Generalized Models of 2–D Linear Discrete Systems and Computation of its Transfer Function.

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Abstract.
A generalized model of 2–D linear discrete systems with constant coefficients is defined. The proposed model is the more general one and encompasses all the known 2–D linear models. An algorithm is developed for the computation of the transfer function matrix of a two dimensional system, which is given in its generalized form, without inverting a polynomial matrix.

Keywords : 2–D systems, modelling, transfer function matrix.

1. Introduction.

The two–dimensional (2–D) systems have drawn considerable attention the last years since they provide the mathematical framework for the study of 2–D digital filters which find numerous applications in image processing (medical imaging, processing of geophysical and seismic data, processing of satellite photos and video images), electrical networks with variable elements e.t.c. .

Roesser (1975), Attassi (1975), Fornasini–Marchesini (1978), J.Kurek (1985), and T.Kaczorek (1988) have presented models for (2–D) linear discrete systems with constant coefficients. The most general among them was the singular model of T.Kaczorek.

The main purpose of this paper is to present a generalized model for 2–D linear discrete systems and to establish a new algorithm for the direct computation of the transfer function of this model without inverting a polynomial matrix of two variables. This problem has been considered by Koo and Chen (1977) who extended the Fadeeva's algorithm (Zadeh and Desoer (1963)) for the inversion of the resolvent matrix sI_n-A in the 2–D case. B.G.Mertzios (1984) presented a generalized algorithm for the inversion of the generalized pencil $sE-A$ where both E and A may be singular matrices and later (1986) applied this algorithm to 2–D state space systems producing an alternative to the algorithm in Koo and Chen. Moreover, explicit expressions for the transfer function matrices of polynomial matrix descriptions (PMDs) and 2–D state–space systems in terms of the system matrices have been derived by Buslowicz (1980), Fragulis *et al.* (1991) and Mertzios *et al.* (1981, 1987, 1988). In particular Fragulis *et al.* (1991) extended the Mertzios algorithm to the case of polynomial dynamical system.

2. Generalized Model of 2–D Linear Discrete Systems.

A model described by the equations :

$$A(\rho,w)x(i,j) = B(\rho,w)u(i,j) \qquad (2.1)$$
$$y(i,j) = C(\rho,w)x(i,j) + D(\rho,w)u(i,j)$$

will be called *generalized model* (GM) of 2–D linear discrete systems, where $A(\rho,w) \in \mathbb{R}[s_1,s_2]^{n\times n}$ with $\det|A(\rho,w)|\neq 0$, $B(\rho,w) \in \mathbb{R}[s_1,s_2]^{n\times m}$, $C(\rho,w) \in \mathbb{R}[s_1,s_2]^{\ell\times n}$, $D(\rho,w) \in \mathbb{R}[s_1,s_2]^{\ell\times m}$ i,j are integer–valued vertical and horizontal coordinates respectively ; $x(i,j) \in \mathbb{R}^n$ is the local state vector at (i,j), $u(i,j) \in \mathbb{R}^m$ is the input vector, $y(i,j) \in \mathbb{R}^\ell$ is the output vector and ρ,w are backward shift operators such that :

$$\rho^k w^\ell x(i,j) = x(i+k,j+\ell) \qquad k=0,1,\ldots \quad \ell=0,1,\ldots \qquad (2.2)$$

The polynomial matrix of two variables $A(\rho,w)$ may be written as

$$A(\rho,w) = \sum_{i=0}^{q_1}\sum_{j=0}^{q_2} A_{ij}\rho^i w^j \qquad (2.3)$$

The global boundary conditions for (2.1) are given by

$$x(i,k) = x_{ik} \text{ for } i=0,1,\ldots \text{ and } k=0,1,\ldots,q_2-1$$
$$x(m,j) = x_{mj} \text{ for } m=0,1,\ldots,q_1-1 \text{ and } j=0,1,\ldots \qquad (2.4)$$

where x_{ik} and x_{mj} are known vectors. The model (2.1) may be written in the following form:

