On the Behaviour of Discrete Time AR Representations.

by

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Abstract. The behaviour of discrete time autoregressive (AR) representations (i.e. singular homogeneous discrete systems of linear difference equations) is determined in relation to the structural invariants of the polynomial matrix which describe the autoregressive representation.

Keyword : autoregressive representation, discrete homogeneous systems, behaviour, structural invariants.

1. **Introduction.** Consider a system of linear homogeneous difference (differential) and algebraic equations described in matrix form by :

$$R(\sigma)\, w(t) = 0 \qquad (1.1)$$

where σ is the backwards shift (differential) operator i.e. $\sigma^i w(t)=w(t+i)$ ($\sigma=d/dt$), $R(\sigma)=R_0+R_1\sigma+ + \dots + R_k\sigma^k \in \mathbb{R}[\sigma]^{p\times m}$ with $\mathrm{rank}_{\mathbb{R}(\sigma)}R(\sigma)=r$ and $w(t) : \mathbb{Z}^+ \longrightarrow \mathbb{R}^m$ ($w(t) : \mathbb{R} \longrightarrow \mathbb{R}^m$). Following the terminology of Willems [8] we call the set of equations (1.1) an *AR representation* (*AutoRegressive representation*) of $\mathcal{B}$ (*behaviour*), where $\mathcal{B}$ is the *solution vector space* of equations (1.1) defined by

$$\mathcal{B}(R) := \{w(t) : \mathbb{Z}^+ \longrightarrow \mathbb{R}^m \ (w(t) : \mathbb{R} \longrightarrow \mathbb{R}^m) \mid (AR) \text{ is satisfied } \forall t\in\mathbb{Z}^+ \ (\forall t\in\mathbb{R}) \}$$

In the case when $p=m=r$ i.e. $R(\sigma)$ is a square and nonsingular polynomial matrix (*regular*), $\sigma=d/dt$ is the differential operator and $w(t) : [0-, +\infty) \longrightarrow \mathbb{R}^m$ i.e.

$$R\left(\frac{d}{dt}\right) w(t) = 0 \qquad (1.2)$$

the solution vector space of the homogeneous matrix differential equations (1.2) has been studied by many authors among them Gantmacher [4], Gohberg et.al. [3], Verghese [7], Vardulakis [6] e.t.c. . Through these studies it has been shown that if $w(t) : [0+, \infty) \longrightarrow \mathbb{R}^m$ then the solution vector space $\mathcal{B}(R)$ has dimension equal to the sum of the finite zeros of $R(\sigma)$, while when $w(t)\in[0-, +\infty) \longrightarrow \mathbb{R}^m$ then the solution vector space $\mathcal{B}(R)$ is the direct sum of two vector spaces known as the *smooth* or finite and *impulsive* or infinite solution spaces, each vector space related respectively with the finite and infinite frequency behaviour of $R(\sigma)$ and has dimension equal to the sum of the finite and infinite zeros of $R(\sigma)$ (order accounted for). However no similar results for the above cases has been obtained until now for the discrete analog case i.e. σ is the backwards shift operator $\sigma\, w(t) = w(t+1)$.

In system theory we need sometimes descriptions of dynamical systems where there is no distinction between inputs and outputs i.e. interconnection of systems. In such cases the model (1.1) with $R(\sigma)$ *singular* i.e. $p\neq m$ or $p=m$ with $\det|R(\sigma)|\neq 0$, is very useful [1], [5], [8] and [9]. An open problem noted by Willems [8] concerning those singular systems of the form (1.1) is the investigation of their behaviours i.e. which is the Ker $R(\sigma)$. In this paper an extension in two separate senses of the above ideas of the regular case is presented. The first extension is the study of the solution vector space of *discrete regular* AR representations when $w(t) : \mathbb{Z}^+ \longrightarrow \mathbb{R}^m$ i.e. $R(\sigma)$ is a regular polynomial matrix. The second extension concerns the study of the solution vector space $\mathcal{B}(R)$ of *discrete singular* AR representations of the form (1.1) when $w(t) : \mathbb{Z}^+ \longrightarrow \mathbb{R}^m$ i.e. $R(\sigma)$ is a singular polynomial matrix.

