Structural Properties of Inverse Linear Systems

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Abstract.
Some of the finite and infinite properties of a square inverse system are derived here from the corresponding properties of its original system. An extension of Morf's system equivalence transformation is considered which has an additional property of preserving the infinite frequency structure of a system.

Keywords: *Square inverse systems, system transformations.*

1. Introduction. Inverse systems have long held an interest (Rosenbrock, 1970; Rosenbrock and Van Der Weiden, 1977) for the design of linear systems, since it was noted that a form of polynomial system matrix realisation of the inverse of a square invertible transfer function matrix $G(s)$ is directly derivable from a polynomial realisation of $C(s)$. The relationship between certain properties of the inverse system and the given system could then be easily derived. This relationship was described in the case of the finite frequency aspects and Kailath (1980) furthered the description. Specifically Kailath established that the the same form of equivalence which relates two given realisations of $G(s)$ is induced between the derived realisations of its inverse. In all of this work the focus was on the finite frequency aspects of the system's behaviour, typical of the conventional study of linear systems. In this paper the extension of these results to the case of the infinite frequency invariants and the so-called generalised theory of linear systems will be given.

2. Preliminaries
Consider a linear time invariant multivariable system Σ described by a polynomial matrix description (PMD)

$$\begin{aligned} A(\rho)\beta(t) &= B(\rho)u(t) \\ y(t) &= C(\rho)\beta(t) + D(\rho)u(t) \end{aligned} \qquad (2.1)$$

where $(\rho = d/dt)$ $A(\rho) \in \mathbb{R}[\rho]^{r\times r}$ with $|A(\rho)| \neq 0$, $B(\rho) \in \mathbb{R}[\rho]^{r\times m}$, $C(\rho) \in \mathbb{R}[\rho]^{p\times r}$, $D(\rho) \in \mathbb{R}[\rho]^{p\times m}$, $\beta(t) : (0-,\infty) \to \mathbb{R}^r$ the *pseudo state* of Σ, $u(t) : (0-,\infty) \to \mathbb{R}^r$ the *control input* and $y(t)$ the *output* of Σ and let

$$P(s) = \begin{bmatrix} A(s) & B(s) \\ -C(s) & D(s) \end{bmatrix} \in \mathbb{R}[s]^{(r+p)\times(r+m)} \qquad (2.2)$$

be its Rosenbrock system matrix and

$$\mathcal{P}(s) \left[\begin{array}{ccc|c} A(s) & B(s) & 0 & 0 \\ -C(s) & D(s) & I_p & 0 \\ 0 & -I_m & 0 & I_m \\ \hline 0 & 0 & -I_p & 0 \end{array}\right] =: \begin{bmatrix} \mathcal{T}(s) & \mathcal{U} \\ -\mathcal{V} & 0 \end{bmatrix} \qquad (2.3)$$

its *normalized form* (Verghese, 1979). If we take Laplace transforms in (2.1) we obtain

$$\begin{aligned} A(s)\beta(s) &= B(s)u(s) + \alpha_0(s) \\ y(s) &= C(s)\beta(s) + D(s)u(s) + \beta_0(s) \end{aligned} \qquad (2.4)$$

where $\alpha_0(s), \beta_0(s)$ are polynomial vectors, whose coefficients are determined by the initial values of $\{\beta(\cdot), u(\cdot)\}$ and their derivatives. The transfer function of the system (2.1) is

$$G(s) = C(s)A^{-1}(s)B(s) + D(s) \qquad (2.5)$$

In connection with (2.3) it is then possible to introduce (Morf 1975, Kailath 1980) an *extended* or *generalized transfer function* $H(s)$

$$\begin{bmatrix} y(s) \\ \beta(s) \\ u(s) \end{bmatrix} = H(s) \begin{bmatrix} \beta_0(s) \\ \alpha_0(s) \\ u(s) \end{bmatrix}$$

$$H(s) = \begin{bmatrix} I_p & -C(s)A(s)^{-1} & G(s) \\ 0 & A(s)^{-1} & A(s)^{-1}B(s) \\ 0 & 0 & I_m \end{bmatrix} \qquad (2.6)$$

An important feature is that $H(s)$ is invertible and the inverse $P_M(s) := H(s)^{-1}$ is a polynomial matrix such that

$$\begin{bmatrix} \beta_0(s) \\ \alpha_0(s) \\ u(s) \end{bmatrix} = P_M(s) \begin{bmatrix} y(s) \\ \beta(s) \\ u(s) \end{bmatrix} \qquad (2.7)$$

$$P_M(s) = \begin{bmatrix} I_p & -C(s) & -D(s) \\ 0 & A(s) & -B(s) \\ 0 & 0 & I_m \end{bmatrix}$$