$$\begin{bmatrix} A(\rho,w) & B(\rho,w) & 0 \\ -C(\rho,w) & D(\rho,w) & I_\ell \\ 0 & -I_m & 0 \end{bmatrix} \begin{bmatrix} x(i,j) \\ -u(i,j) \\ y(i,j) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ I_m \end{bmatrix} u(i,j)$$

$$y(i,j) = [0 \ \ 0 \ \ I_\ell] \begin{bmatrix} x(i,j) \\ -u(i,j) \\ y(i,j) \end{bmatrix} \tag{2.5}$$

which will be called *normalized form* of the generalized model (2.1). A model described by the equations :

$$Ex(i+1,j+1) = A_0x(i,j)+A_1x(i+1,j) + +A_2x(i,j+1)+B_0u(i,j)+B_1u(i+1,j)+B_2u(i,j+1)$$

$$y(i,j) = Cx(i,j) + Du(i,j) \tag{2.6}$$

will be called *singular general model* and has been presented by T. Kaczorek (1988). We can easily see that this model may be rewritten as

$$(E\rho w-A_0-A_1\rho-A_2w)x(i,j) = (B_0+B_1\rho+B_2w)u(i,j)$$
$$y(i,j) = Cx(i,j) + Du(i,j) \tag{2.7}$$

So for $A(\rho,w)=E\rho w-A_0-A_1\rho-A_2w$, $B(\rho,w)=B_0+B_1\rho+B_2w$, $C(\rho,w)=C$ and $D(\rho,w)=D$ we obtain that the *singular generalized model* which has been the most general model until now is a particular model of the class of generalized models. (see T. Kaczorek 1987).

3. Calculation of the Inverse of the Polynomial Matrix $A(\rho,w)$.

Theorem 1 (Cayley Hamilton)

Consider $A \in \mathbb{R}^{n\times n}$ a constant matrix and let

$$\Delta(\lambda) = \det|\lambda I_n-A| = =\lambda^n+a_1\lambda^{n-1}+a_2\lambda^{n-2}+\cdots+a_{n-1}\lambda+a_n \tag{3.1}$$

be the characteristic polynomial of A. Then

$$\Delta(A)=A^n+a_1A^{n-1}+a_2A^{n-2}+\cdots+a_{n-1}A+a_nI_n=0 \tag{3.2}$$ □

We may replace the matrix A by the polynomial matrix with two variables $A(\rho,w)$ and introduce the following theorem :

Theorem 2

Consider a polynomial matrix with two variables $A(\rho,w) \in \mathbb{R}[s_1,s_2]^{n\times n}$ and let

$$\Delta(z)=\det|zI_n-A(\rho,w)| = = z^n+a_1(\rho,w)z^{n-1}+\cdots+a_{n-1}(\rho,w)z+a_n(\rho,w) \tag{3.3}$$

be the characteristic polynomial of $A(\rho,w)$. Then

$$\Delta(A(\rho,w))=A(\rho,w)^n+a_1(\rho,w)A(\rho,w)^{n-1}+\cdots +a_n(\rho,w)I_n=0 \tag{3.4}$$ □

Consider the generalized model of 2–D linear discrete systems (2.1). Taking the z–transform with zero initial conditions i.e. $x(i,k)=0$ for $i=0,1,\ldots$ and $k=0,1,\ldots,q_2-1$, $x(m,j)=0$ for $m=0,1,\ldots,q_1-1$ and $j=0,1,\ldots$ of (2.1), we obtain

$$A(z_1,z_2)x(z_1,z_2) = B(z_1,z_2)u(z_1,z_2)$$
$$y(z_1,z_2) = C(z_1,z_2)x(z_1,z_2)+D(z_1,z_2)u(z_1,z_2) \tag{3.5}$$

which gives rise to the transfer function matrix :

$$H(z_1,z_2) = C(z_1,z_2)A^{-1}(z_1,z_2)B(z_1,z_2) + D(z_1,z_2) \tag{3.6}$$

or equivalently according (2.5)