More specifically in section 2 following similar lines with Vardulakis [6] and Cohberg et al. [3] we develop $\mathbf{n}$ (=the total number of finite zeros of $R(\sigma)$, order accounted for) independent solutions $w(t)$ of the system (1.1) which are due to the finite zero structure of $R(\sigma)$ while in section 3 we develop $\hat{q}$ (=the total number of infinite zeros of $R(\sigma)$, order accounted for) independent polynomial solutions $w(z)$ of the following system

$$Z[R(\sigma)w(t)] = Z[0] \Leftrightarrow$$

(1.3)
$$R(z)\, w(z) = [z^k I_p \ldots z^2 I_p\ z I_p] \begin{bmatrix} R_k & 0 & \cdots & 0 \\ R_{k-1} & R_k & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ R_1 & R_2 & \cdots & R_k \end{bmatrix} \begin{bmatrix} w(0) \\ w(1) \\ \vdots \\ w(k-1) \end{bmatrix}$$

where $z[w(t)] = \sum_{i=0}^{\infty} w(i)\ z^{-i}$ under a specific set of initial conditions $\mathcal{X}_0^{\mathbf{n}}$. Our assumption that $R(\sigma)\in\mathbb{R}[s]^{p\times m}$ and $\mathrm{rank}_{\mathbb{R}(\sigma)}R(\sigma)=r$ i.e. $R(\sigma)$ is a *singular* polynomial matrix, (where not necessary $p=m=r$), provides the polynomial matrix $R(\sigma)$ with extra structural invariants i.e. left and right kernel. As we show in section 4 the role of the right kernel of $R(\sigma)$ is to provide the system (1.1) with extra $\hat{\epsilon}$ (=the total number of right minimal indices of $R(\sigma)$, order accounted for) independent solutions while the left kernel (Section 5) gives rise to $\hat{\eta}$ (=the total number of left minimal indices of $R(\sigma)$, order accounted for) linearly independent certain constraints between the initial conditions $w(0),w(1),..,w(k-1)$ which must not be satisfied so that the system (1.1) has a solution and we denote this set of initial conditions by $\mathcal{X}_0^{\eta}$. Finally in Section 6 we give conditions for the uniqueness of solution and we present the whole vector space of (1.1) and its relation with the structural invariants of the polynomial matrix $R(\sigma)$. It is shown that in case where $R(\sigma)$ is nonsingular each element of the solution vector space of (1.1) is a specific discrete vector valued function and the dimension of the vector space is $\mathbf{n}$, while in the singular case the elements of its solution vector space are equivalence classes and the dimension of the solution vector space is $f=n+\hat{\epsilon}$.

2. Finite elementary divisors and solutions of AR representations. Consider the AR representation (1.1) and let $w(0),w(1),..,w(k-1)$ be the "initial values" of $w(t)$. Let us assume that $R(\sigma)\in\mathbb{R}[s]^{p\times m}$ has ℓ distinct eigenvalues $\lambda_1,\lambda_2,\ldots,\lambda_\ell$ where for simplicity of notation we assume that $\lambda_i\in\mathbb{R}$, $i\in\ell$ and let

(2.1)
$$U_L(\sigma)\, S^{\mathbb{C}}_{R(\sigma)}\, U_R(\sigma) = \text{block diag}[1,1,..,1,f_\rho(\sigma),f_{\rho+1}(\sigma),\ldots,f_r(\sigma),0_{p-r,m-r}]$$

$1\leq\rho\leq r$ be the Smith form of $R(\sigma)$ (in $\mathbb{C}$) where $f_i(\sigma)\in\mathbb{R}[\sigma]$ are the invariant polynomials of $R(\sigma)$ and $f_i(\sigma)/f_{i+1}(\sigma)$ $i=\rho,\rho+1,\ldots,r-1$. Assume that the partial multiplicities of the eigenvalue λ_i, $i\in\ell$ are $0\leq\sigma_{i\rho}\leq\sigma_{i\rho+1}\leq\cdots\leq\sigma_{ir}$. Let $u_j(\sigma)\in\mathbb{R}[\sigma]^{m\times 1}$ $j\in r$ be the columns of $U_R(\sigma)$ and $u_j^{(q)}(\sigma) = (d^q/d\sigma^q)u_j(\sigma)$, $q=0,1,\ldots,(\sigma_{ij}-1)$. Let also

$$w^i_{jq} := \frac{1}{q!} u_j^{(q)}(\lambda_i) \qquad i\in\ell \text{ and } j=\rho,\rho+1,\dots,r$$

Define the vector valued functions

$$w^i_{jq}(t) := \begin{cases} \lambda_i^t w^i_{jq} + t\,\lambda_i^{t-1} w^i_{j,q-1} + \cdots + \begin{bmatrix} t \\ q \end{bmatrix} \lambda_i^{t-q} w^i_{j0} & \text{if } \lambda_i\neq 0 \\ w^i_{jq}\,\Delta(t) + w^i_{j,q-1}\,\Delta(t-1) + \cdots + w^i_{j0}\,\Delta(t-q) & \text{if } \lambda_i=0 \end{cases}$$

$$i\in\ell\ ;\ j=\rho,\rho+1,\dots,r\ ;\ q=0,1,\dots,\sigma_{ij}-1$$

where $\Delta(t-q)=1$ if $t=q$ and $\Delta(t-q)=0$ for $t\neq q$, is the unit pulse function. Let

$$\Psi_{ij}(t) := [w^i_{j0}(t),\ w^i_{j1}(t),\ \dots,\ w^i_{j(\sigma_{ij}-1)}(t)]$$

$$C_{ij} := [w^i_{j0},\ w^i_{j1},\dots,\ w^i_{j(\sigma_{ij}-2)},\ w^i_{j(\sigma_{ij}-1)}]\in\mathbb{R}^{r\times\sigma_{ij}}$$