For the various definitions of poles and zeros and decoupling zeros of a system matrix the reader is referred to Rosenbrock, 1970; Verghese (1979), but we introduce the following

Definition 1. (Karampetakis & Vardulakis 1992) The *input (output) dynamical indices* of Σ are the right (left) minimal indices of the compound matrix $[\mathcal{T}(s)\ \mathcal{U}]\left[\begin{pmatrix}\mathcal{T}(s)\\ -\mathcal{V}\end{pmatrix}\right]$. □

Now let:

$$S^{\infty}_{\mathcal{T}(s)} = \text{diag}\ [s^{q_1}, \dots, s^{q_v}, 1, \dots, 1, \frac{1}{s^{\hat{q}_{\kappa-1}}}, \dots, \frac{1}{s^{\hat{q}_{r+p+m}}}] \quad (2.8)$$

where $q_1 \geq q_2 \geq \dots \geq q_v > 0$ and $\hat{q}_{r-p-m} \geq \dots \geq \hat{q}_{\kappa-2} \geq \hat{q}_{\kappa-1} > 0$ are respectively the orders of the poles and zeros at $s = \infty$ of $\mathcal{T}(s)$. Then we have the following:

Definition 2. (Verghese 1979) The *order* n of Σ is defined as the total number of finite system poles of Σ or equivalently as $n := \deg \mathcal{T}(s) \equiv \deg T(s)$. The *generalized order* f of Σ is defined as the total number of system poles of Σ in $\mathbb{C} \cup \{\infty\}$ or equivalently as the total number of zeros of $\mathcal{T}(s)$ in $\mathbb{C} \cup \{\infty\}$ i.e.

$$f := n + \sum_{i=\kappa-1}^{r-p-m} \hat{q}_i(\mathcal{T}(s)) = \delta_M(\mathcal{T}(s)) \quad (2.9)$$

where δ_M denotes McMillan degree □

A system matrix transformation with many important system theory implications is the following (Hayton et.al. 1990).

Definition 3. Two Rosenbrock system matrices $P_1(s), P_2(s)$ are said to be *full system equivalent* (FSE) if $\exists$ polynomial matrices $M(s), N(s), X(s)$ and $Y(s)$ such that

$$\begin{bmatrix} M(s) & 0 \\ X(s) & I \end{bmatrix}\begin{bmatrix} A_1(s) & B_1(s) \\ -C_1(s) & D_1(s)\end{bmatrix} = \begin{bmatrix} A_2(s) & B_2(s) \\ -C_2(s) & D_2(s)\end{bmatrix}\begin{bmatrix} N(s) & Y(s) \\ 0 & I\end{bmatrix} \quad (2.10)$$

where the compound matrices

$$\begin{bmatrix} M(s) & 0 & A_2(s) & B_2(s) \\ X(s) & I & -C_2(s) & D_2(s)\end{bmatrix} \text{and} \begin{bmatrix} A_1(s) & B_1(s) \\ -C_1(s) & D_1(s) \\ -N(s) & -Y(s) \\ 0 & -I\end{bmatrix} \quad (2.11)$$

satisfy the following conditions:

(i) they have full normal rank (2.12a)
(ii) they have no finite nor infinite zeros (2.12b)
(iii) the following McMillan degree conditions hold

$$\delta_M \begin{bmatrix} M(s) & 0 & A_2(s) & B_2(s) \\ X(s) & I & -C_2(s) & D_2(s)\end{bmatrix} = \delta_M(P_2)$$

and

$$\delta_M \begin{bmatrix} A_1(s) & B_1(s) \\ -C_1(s) & D_1(s) \\ -N(s) & -Y(s) \\ 0 & -I\end{bmatrix} = \delta_M(P_1) \quad (2.12c) \quad □$$

Some important properties of (FSE) are given by (Hayton et.al. 1990, Pugh & Shelton 1978, Karampetakis & Vardulakis 1992).

Theorem 1. Under *full system equivalence* the following are invariant:

(i) generalized order f, the order n and the Rosenbrock degree d_R,
(ii) transfer function and so the finite and infinite transmission poles and zeros,
(iii) finite and infinite system poles and zeros,
(iv) finite and infinite invariant zeros,
(v) sets of finite and infinite input (output) decoupling zeros.
(vi) set of input (output) dynamical indices. □

In the sequel we shall examine the properties of a system Σ (decoupling zeros, system poles and zeros, etc.) in relation to those of its inverse system, in the case where the transfer function matrix $G(s)$ of Σ is square and invertible. For the finite case these results have been proposed by Rosenbrock & Van Der Weiden (1977), and in what follows we extend these results to the infinite frequency aspects of the system.