$$H(z_1,z_2) = \mathcal{V}\,\mathcal{T}^{-1}(z_1,z_2)\,\mathcal{U} \tag{3.7}$$

where

$$\mathcal{T}(z_1,z_2) = \begin{bmatrix} A(z_1,z_2) & B(z_1,z_2) & 0 \\ -C(z_1,z_2) & D(z_1,z_2) & I_\ell \\ 0 & -I_m & 0 \end{bmatrix} ; \quad \mathcal{U} = \begin{bmatrix} 0 \\ 0 \\ I_m \end{bmatrix}$$

$$\mathcal{V} = [0 \ \ 0 \ \ I_\ell] \tag{3.8}$$

Therefore actually the problem of computing $H(z_1,z_2)$ is reduced to the calculation of $A^{-1}(z_1,z_2)$ or equivalently of $\mathcal{T}^{-1}(z_1,z_2)$ in the most efficient way.

In the sequel a generalization of the Leverrier algorithm in polynomial matrices with two variables is proposed. More specifically if we substitute the constant matrix A by $A(z_1,z_2)$, in the known Leverrier algorithm where

$$A(z_1,z_2) = \sum_{j=0}^{q_1}\sum_{k=0}^{q_2} A_{jk}\, z_1^j z_2^k \tag{3.9}$$

the following algorithm results

Algorithm 1. (Inverse of $A(z_1, z_2)$)

Define

$$(zI_n-A(z_1,z_2))^{-1} = \frac{1}{\Delta(z)}\{R_0(z_1,z_2)z^{n-1}+\cdots +R_{n-2}(z_1,z_2)z+R_{n-1}(z_1,z_2)\} \tag{3.10}$$

where

$$\Delta(z) = \det|zI_n-A(z_1,z_2)| = =z^n+a_1(z_1,z_2)z^{n-1}+\cdots+a_n(z_1,z_2) \tag{3.11}$$

and $R_0(z_1,z_2)$, $R_1(z_1,z_2)$, .., $R_{n-1}(z_1,z_2)$ are polynomial matrices with two variables. Using the Leverrier algorithm for (3.3) we obtain :

$$R_0(z_1,z_2) = I_n$$

$$R_1(z_1,z_2) = A(z_1,z_2)R_0(z_1,z_2)+a_1(z_1,z_2)I_n = A(z_1,z_2)+a_1(z_1,z_2)I_n$$

$$R_2(z_1,z_2) = A(z_1,z_2)R_1(z_1,z_2)+a_2(z_1,z_2)I_n = A(z_1,z_2)^2+a_1(z_1,z_2)A(z_1,z_2)+a_2(z_1,z_2)I_n$$

. . . .

$$R_{n-1}(z_1,z_2)=A(z_1,z_2)R_{n-2}(z_1,z_2)+a_{n-1}(z_1,z_2)I_n = A(z_1,z_2)^{n-1}+\cdots+a_{n-1}(z_1,z_2)I_n$$

$$0 = A(z_1,z_2)R_{n-1}(z_1,z_2)+a_n(z_1,z_2)I_n = A(z_1,z_2)^n+a_1(z_1,z_2)A(z_1,z_2)^{n-1}+\cdots+a_n(z_1,z_2)I_n \quad (3.12)$$

and

$$a_1(z_1,z_2)=-\frac{1}{1}\,tr[A(z_1,z_2)R_0(z_1,z_2)]$$

$$a_2(z_1,z_2)=-\frac{1}{2}\,tr[A(z_1,z_2)R_1(z_1,z_2)]$$

$$a_3(z_1,z_2)=-\frac{1}{3}\,tr[A(z_1,z_2)R_2(z_1,z_2)]$$

$$\cdots\cdots\cdots\cdots$$

$$a_{n-1}(z_1,z_2)=-\frac{1}{n-1}\,tr[A(z_1,z_2)R_{n-1}(z_1,z_2)]$$

$$a_n(z_1,z_2)=-\frac{1}{n}\,tr[A(z_1,z_2)R_n(z_1,z_2)] \quad (3.13)$$

□

From the above algorithm, which is a generalization of the known Leverrier algorithm, we obtain that :

$$[zI_n-A(z_1,z_2)]\times [R_0(z_1,z_2)z^{n-1}+\cdots+R_{n-2}(z_1,z_2)z+R_{n-1}(z_1,z_2)] = \Delta(z)I_n=\det|zI_n-A(z_1,z_2)| \quad (3.14)$$