$i\in\ell$, $j=\rho,\rho+1,\dots,r$ and

$$J_{ij} := \begin{bmatrix} \lambda_i & 1 & \cdots & 0 \\ 0 & \lambda_i & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ \vdots & & & 1 \\ 0 & 0 & \cdots & \lambda_i \end{bmatrix}\in\mathbb{R}^{\sigma_{ij}\times\sigma_{ij}} \qquad \begin{matrix} i\in\ell \\ j=\rho,\rho+1,\dots,r \end{matrix}$$

$$\Psi_i(t) := [\underbrace{\Psi_{i\rho}(t)}_{\leftarrow\sigma_{i\rho}\rightarrow},\ \underbrace{\Psi_{i,\rho+1}(t)}_{\leftarrow\sigma_{i,\rho+1}\rightarrow},\dots,\ \underbrace{\Psi_{ir}(t)}_{\leftarrow\sigma_{ir}\rightarrow}] \qquad i\in\ell$$

$$C_i := [C_{i\rho},\ C_{i,\rho+1}\ \dots,C_{ir}]\in\mathbb{R}^{r\times m_i} \qquad i\in\ell$$

$$J_i := \text{block diag}[J_{i\rho},\ J_{i,\rho+1},\ \dots,\ J_{ir}]\in\mathbb{R}^{m_i\times m_i} \qquad i\in\ell$$

where $m_i = \sigma_{i\rho}+\sigma_{i\rho+1}+\cdots+\sigma_{ir}$, $i\in\ell$. Finally let

$$\Psi(t) := [\Psi_1(t),\ \Psi_2(t),\dots,\Psi_\ell(t)] \qquad i\in\ell$$

$$C := [C_1,\ C_2,\dots,C_\ell]\in\mathbb{R}^{r\times n}$$

$$J := \text{block diag}[J_1,\ J_2,\ \dots,\ J_\ell]\in\mathbb{R}^{\hat{n}\times\hat{n}}$$

where

$$n := m_1 + m_2 + \cdots + m_\ell = \deg\left[\prod_{j=\rho}^{r} f_j(s)\right]$$

Then we have the following

DEFINITION 2.1. We denote by $X^{\mathbb{C}}$ the solution vector space of (1.1) which is spanned by the following vectors

$$X^{\mathbb{C}} := \langle C\,J^t\rangle = \langle\Psi_1(t),\ \Psi_2(t),\dots,\Psi_\ell(t)\rangle \subseteq \mathcal{B}$$ □

THEOREM 2.2. The dimension of $X^{\mathbb{C}}$ is equal to

$$\boxed{\dim X^{\mathbb{C}} = n := \text{total number of finite zeros of } R(\sigma)\in\mathbb{R}[\sigma]^{p\times m}}$$

In the case when $R(\sigma)$ is square and nonsingular then $X^{\mathbb{C}} \equiv \mathcal{B}$. □

An interesting result concerning the solutions of (1.1) which are due to the finite zero structure of $R(\sigma)$ is given by the following :

PROPOSITION 2.3. Let $\Psi(t)$ be a basis matrix for the solution space $X^{\mathbb{C}}$ of (1.1). If $p\geq m$, $\text{rank}_{\mathbb{R}} R_k = m$ and $\eta_i=0$ $i=r+1,..,p$ (where η_i are the left minimal indices of $R(\sigma)$) then for *every* set of initial conditions $w(q) \in \mathbb{R}^m$, $q=0,1,..,k-1$ there exist a *unique* $x(0) = [x_1(0),x_2(0),...,x_n(0)]^T \in \mathbb{R}^n$, such that

$$w(t) = \Psi(t)\, x(0)$$

is a solution of (1.1) satisfying the given initial conditions. □

3. Infinite elementary divisors and solutions of AR representations. Consider (1.1) and define the "*dual*" polynomial matrix $\tilde{R}(d)$ of $R(\sigma)$ as

$$\tilde{R}(d) := R_k + R_{k-1}d + \cdots + R_1 d^{k-1} + R_0 d^k := d^k\, R\left(\frac{1}{d}\right)$$

