}$$

However, the r.h.s. of (3.24) is polynomial, thus

$$s.p.(\hat{\xi}(s) - \hat{\xi}'(s)) = 0 \tag{3.25}$$

where s.p. denotes the strictly proper part of the indicated function. The condition (3.25) is a necessary condition which two solutions $\xi(t), \xi'(t)$ of (3.14) must satisfy if they are to map onto the same element $\hat{\xi}_1(s)$, under (3.15).

Now $T(s)$ and $M(s)$ have no common infinite zeros. A necessary and sufficient condition for this is that $\exists$ proper rational matrices $R_1(s)$, $R_2(s)$ such that

$$R_1(s)\mathcal{T}(s) - R_2(s)M(s) = I, \tag{3.26}$$

Since $R_1(s)$ and $R_2(s)$ are proper they possess Laurent expansions in s^{-1} which consist only of positive powers i.e. for $i = 1, 2$

$$R_i(s) = R_{0i} + R_{1i}\frac{1}{s} - R_{2i}\frac{1}{s^2} + \ldots \tag{3.27}$$

Substituting from (3.10) and (3.27) into (3.26) then gives, on multiplying out and comparing positive powers of s

$$\begin{aligned}
s^k &: \quad R_{01}\mathcal{T}_k + R_{02}M_k = 0 \\
s^{k-1} &: \quad R_{11}\mathcal{T}_k + R_{01}\mathcal{T}_{k-1} - R_{12}M_k - R_{02}M_{k-1} = 0 \\
&\vdots \quad \cdots\cdots \\
s &: \quad R_{k-1,1}\mathcal{T}_k + R_{k-2,1}\mathcal{T}_{k-1} - \ldots - R_{01}\mathcal{T}_1 - \\
&\qquad - R_{k-1,2}M_k + R_{k-2,2}M_{k-1} + \ldots - R_{02}M_1 =
\end{aligned}$$

which may be written more concisely as

$$X_{R_1}X_T + X_{R_2}X_M = 0 \tag{3.28}$$

where

$$X_{R_i} = \begin{bmatrix} R_{0i} & 0 & \ldots & 0 \\ R_{1i} & R_{0i} & & 0 \\ \vdots & & & \vdots \\ R_{k-1,i} & R_{k-2,i} & \ldots & R_{0i} \end{bmatrix}$$

If pol denotes "polynomial part" then

$$\begin{aligned}
pol(R_i(s)[s^{k-1}I, s^{k-2}I, \ldots, I]) &= \\
&= [R_{0i}s^{k-1} - R_{1i}s^{k-2} + \ldots - R_{k-1,i}, \\
&\qquad R_{0i}s^{k-2} + R_{1i}s^{k-3} + \ldots + R_{k-2,i}, \ldots, R_{0i}] \\
&= [s^{k-1}I, s^{k-2}I, \ldots, I]X_{R_i} \qquad i = 1, 2
\end{aligned} \tag{3.29}$$

Premultiplying (3.28) by $[s^{k-1}I, s^{k-2}I, \ldots, I]$ and postmultiplying by $(\bar{\xi}(0-) - \bar{\xi}'(0-))$ yields, by virtue of (3.29), that

$$pol(R_1(s)\{\hat{\alpha}_T(s) - \hat{\alpha}'_T(s)\} + R_2(s)\{\hat{\alpha}_M(s) - \hat{\alpha}'_M(s)\}) = 0 \tag{3.30}$$

Now from (3.26), postmultiplication by $\hat{\xi}(s) - \hat{\xi}'(s)$ yields

$$\hat{\xi}(s) - \hat{\xi}'(s) = R_1(s)\mathcal{T}(s)\{\hat{\xi}(s) - \hat{\xi}'(s)\} - R_2(s)M(s)\{\hat{\xi}(s) - \hat{\xi}'(s)\}$$

Substituting from (3.21) gives

$$\hat{\xi}(s) - \hat{\xi}'(s) = R_1(s)\{\hat{\alpha}_T(s) - \hat{\alpha}'_T(s)\} - R_2(s)\{\hat{\alpha}_M(s) - \hat{\alpha}'_M(s)\}$$

Now (3.30) indicates that the polynomial part of the right hand side of this is zero and so

$$pol(\hat{\xi}(s) - \hat{\xi}'(s)) = 0 \tag{3.31}$$

It thus follows that (3.31) is the necessary condition which $\xi(s), \xi'(s)$ must satisfy if they are to map onto the same element $\xi_1(s)$ under (3.15) when $T(s)$, $M(s)$ have no common infinite zeros.