Therefore for $z=0$ we obtain :

$$A(z_1,z_2)R_{n-1}(z_1,z_2)=(-1)^{n-1}\det A(z_1,z_2) \quad (3.15)$$

Using (3.12) and (3.13) we also obtain that :

$$A(z_1,z_2)R_{n-1}(z_1,z_2)=-a_n(z_1,z_2)I_n \quad (3.16)$$

Equating the second–right terms of (3.15) and (3.16) we obtain that :

$$a_n(z_1,z_2)=(-1)^n\det A(z_1,z_2) \quad (3.17)$$

and

$$A^{-1}(z_1,z_2)=-\frac{R_{n-1}(z_1,z_2)}{a_n(z_1,z_2)} \quad (3.18)$$

It is seen from (3.9), (3.12) and (3.13) that $R_i(z_1,z_2)$ and $a_i(z_1,z_2)$ may be written as :

$$R_i(z_1,z_2)=\sum_{j=0}^{iq_1}\sum_{k=0}^{iq_2} R_{ijk}\, z_1^j z_2^k \quad i=0,1,\ldots,n-1 \quad (3.19)$$

and

$$a_i(z_1,z_2)=\sum_{j=0}^{iq_1}\sum_{k=0}^{iq_2} a_{ijk}\, z_1^j z_2^k \quad i=0,1,\ldots,n \quad (3.20)$$

where R_{ijk}, a_{ijk} are constant coefficient matrices and scalar of the powers $z_1^j z_2^k$. It is seen from (3.18) that for the computation of the inverse of $A(z_1,z_2)$ and therefore for the transfer function we need only the quantities $R_{n-1}(z_1,z_2)$ and $a_n(z_1,z_2)$ i.e. the coefficient matrices $R_{n-1,jk}$ and the coefficients $a_{n,jk}$ defined by :

$$R_{n-1}(z_1,z_2)=(-1)^{n-1}\,adj(A(z_1,z_2)) = \sum_{j=0}^{(n-1)q_1}\sum_{k=0}^{(n-1)q_2} R_{n-1,jk}\, z_1^j z_2^k \quad (3.21)$$

and

$$a_n(z_1,z_2)=(-1)^n\det A(z_1,z_2) = \sum_{j=0}^{nq_1}\sum_{k=0}^{nq_2} a_{n,jk}\, z_1^j z_2^k \quad (3.22)$$

Taking into account that :

$$A(z_1,z_2)\,R_i(z_1,z_2) = \left(\sum_{j=0}^{q_1}\sum_{k=0}^{q_2} A_{jk}\, z_1^j z_2^k\right)\times\left(\sum_{j=0}^{iq_1}\sum_{k=0}^{iq_2} R_{i,jk}\, z_1^j z_2^k\right) = \sum_{j=0}^{(i+1)q_1}\sum_{k=0}^{(i+1)q_2}\left(\sum_{l=0}^{j}\sum_{m=0}^{k} A_{lm} R_{i,j-l,k-m}\right) z_1^j z_2^k \quad (3.23)$$

and substituting (3.19), (3.20) and (3.23) in the recursive relations (3.12) and (3.13) , we obtain the following recursive algorithm that determine $a_{i+1,jk}$, $R_{i+1,jk}$ for $j=0,1,\ldots,(i+1)q_1$ and $k=0,1,\ldots,(i+1)q_2$

Algorithm 2. (Inverse of $A(z_1,z_2)$)

Initial Conditions

$$R_{000}=I_n \quad (3.24)$$

Boundary Conditions

$$R_{0jk}=0 \quad \forall\, j,k>0 \quad (3.25)$$

$$R_{ijk} = 0 \quad j = iq_1+1, \ldots, (n-1)q_1 \quad k = iq_2+1, \ldots, (n-1)q_2 \tag{3.26}$$