Let $\tilde{U}_L(d)\in\mathbb{R}(d)^{p\times p}$, $\tilde{U}_R(d)\in\mathbb{R}(d)^{m\times m}$ be rational matrices having no poles and zeros at $d=0$ and such that

$$\tilde{U}_L(d)\, \tilde{R}(d)\, \tilde{U}_R(d) = S^{0}_{\tilde{R}(d)}(d)$$

where $S^{0}_{\tilde{R}(d)}(d)$ is the Smith form of $\tilde{R}(d)$ at $d=0$. Let now $\tilde{U}_R(d) = [\tilde{u}_1(d),\tilde{u}_2(d),...,\tilde{u}_m(d)]$ where $\tilde{u}_j(d)\in\mathbb{R}(d)^{m\times 1}$ and $\tilde{u}_j^{(q)}(d)$, $\tilde{R}^{(q)}(d)$ be the qth derivatives of $\tilde{u}_j(d)$ and $\tilde{R}(d)$ with respect to d for $q=0,1,..,\mu_j-1$ i.e. $\mu_j=q_1+\hat{q}_j$ where $q_1=k$ [6] and $\hat{q}_r\geq\hat{q}_{r-1}\geq\cdots\geq\hat{q}_{\kappa+1}$ are the orders of the zeros of $R(\sigma)$ at $s=\infty$,and $j=\kappa+1,..,r$. Define

$$x_{jq} := \frac{1}{q!}\, \tilde{u}_j^{(q)}(0)$$

for $q=0,1,..,\mu_j-1$ and $j=\kappa+1,..,r$. Then for initial conditions

$$\text{(3.1)}\qquad \begin{bmatrix} w(0) \\ w(1) \\ \vdots \\ w(k-1) \end{bmatrix} = - \begin{bmatrix} x_{jq+1} \\ x_{jq+2} \\ \\ x_{jq+q_1} \end{bmatrix} \qquad \begin{matrix} q=0,1,\ldots,\hat{q}_j-1 \\ j=\kappa+1,\ldots,r \end{matrix}$$

we shall obtain respectively the linearly independent polynomial solutions of (1.3)

$$\text{(3.2)}\qquad w^{\infty}_{jq}(z) = x_{j0}\, z^{q+1} + x_{j1}\, z^{q} + \cdots + x_{jq}\, z \quad \begin{matrix} j=\kappa+1,..,r \\ q=0,1,..,\hat{q}_j-1 \end{matrix}$$

According to the natural definition of the Z—transform we must have that $w(z)=Z[w(t)]= \sum_{i=0}^{\infty} w(i)z^{-i}$, and thus the polynomial solutions of (3.2) doesn't gives rise to discrete vector solutions of the system (1.1). However for our simplicity we give the following definition.

DEFINITION 3.1. We denote by $\mathcal{X}_0^{\infty}$ the space which is spanned by the initial conditions (3.1)

$$\mathcal{X}_0^{\infty} = \left\langle \begin{bmatrix} w(0) \\ w(1) \\ \vdots \\ w(k-1) \end{bmatrix} = - \begin{bmatrix} x_{jq+1} \\ x_{jq+2} \\ \\ x_{jq+k} \end{bmatrix} \quad \begin{matrix} q=0,1,\ldots,\hat{q}_j-1 \\ j=\kappa+1,\ldots,r \end{matrix} \right\rangle$$

which gives rise to solutions of the form (3.2) and by X_z^{∞} the solution space of (1.3) which is spanned by the following vectors

$$X_z^{\infty} := < w^{\infty}_{\kappa+1,0}(z),\ldots,w^{\infty}_{\kappa+1,\hat{q}_{\kappa+1}-1}(z),\ldots,w^{\infty}_{r,\hat{q}_r-1}(z) >$$ □

THEOREM 3.2. The dimension of the solution space of (1.3) X_z^{∞}, which is due to the infinite zero structure of $R(\sigma)$ is

$$\boxed{\dim X_z^{\infty} := \hat{q} = \hat{q}_{\kappa+1}+\hat{q}_{\kappa+2}+\cdots\hat{q}_r}$$ □

COROLLARY 3.3. In the case when $R(\sigma)$ is square and nonsingular the AR representation (1.1) is incompatible under initial conditions $\bar{w}(0)=[w(0)^T, w(1)^T, \ldots, w(k-1)]^T \in \mathcal{X}_0^{\infty}$ and we shall call the set $\mathcal{X}_0^{\infty}$ as the set of *nonadmissible initial conditions* of the AR representation (1.1). □

4. Right minimal indices and solutions of AR representations. $R(\sigma) \in \mathbb{R}[\sigma]^{p\times m}$ according to our assumption has rank r and therefore the dimension of the right null space of $R(\sigma)$ is equal to $m-r$. Consider a minimal basis of the right null space of $R(\sigma)$, let

$$[\bar{u}_{r+1}(\sigma), \bar{u}_{r+2}(\sigma),\ldots,\bar{u}_m(\sigma)]$$

where

$$\bar{u}_i(\sigma) = \bar{u}_{i0} + \bar{u}_{i1}\sigma + \cdots + \bar{u}_{i\epsilon_i}\sigma^{\epsilon_i} \in \mathbb{R}[s]^{m\times 1}$$