Taking (3.19), (3.25) and (3.31) together yields the necessary condition under which the relation (3.7) is an injective mapping. The sufficiency of the conditions is more simply established and the details may be found in [9].

Corollary 1. The absence of finite zeros in the polynomial matrix $[\mathcal{T}(s)^T \; M(s)^T]^T$ is a necessary and sufficient condition for the strictly proper part of the inverse map of (3.7) to be uniquely determined. In the same way the polynomial part of the inverse map of (3.7) is uniquely determined iff the polynomial matrix $\{\mathcal{T}(s)^T \; M(s)^T\}^T$ has no infinite zeros and the McMillan degree condition (3.13) is satisfied. □

Corollary 2. In the case where $M(p) = M$ a constant matrix in relation (3.7) then no initial conditions are embedded in the relation and so (3.7) always represents a map in the formal sense. □

4. Fundamental Equivalence of Polynomial Matrix Descriptions.

If $T(p)$ and $M(p)$ have no common finite or infinite zeros then (3.7) is particularly interesting. This is because the set of solutions χ_0 of the homogeneous equations (3.6) forms a vector space with dimension $f := \delta_M(T(s))$. Since (3.7) is a vector space isomorphism between χ_0 and χ_0^1, it will preserve the generalized order f. There is a subspace of χ_0 which can be defined as the *strongly controllable* subspace of the system (3.3). Such subspaces are preserved by this isomorphism.

Suppose Σ_1 is a normalized system which has χ_0^1 as its set of solutions corresponding to a given $u(t)$ for all possible initial conditions, and has an output relation such that the diagram 4.1 commutes. Then the system Σ_1 has not only the same generalized order and input-output relations as Σ but also the same controllability and observability properties. It is then reasonable to call these two systems equivalent and this is the basis of the definition

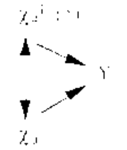

Diagram 4.1 Y_u **is the set of outputs corresponding to u(t).**

Definition 6. Let $P_1(s), P_2(s)$ be the normalized forms of $(P_1(s), P_2(s)) \in P_0 \times P_0$. $P_1(s)$ and $P_2(s)$ are said to be *fundamentally equivalent* iff the following hold

(i) $\exists$ a *bijective* map between the pseudostates $\xi_i(t)$, $i = 1, 2$ of $P_1(s)$ and $P_2(s)$.

$$\xi_2(t) = N(p)\xi_1(t) \quad (4.1)$$

and

(ii) they have the same output $y(t)$. □

It will be seen that Definition 6 is the complete extension of the definition of fundamental equivalence of Pernebo [7]. The main difference is that, this new definition additionally refers to the impulsive solution set of the system of differential equations (3.3a). It is also possible to define the notion of fundamental equivalence for PMDs in a similar way to that of [3]. More specifically consider two PMDs with the same output $y(t)$ and let $P_1(s)$ and $P_2(s)$ be their normalized forms

$$T_i(p)\xi_i(t) = \mathcal{U}_i u(t) \quad (4.4a)$$

$$y(t) = V_i\xi_i(t) \quad i = 1, 2 \quad (4.4b)$$

Following [3] and the remarks of section 2, an intuitively attractive way of defining equivalence is in terms of maps between

(i) the solution/input vector pairs

$$\begin{bmatrix} \xi_1(s) \\ u(s) \end{bmatrix} \text{ and } \begin{bmatrix} \xi_2(s) \\ u(s) \end{bmatrix} \quad (4.5)$$

and

(ii) the restricted initial condition/output vector pairs

$$\begin{bmatrix} \mathcal{X}_1\xi_1(0-) \\ y(s) \end{bmatrix} \text{ and } \begin{bmatrix} \mathcal{X}_2\xi_2(0-) \\ y(s) \end{bmatrix} \quad (4.6)$$