Recursive Relations for $a_i(s_1,s_2)$

$$a_{i+1,j,k} = -\frac{1}{i+1}\,\mathrm{tr}\left(\sum_{l=0}^{j}\sum_{m=0}^{k} A_{lm} R_{i,j-l,k-m}\right)$$

$$j=0,1,\ldots,(i+1)q_1 \quad k=0,1,\ldots,(i+1)q_2 \quad i=0,1,\ldots,n-1 \tag{3.27}$$

Recursive Relation for $R_i(s_1,s_2)$

$$R_{i+1,jk} = \left(\sum_{l=0}^{j}\sum_{m=0}^{k} A_{lm} R_{i,j-l,k-m}\right) + a_{i+1,jk} I_n$$

$$j=0,1,\ldots,(i+1)q_1 \quad k=0,1,\ldots,(i+1)q_2 \quad i=0,1,\ldots,n-2 \tag{3.28}$$

Terminate

$$R_{jk} = R_{n-1,jk} \quad j=0,1,..,(n-1)q_1 \quad k=0,1,\ldots,(n-1)q_2$$

$$a_{jk} = a_{n,jk} \quad j=0,1,\ldots,nq_1 \quad k=0,1,\ldots,nq_2 \tag{3.29}$$

□

It's readily seen that the inversion algorithm is a three–dimensional algorithm since it depends of three independent variables i,j,k.

Note that if the polynomial matrix has no inverse ($\det|A(z_1,z_2)|=0$), then from the formula (3.27) we obtain coefficients $a_{n,jk}$ for $j=0,1,..,nq_1$ and $k=0,1,..,nq_2$ in (3.22) equal to zero.

The formulae (3.27) and (3.28) constitute the ...nsion of the algorithm for the inversion of the ...ncil $sE-A$ (Mertzios 1984) for the proposed 2–D generalized model. Indeed the 1–D case results from algorithm 2 for $q_1=1$ and $q_2=0$ since for this case the polynomial matrix is a singular pencil $A(s)=A_1s+A_0$. Moreover the formulae (3.27) and (3.28) are reduced to the inversion algorithm for generalized dynamical systems (Fragulis *et al.* 1991), if we assume that $q_2=0$ and q_1 is an arbitrary constant; this is the case where the polynomial matrix has only one variable, $A(s)=A_0+A_1s+\cdots+A_{q_1}s^{q_1}$.

4. Computation of the transfer function matrix.

Consider the model (2.1) where

$$C(z_1,z_2) = \sum_{i=0}^{c_1}\sum_{j=0}^{c_2} C_{ij} z_1^i z_2^j \tag{4.1a}$$

$$B(z_1,z_2) = \sum_{i=0}^{b_1}\sum_{j=0}^{b_2} B_{ij} z_1^i z_2^j \tag{4.1b}$$

$$D(z_1,z_2) = \sum_{i=0}^{d_1}\sum_{j=0}^{d_2} D_{ij} z_1^i z_2^j \tag{4.1c}$$

Now substituting (4.1) and (3.18) in (3.6) we obtain that the transfer function matrix of (2.1), as follows

$$H(z_1,z_2) = \frac{-1}{a_n(z_1,z_2)} \times$$

$$\times \sum_{j=0}^{m_1}\sum_{k=0}^{m_2}\left[\left(\sum_{l=0}^{j}\sum_{m=0}^{k} F_{lm} B_{j-l,k-m}\right) + D_{jk}\right] z_1^j z_2^k \tag{4.2}$$

where $m_1=(n-1)q_1+c_1+b_1$, $m_2=(n-1)q_2+c_2+b_2$ and

$$F_{ij} = \sum_{l=0}^{i}\sum_{m=0}^{j} C_{lm} R_{n-1,i-l,j-m} \tag{4.3}$$

The procedure for the computation of the inverse of a polynomial matrix in two variables provides the means for the computation of the inversion of square rational matrices.