The greatest degrees of the columns $\bar{u}_i(\sigma)$, $i=r+1,..,m$ let $\{\epsilon_{r+1}, \epsilon_{r+2}, \ldots, \epsilon_m\}$ are called *right minimal indices* of $R(\sigma)$. For initial conditions

$$\begin{bmatrix} w(0) \\ w(1) \\ \vdots \\ w(k-1) \end{bmatrix} = \begin{bmatrix} \bar{u}_{jq} \\ \bar{u}_{jq-1} \\ \\ \bar{u}_{jq-k} \end{bmatrix} \quad \begin{matrix} q=0,1,\ldots,\epsilon_j-1 \\ j=r+1,\ldots,m \end{matrix}$$

with $\bar{u}_{jf}=0$ for $f<0$, we obtain the linearly independent solutions

$$w^{\epsilon}_{jq}(t) := \bar{u}_{jq}\,\Delta(t) + \cdots + \bar{u}_{j1}\,\Delta(t-q+1) + \bar{u}_{j0}\,\Delta(t-q) \quad q=0,1,..,\epsilon_j-1 \tag{4.1}$$

where $\Delta(t-q)=1$ for $t=q$ and $\Delta(t-q)=0$ for $t\neq q$ is the unit pulse function.

DEFINITION 4.1. We shall denote by X^{ϵ} the solution space of the AR representation (1.1) which is spanned by the following vectors :

$$X^{\epsilon} := \langle w^{\epsilon}_{r+1,0}(t),...,w^{\epsilon}_{r+1,\epsilon_{r+1}-1}(t),...,w^{\epsilon}_{m,\epsilon_m-1}(t)\rangle \subset \mathcal{B}$$ □

THEOREM 4.2. The dimension of the solution space X^{ϵ} is

$$\boxed{\dim X^{\epsilon} := \hat{\epsilon} = \epsilon_{r+1} + \epsilon_{r+2} + \cdots + \epsilon_m}$$

with $X^{\mathbb{C}} \cap X^{\epsilon} = \{0\}$. □

5. Left minimal indices and solutions of AR representations. Contrary to regular AR representations i.e. $R(\sigma)\in\mathbb{R}[\sigma]^{p\times p}$ and $\mathrm{rank}_{\mathbb{R}(\sigma)}R(\sigma) = p$, a singular AR representation of the form (1.3) (under Z–transforms) does not always have a solution or more specifically there exist some initial conditions which do not gives rise to a solution (the system (1.3) or equivalently the system (1.1) is incompatible). As we shall see in the sequel this strange behaviour is due to the left kernel of $R(\sigma)$ in (1.1). More analytically let a left minimal basis of the left kernel of $R(\sigma)$ be

$$\{v_{r+1}(\sigma),\ v_{r+2}(\sigma),...,v_p(\sigma)\}$$

where

$$v_i(\sigma) = v_{i0} + v_{i1}\sigma + \cdots + v_{i\eta_i}\sigma^{\eta_i} \in\mathbb{R}[\sigma]^{1\times p} \qquad i=r+1,..,p$$

and $\{\eta_{r+1},\ \eta_{r+2},...,\ \eta_p\}$ be the *left minimal indices* of $R(\sigma)$. Then a necessary and sufficient condition for the existence of a solution in (1.3) and consequently in (1.1) is given by the following

THEOREM 5.1. The system (1.3) has a solution $w(z)$ if and only if the following $\hat{\eta} = \eta_{r+1} + \eta_{r+2} + \cdots + \eta_p$ linearly independent conditions between $w(q)$, $q=0,1,...,k-1$, are satisfied

$$\begin{bmatrix} v_{i\eta_i} & 0 & \cdots & 0 \\ v_{i\eta_i-1} & v_{i\eta_i} & \cdots & 0 \\ \vdots & \vdots & & v_{i\eta_i} \\ v_{i1} & v_{i2} & \cdots & v_{ik} \end{bmatrix} \begin{bmatrix} R_0 & R_1 & \cdots & R_{k-1} \\ 0 & R_0 & \cdots & R_{k-2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & R_0 \end{bmatrix} \begin{bmatrix} w(0) \\ w(1) \\ \vdots \\ w(k-1) \end{bmatrix} = 0 \qquad \begin{matrix} u_{if}=0 \text{ for } f>\eta_i \\ \\ i=r+1,..,p \end{matrix} \tag{5.1}$$ □

DEFINITION 5.2 We shall denote by $\mathcal{X}^{\eta}_0$ the set of the initial conditions which doesn't satisfy satisfy (5.1) :

$$\mathcal{X}^{\eta}_0 := \langle\ \tilde{w}(0)=[w(0)^T,\ w(1)^T,...,w(k-1)^T]^T \mid \tilde{w}(0) \text{ doesn't satisfy (5.1)}\rangle$$ □