Assuming nonzero initial conditions, Laplace transformation of (4.4a), (4.4b) gives

$$\begin{bmatrix} T_i(s) & -\mathcal{U}_i \\ V_i & 0 \end{bmatrix} \begin{bmatrix} \xi_i(s) \\ u(s) \end{bmatrix} = \begin{bmatrix} S_{k-1}\mathcal{X}_i\tilde{\xi}_{T_i}(0-) \\ y(s) \end{bmatrix} \quad (4.7)$$

It is noted as in section 2 that the pairs referred to in (4.5) and (4.6) are fundamental to the system. The system equations themselves are in a certain sense mapping to an appropriate set of initial conditions/output pairs from an appropriate set of solution/input pairs. It is the nature of these two sets which will typically characterize a system, for the solution/input pairs (4.5) have encoded within them the controllability properties of the system, while the restricted initial condition/output pairs (4.6) encapsulate the observability properties. It would appear that to preserve the structure of the pairs in (i) and (ii) above (and hence to preserve the controllability and observability properties) it would be necessary for there to be two bijections, one relating the solution/input vector pairs and the other relating the restricted initial condition/output vector pairs. However this is not necessary, and a subsequent result shows that the following definition suffices.

Definition 7. Let $P_1(s), P_2(s)$ be the normalized forms of $(P_1(s), P_2(s)) \in P_0 \times P_0$. $P_1(s)$, $P_2(s)$ are *extended fundamentally equivalent* if $\exists$

(i) an *injective* map

$$\begin{bmatrix} \xi_2(s) \\ u(s) \end{bmatrix} = \begin{bmatrix} X(s) & Y(s) \\ 0 & I \end{bmatrix} \begin{bmatrix} \xi_1(s) \\ u(s) \end{bmatrix} \quad (4.8)$$

or equivalently from (3.6) an injective map

$$\xi_2(s) = X(s)\xi_1(s) \quad (4.9)$$

and

(ii) a *surjective* map

$$\begin{bmatrix} \mathcal{X}_2\tilde{\xi}_2(0-) \\ y(s) \end{bmatrix} = \begin{bmatrix} M & 0 \\ X & I \end{bmatrix} \begin{bmatrix} \mathcal{X}_1\tilde{\xi}_1(0-) \\ y(s) \end{bmatrix} \quad (4.10)$$

where $XV_1 = 0$. □

A theorem analogous to Lemma 1(i) holds for this equivalence.

Theorem 3. Let $P_1(s), P_2(s)$ be *extended fundamentally equivalent*. Then the maps (4.8) (or (4.9)) and (4.10) are both bijective.

Proof See [9] □

It is to be emphasized that the solutions $\xi_i(s)$ of the normalized systems (4.4) under some input $u(s)$ form a subset of the relevant space $\mathbb{R}(s)^{r}$. This subset is determined by the associated system equations and it is between these subsets that the map (4.9) is a bijection. Accordingly it follows that extended fundamental equivalence (definition 7) *implies* fundamental equivalence (definition 6). In fact more than this can be said, and the following result, whose proof can be found in [9] indicates that definitions 6 and 7 are identical notions.

Theorem 4. The PMDs of Σ_1, Σ_2 are *fundamentally equivalent* iff they are *extended fundamentally equivalent*. □

The above theorem indicates that fundamental equivalence and extended fundamental equivalence are different characterizations of the same notion of equivalence. Consequently in the sequel only the term fundamental equivalence will be attached to this notion of equivalence although either of the characterizations of definitions 6 and 7 will be used. It is noted that the usefulness of the particular characterization given in definition 7 lies in its ready indication that all controllability and observability properties of the system will be invariants under such a notion of equivalence.

It is now possible to establish the connection between the equivalence expressed in definitions 6, 7 and that of definitions 4, 5.