Let $T(z_1,z_2)\in\mathbb{R}(z_1,z_2)^{n\times n}$ then $T(z_1,z_2)$ may be written in the following form

$$T(z_1,z_2) = \frac{A(z_1,z_2)}{d(z_1,z_2)} \tag{4.4}$$

where $d(z_1,z_2)$ is the least common denominator of all rational functions of $T(z_1,z_2)$ and $A(z_1,z_2)$ is a two–variable polynomial matrix. Therefore

$$T^{-1}(z_1,z_2) = -d(z_1,z_2)\frac{R_{n-1}(z_1,z_2)}{a_n(z_1,z_2)} \tag{4.5}$$

where

$$A^{-1}(z_1,z_2) = -\frac{R_{n-1}(z_1,z_2)}{a_n(z_1,z_2)} \tag{4.6}$$

5. Example.

Let a generalized 2–D linear discrete system have the form (2.1), or under z–transforms have the form (3.5) where

$$C(z_1,z_2)=\begin{bmatrix} z_1 & 0 & 1 \\ 0 & z_1-z_2 & 0 \end{bmatrix} ;\ B(z_1,z_2)=\begin{bmatrix} z_1 \\ z_2 \\ 1 \end{bmatrix} \quad (5.1a)$$

$$A(z_1,z_2)=\begin{bmatrix} z_1 & z_1^2 & 0 \\ z_1-z_2 & 0 & z_2 \\ 0 & z_2^2 & z_1z_2 \end{bmatrix} ;\ D(z_1,z_2)=0 \quad (5.1b)$$

For this system we have $q_1=2$, $q_2=2$, $c_1=c_2=1$, $b_1=b_2=1$ and $n=3$. We evaluate $R_2(z_1,z_2)$ and $a_3(z_1,z_2)$ via the presented algorithm in order to determine $A^{-1}(z_1,z_2)$

$$R_2(z_1,z_2)=\sum_{j=0}^{6}\sum_{k=0}^{6} R_{2,jk}z_1^jz_2^k=$$

$$=\begin{bmatrix} -z_2^3 & -z_1^3z_2 & z_1^2z_2 \\ z_1z_2^2-z_1^2z_2 & z_1^2z_2 & -z_1z_2 \\ z_1z_2^2-z_2^3 & -z_1z_2^2 & -z_1^3+z_1^2z_2 \end{bmatrix} \quad (5.2)$$

and

$$a_3(z_1,z_2)=\sum_{j=0}^{6}\sum_{k=0}^{6} a_{3,jk}\,z_1^jz_2^k=z_1z_2^3-z_1^3z_2^2+z_1^4z_2 \quad (5.3)$$

Finally, it is found that

$$A(z_1,z_2)^{-1}=\frac{1}{z_1z_2^3 - z_1^3z_2^2 + z_1^4z_2}\times$$

$$\times\begin{bmatrix} z_2^3 & z_1^3z_2 & -z_1^2z_2 \\ -z_1z_2^2+z_1^2z_2 & -z_1^2z_2 & z_1z_2 \\ -z_1z_2^2+z_2^3 & z_1z_2^2 & z_1^3-z_1^2z_2 \end{bmatrix} \quad (5.4)$$

and thus from (4.3) we obtain finally that

$$H(z_1,z_2)=\frac{1}{z_1z_2^3 - z_1^3z_2^2 + z_1^4z_2}\times$$

$$\left\{\begin{bmatrix} z_1^4z_2^2-z_1^3z_2+z_1^3 \\ z_1^4z_2-3z_1^3z_2^2 \end{bmatrix}+\begin{bmatrix} z_1^2z_2^3-z_1^2z_2^2-z_1^2z_2+2z_2^3 \\ 2z_1^2z_2^3+z_1^2z_2-z_1z_2^2 \end{bmatrix}\right\} \quad (5.5)$$

6. Conclusions

Generalized models of 2–D linear discrete systems has defined which encompasses all the known models of 2–D linear discrete systems. A recursive algorithm for the computation of the inverse of a two–variable polynomial matrix is presented. Also the transfer function of 2–D generalized dynamical systems have also been computed. The proposed algorithm may be used for the derivation of the Laurent expansion of the inverse of two–variable polynomial matrices.

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