6. The whole solution space of a discrete time AR representation. In this section we show that the right null space of $R(\sigma)$ determines the uniqueness of the solutions of (1.1). Consider the minimal basis of the right kernel of $R(\sigma)$ as in section 4

$$\{\bar{u}_{r+1}(\sigma),\ \bar{u}_{r+2}(\sigma),\dots,\bar{u}_m(\sigma)\}$$

It is obvious that the following vectors

$$[\hat{u}_{r+1}(\sigma),\ \hat{u}_{r+2}(\sigma),\dots,\hat{u}_m(\sigma)] = [\bar{u}_{r+1}(\sigma),\ \bar{u}_{r+2}(\sigma),\dots,\bar{u}_m(\sigma)] \begin{bmatrix} \sigma^{\epsilon_{r+1}} & 0 & \cdots & 0 \\ 0 & \sigma^{\epsilon_{r+2}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma^{\epsilon_m} \end{bmatrix}^{-1}$$

where

$$\hat{u}_i(\sigma) = \hat{u}_{i\epsilon_i} + \hat{u}_{i\epsilon_i-1}\frac{1}{\sigma} + \cdots + \hat{u}_{i0}\frac{1}{\sigma^{\epsilon_i}}$$

constitutes a minimal MacMillan degree proper basis of the right kernel of $R(\sigma)$. Then one can prove the following

PROPOSITION 6.1. The AR representation (1.1) with initial conditions

$$\begin{bmatrix} w(0) \\ w(1) \\ \vdots \\ w(k-1) \end{bmatrix} \in \text{Kernel} \begin{bmatrix} R_{q_1} & 0 & \cdots & 0 \\ R_{q_1-1} & R_{q_1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ R_1 & R_2 & \cdots & R_{q_1} \end{bmatrix} \tag{6.1}$$

has the solution

$$w(t) = \sum_{k=0}^{t} \hat{u}_{r+1}(k) z_1(t-k) + \sum_{k=0}^{t} \hat{u}_{r+2}(k) z_2(t-k) + \cdots + \sum_{k=0}^{t} \hat{u}_m(k) z_{m-r}(t-k)$$

where $z_1(t), z_2(t), \cdots, z_{m-r}(t)$ are arbitrary discrete functions and

$$\hat{u}_i(t) := Z^{-1}[\hat{u}_i(z)] = \hat{u}_{i\epsilon_i}\,\Delta(t) + \hat{u}_{i\epsilon_i-1}\,\Delta(t-1) + \cdots + \hat{u}_{i0}\,\Delta(t-\epsilon_i)$$

where $\Delta(t-q)=1$ for $t=q$ and $\Delta(t-q)=0$ for $t\neq q$ is the unit pulse function. □

In the regular case i.e. $R(\sigma)\in\mathbb{R}[\sigma]^{p\times p}$ and $\text{rank}_{\mathbb{R}(\sigma)}R(\sigma)=p$, and if condition (6.1) is satisfied then the only solution is $w(t)=0$. A consequence of Theorem 5.1 and Proposition 6.1 is the following theorem of uniqueness of solution.

THEOREM 6.2. The AR representation (1.1) *has* a *unique* solution if and only if the following two conditions are satisfied

i) $p\geq m$ and $\text{rank}_{\mathbb{R}(\sigma)}R(\sigma)=m$ where $R(\sigma)\in\mathbb{R}[\sigma]^{p\times m}$,

ii) $\tilde{w}(0) = [w(0)^T,\ w(1)^T,\dots,w(k-1)^T]^T \notin \mathcal{X}_0^m \cup \mathcal{X}_0^\epsilon$ □

DEFINITION 6.3. The set of initial conditions $\mathcal{X}_0 = \mathcal{X}_0^{\mathbb{C}} \cup \mathcal{X}_0^{\epsilon}$ gives rise to no solution and for this reason will be called *nonadmissible conditions* of the AR representation (1.1). □

COROLLARY 6.4. It follows from Theorem 6.2 that in the case when the right kernel of $R(\sigma)$ is not the null space and $w_0(t)$ is a solution of (1.1) then

$$w(t) = w_0(t) + \sum_{k=0}^{t} \hat{u}_{r+1}(k) z_1(t-k) + \sum_{k=0}^{t} \hat{u}_{r+2}(k) z_2(t-k) + \cdots + \sum_{k=0}^{t} \hat{u}_m(k) z_{m-r}(t-k)$$

is also a solution of (1.1), where $z_1(t), z_2(t), \cdots, z_{m-r}(t)$ are arbitrary discrete functions. □

Let

$$\mathcal{X} = \{x(t) \mid x(t) := \sum_{k=0}^{t} \hat{u}_{r+1}(k) z_1(t-k) + \sum_{k=0}^{t} \hat{u}_{r+2}(k) z_2(t-k) + \cdots + \sum_{k=0}^{t} \hat{u}_m(k) z_{m-r}(t-k)\}$$