Theorem 5. Let $\mathcal{P}_1(s), \mathcal{P}_2(s)$ be the normalized forms (4.4) of two systems Σ_1 and Σ_2 respectively. If $\mathcal{P}_1(s)$ and $\mathcal{P}_2(s)$ are (N.F.S.E.) then they are fundamentally equivalent. □

Proof. Let $\mathcal{P}_1(s), \mathcal{P}_2(s)$ be (N.F.S.E.) then according to [5] $\exists$ polynomial matrices $N(\rho)$ and $M(\rho)$ such that

$$\begin{bmatrix} M & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \mathcal{T}_1(\rho) & -\mathcal{U}_1 \\ \mathcal{V}_1 & 0 \end{bmatrix} = \begin{bmatrix} \mathcal{T}_2(\rho) & -\mathcal{U}_2 \\ \mathcal{V}_2 & 0 \end{bmatrix} \begin{bmatrix} N & 0 \\ 0 & I \end{bmatrix} \qquad (4.11)$$

is a (F.E.) transformation. Now (N.F.S.E.) preserves the transfer function matrix and so the two systems have the same output $y(t)$. Accordingly it is only necessary to establish the condition (i) of fundamental equivalence. Postmultiplying both sides of (4.18) with $[\xi_1^T(t)\ u(t)^T]^T$ gives from (4.4) for $i = 1$.

$$\begin{bmatrix} M(\rho) & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} 0 \\ y(t) \end{bmatrix} = \begin{bmatrix} \mathcal{T}_2(\rho) & -\mathcal{U}_2 \\ \mathcal{V}_2 & 0 \end{bmatrix} \begin{bmatrix} N(\rho)\xi_1(t) \\ u(t) \end{bmatrix} \Longrightarrow$$

$$\begin{bmatrix} 0 \\ y(t) \end{bmatrix} = \begin{bmatrix} \mathcal{T}_2(\rho) & -\mathcal{U}_2 \\ \mathcal{V}_2 & 0 \end{bmatrix} \begin{bmatrix} N(\rho)\xi_1(t) \\ u(t) \end{bmatrix} \qquad (4.12)$$

and so the relation

$$\xi_2(t) = N(\rho)\xi_1(t) \qquad (4.13)$$

gives a solution of the system Σ_2. However, from the (F.E.) conditions governing (4.11)

$$\delta_M \begin{bmatrix} \mathcal{T}_1(\rho) \\ N(\rho) \end{bmatrix} = \delta_M(\mathcal{T}_1(\rho)) \qquad (4.21)$$

and so Theorem 1 implies that (4.13) is a map in the formal sense. The fact that the compound matrix

$$\begin{bmatrix} \mathcal{T}_1(\rho) \\ N(\rho) \end{bmatrix} \qquad (4.15)$$

has no finite nor infinite zeros implies, by Theorem 2, that the map (4.13) is injective and thus a monomorphism between the solution (vector) spaces $\mathcal{X}_u^1, \mathcal{X}_u^2$ resectively of the systems Σ_1 and Σ_2. In the same way the symmetry property of (N.F.S.E.) to establish a monomorphism between $\mathcal{X}_u^2, \mathcal{X}_u^1$ respectively. It then follows that the solution spaces $\mathcal{X}_u^1, \mathcal{X}_u^2$ are isomorphic and so $\exists$ a bijective map of the form (4.1), as required. □

The converse of the above theorem is also true [9], although the details are not included here.

5. Conclusions

A neat characterization of the transformation of full system equivalence has been given in terms of the existence of a bijective map between the finite and infinite solution sets of PMDs under under a fixed input $u(t)$. The characterization has enabled the true nature and role of the conditions of full system equivalence to emerge. Thus for example the McMillan degree conditions (2.6c), which have previously appeared somewhat arbitrarily attached to the transformation, are seen to be vital in a quite fundamental way. The importance of the appropriately defined restricted initial conditions of a general PMD in this new characterization has also been noted.

Fundamental system equivalence is therefore the complete mapping interpretation of the matrix transformation of (F.S.E.). Thus whereas (F.S.E.) is a closed form matrix expression of this notion of equivalence which applies to the system matrix representation (or what is the same thing, their forms), fundamental system equivalence gives a [illegible] tion in terms of bijective mapping between the s[illegible] the defining differential equations. It is concluded [illegible] the transformation of full system equivalence, wi[illegible] characterisations, is the basic transformational to[illegible] multaneous study of the finite and infinite frequen[illegible] of general linear multivariable systems.

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