Define the following set of solutions

$$\mathcal{B}^{\mathbb{C}} = X^{\mathbb{C}} \oplus \mathcal{X} = \{w(t) \mid w(t)=w_1(t)+w_2(t) \text{ where } w_1(t) \in X^{\mathbb{C}} \text{ and } w_2(t) \in \mathcal{X}\}$$

$$\mathcal{B}^{\epsilon} = X^{\epsilon} \oplus \mathcal{X} = \{w(t) \mid w(t)=w_1(t)+w_2(t) \text{ where } w_1(t) \in X^{\epsilon} \text{ and } w_2(t) \in \mathcal{X}\}$$

where $X^{\mathbb{C}} \cap X^{\epsilon} = \{0\}$, $X^{\mathbb{C}} \cap \mathcal{X} = \{0\}$ and $X^{\epsilon} \cap \mathcal{X} = \{0\}$. It is obvious that **the spaces $\mathcal{B}^{\mathbb{C}}$ and $\bar{\mathcal{B}}^{\epsilon}$ are the solution spaces of (1.1) which are due to the finite elementary divisors and the right kernel of $R(\sigma)$ respectively.**

An important question which arises from the above is if the solution sets of the AR representation (1.1) constitutes a vector space and if the answer is yes, which is the dimension of this vector space. The behaviour of the AR representation (1.1) is defined as

$$\mathcal{B} = \{w(t) : \mathbb{Z}^{+} \longrightarrow \mathbb{R}^{m} \mid (\text{AR}) \text{ is satisfied } \forall t \in \mathbb{Z}^{+}\}$$

Define the following relation $\mathcal{R} \subset \mathcal{B} \times \mathcal{B}$ between the solutions of (1.1)

(6.2) $$\mathcal{R}(w_1(t), w_2(t)) = \{ (w_1(t), w_2(t)) \mid w_1(t) - w_2(t) \in \mathcal{X} \text{ where } w_1(t), w_2(t) \in \mathcal{B} \}$$

PROPOSITION 6.5. The relation (6.2) is an equivalence relation. □

We call an **equivalence class** of the element $w(t) \in \mathcal{B}$, and we denote this with $[w(t)]$, the set of all the elements of $\mathcal{B}$ which are equivalent to $w(t)$ or equivalently

(6.3) $$[w(t)] := \{w_1(t) \in \mathcal{B} \mid (w(t), w_1(t)) \in \mathcal{R}\} = w(t) \oplus \mathcal{X} =$$

$$= \{ w(t) + x(t) \text{ where } w(t) \in \mathcal{B} \text{ and } x(t) \in \mathcal{X} \}$$

We can see that any equivalence class of an element $w(t)$ give us the solution of (1.1) under some specific initial conditions. In case where $R(\sigma)$ has not a right kernel then every equivalence class is constituted by a unique element contrary to the singular case where to each equivalence class corresponds an arbitrary number of elements of $\mathcal{B}$. We conclude therefore that the whole solution space of the AR representation (1.1) $\mathcal{B}$, is divided into equivalence classes, which are defined by (6.3). Define now the following "*sum*" between equivalence classes of the form (6.3)

$$[w_1(t)] + [w_2(t)] := (w_1(t) + w_2(t)) \oplus \mathcal{X} =$$
$$= \{ w_1(t) + w_2(t) + x(t) \text{ where } w_1(t) \in \mathcal{B}, w_2(t) \in \mathcal{B} \text{ and } x(t) \in \mathcal{X} \}$$

and the "*product*"

$$\lambda [w(t)] := \lambda w(t) \oplus \mathcal{X} =$$
$$= \{ \lambda w(t) + x(t) \text{ where } \lambda \in \mathbb{R}, w(t) \in \mathcal{B} \text{ and } x(t) \in \mathcal{X} \}$$

It is now easily to show the following

PROPOSITION 6.6. The space which is spanned by the equivalence classes which are defined in (6.3) is a vector space $\hat{\mathcal{B}}=\mathcal{B}/R$ and this is the solution vector space of the AR representation (1.1). □

Consider now the following spaces

$$\hat{\mathcal{B}}^{\mathbb{C}} := \{[w(t)] \mid w(t) \in \mathcal{B}^{\mathbb{C}}\} = \mathcal{B}^{\mathbb{C}}/R \tag{6.4}$$
$$\hat{\mathcal{B}}^{\epsilon} := \{[w(t)] \mid w(t) \in \mathcal{B}^{\epsilon}\} = \mathcal{B}^{\epsilon}/R$$

It is obvious that the above spaces partition the sets $\mathcal{B}^{\mathbb{C}}$, $\mathcal{B}^{\epsilon}$ and are vector spaces.

THEOREM 6.7. The vector space $\hat{\mathcal{B}} := \mathcal{B}/R = \{ [w(t)] \mid w(t) \in \mathcal{B}^{\mathbb{C}} \oplus \mathcal{B}^{\epsilon}\}$ has dimension

$$\boxed{\dim \hat{\mathcal{B}} = f := n + \hat{\epsilon}}$$

□

$f = \dim\hat{\mathcal{B}}$ is called *generalized order* of the AR representation (1.1).

COROLLARY 6.8. One basis of the vector space $\hat{\mathcal{B}}$ is the following :

$$\hat{\mathcal{B}} = < [w^{1}_{\rho 0}(t)], \ldots,[w^{\ell}_{r(\sigma_{\ell r}-1)}(t)], [w^{\epsilon}_{r+1,0}(t)], \ldots, [w^{\epsilon}_{m,\epsilon_m-1}(t)] >$$

where $[w]$ denotes according to (6.3) the equivalence class of the elemnet $w(t)$, the elements $w^{i}_{jq}(t)$ with $i\in\ell$, $j=\rho,\rho+1,\ldots,r$ and $q=0,1,\ldots,\sigma_{ij}-1$ has been defined in section 2 and the elements $w^{\epsilon}_{jq}(t)$ with $j=r+1,\ldots,m$ and $q=0,1,\ldots,\epsilon_j-1$ has been defined in section 4. We can see also from theorem 6.7 that

$$\hat{\mathcal{B}}^{\mathbb{C}} = < [w^{1}_{\rho 0}(t)], \ldots,[w^{\ell}_{r(\sigma_{\ell r}-1)}(t)] >$$

and

$$\hat{\mathcal{B}}^{\epsilon} = < [w^{\epsilon}_{r+1,0}(t)], \ldots, [w^{\epsilon}_{m,\epsilon_m-1}(t)] >$$

which implies that $\dim \hat{\mathcal{B}}^{\mathbb{C}}=n$ and $\dim \hat{\mathcal{B}}^{\epsilon}=\epsilon$. □

7. Conclusions.

In this paper we gave the solution space of discrete time *singular* AR representations i.e. $R(\sigma)w(t)=0$ where $R(\sigma)\in\mathbb{R}[\sigma]^{p\times m}$ and $\text{rank}_{\mathbb{R}(\sigma)}R(\sigma)=r$. It was shown that some differences distinguish the singular from the regular AR representations i.e. the infinite number of solutions or the absence of solutions. Summarizing, in sections 2,3 and 4 we obtain solution vector spaces of the system (1.1) or (1.3) which are due to the finite and infinite elementary divisors and the right kernel of a polynomial matrix while in section 5 we gave the role of the left kernel in the existence of solutions of an AR representation. In section 6 we gave conditions for the uniqueness of a solution of a discrete AR representation. We have shown also that the solution vector space (behavior $\hat{B}$) of (1.1) is composed of equivalence classes and its dimension is equal to $f=\hat{n}+\hat{\epsilon}$ i.e. is equal to the total number of zeros at $\mathbb{C}$ and the sum of the right minimal indices of $R(\sigma)$ (order accounted for). The meaning of the algebraic structure of a polynomial matrix in relation to the solution vector spaces of singular disrete AR representations has thus been elucidated.

The investigation of the solution vector space of *discrete singular* AR representations gives rise to a numerous applications as for example the solution of the zeroing—output problem, the determination of the controllable or uncontrollable and observable or unobservable subspaces of discrete polynomial matrix descriptions (PMDs) e.t.c. .

REFERENCES

[1] BLOMBERG H. and YLINEN R., *Algebraic Theory for Multivariable Linear Systems.* Mathematics in Science and Engineering, Vol.166, Academic Press, London, 1983.

[2] COPPEL W.A. and CULLEN D.J., Strong System Equivalence (II)., *J. Austral. Math. Soc., Ser. B* 27, [1985], pp.223—237.

[3] COHBERG I., LANGASTER P and RODMAN I., *Matrix Polynomials.*, Academic Press, New York, 1982.

[4] GANTMACHER F.R., *The Theory of Matrices.*, Chelsea, New York, 1971.

[5] KUIJPER M., First Order Representations of Linear Systems., Ph.D. Thesis, Center for Mathematics and Computer Science, Amsterdam, The Netherlands, 1992.

[6] VARDULAKIS A.I.G, *Linear Multivariable Control, Algebraic Analysis and Synthesis Methods*, John Wiley and Sons, 1991.

[7] VERGHESE G., *Infinite—Frequency Behaviour in Dynamical Systems*, Ph.D.Dissertation, Department of Electronic Engineering, Stanford University, 1978.

[8] WILLEMS J.C., From Time Series to Linear System — Part I. Finite Dimensional Time Invariant Systems., *Automatica*, 22, , [1986], pp.561—580.

[9] WILLEMS J.C., Paradigms and Puzzles in the Theory of Dynamical Systems., *IEEE Trans.Auto.Control*, 36, [1991], pp.259—294.