3. Square Inverse Linear Systems.

Consider the linear multivariable system Σ of (2.1) and let $p = m$ i.e. the number of inputs are equal to the number of outputs. Let also the transfer function matrix $G(s) \in \mathbb{R}(s)^{p\times p}$ of Σ be invertible. Then a system matrix giving rise to $G^{-1}(s)$ is (Rosenbrock 1970, p 172)

$$P'(s) = \left[\begin{array}{cc|c} A(s) & B(s) & 0 \\ -C(s) & D(s) & -I_p \\ \hline 0 & I_p & 0\end{array}\right] \quad (3.1)$$

The system Σ' which is described by the Rosenbrock system matrix (3.1) is defined as the *inverse system* of Σ. In the present note the finite and infinite frequency behaviour of Σ is considered in relation to that of Σ'.

Theorem 2. Σ and Σ' have the same finite and infinite decoupling zeros, or equivalently the same finite and infinite input, output and input-output decoupling zeros.

Proof. The finite and infinite input decoupling zeros of Σ' are respectively the finite and infinite zeros of the compound matrix:

$$[\mathcal{T}'(s)\ \mathcal{U}'] = \left[\begin{array}{cccc|c} A(s) & B(s) & 0 & 0 & 0 \\ -C(s) & D(s) & -I_p & 0 & 0 \\ 0 & -I_p & 0 & I_p & 0 \\ 0 & 0 & -I_p & 0 & I_p\end{array}\right] \overset{s.e}{\sim} \left[\begin{array}{cc|c} \mathcal{T}(s) & \mathcal{U} & 0 \\ \hline 0 & 0 & I_p\end{array}\right] \quad (3.2)$$

formed from the normalized form system matrix of Σ'. In (3.2), s.e denotes the transformation of strict equivalence which is well known to leave invariant the finite and infinite zero structure. Thus the two systems have the same finite and infinite input decoupling zeros. By similar arguments the result for the finite and infinite output decoupling zeros follows.

Remove all the finite and infinite decoupling zeros from $\sum'$ and so from the compound matrix (3.2) to leave

$$[\mathcal{T}_1'(s)\ \mathcal{U}_1'] \overset{\text{s.e.}}{\sim} \left[\begin{array}{cc|c} \mathcal{T}_1(s) & \mathcal{U}_1 & 0 \\ \hline 0 & 0 & I_p \end{array}\right] \quad (3.3)$$

where $[\mathcal{T}_1(s)\ \mathcal{U}_1]$. is the result of removing all finite and infinite input decoupling zeros from $[\mathcal{T}(s)\ \mathcal{U}]$. Then consider

$$\begin{bmatrix} \mathcal{T}_1'(s) \\ \mathcal{V}' \end{bmatrix} \quad (3.4)$$

The finite and infinite input-output decoupling zeros of $\sum'$ are the finite and infinite zeros of $[\mathcal{T}'(s)^T\ \mathcal{V}^T]^T$ or equivalently $[\mathcal{T}(s)^T\ \mathcal{V}^T]^T$, which are not zeros of the compound matrix (3.4), or equivalently $[\mathcal{T}_1(s)^T\ \mathcal{V}^T]^T$. Hence $\sum$ and $\sum'$ have the same finite and infinite input-output decoupling zeros. □

It can be easily seen from (3.2) that the compound matrices $[\mathcal{T}'(s)\ \mathcal{U}']$ and $[\mathcal{T}(s)\ \mathcal{U}]$ are strict equivalent (s.e.) and therefore have the same finite and infinite zeros and the same right and left minimal indices. Thus

Corollary 1. The systems $\sum$ and $\sum'$ have the same input and output dynamical indices. □

Theorem 3. (Pugh and Ratcliffe 1979) If $G(s)$ is square and invertible then the finite (resp. infinite) poles of $G(s)$ are the finite (resp. infinite)zeros of $G^{-1}(s)$, and vice versa. □

A slight restatement of this result gives

Corollary 2. The finite (resp. infinite) transmission zeros of $\sum'$ are the finite (resp. infinite) transmission poles of $\sum$ and the finite (resp. infinite) transmission poles of $\sum'$ are the finite (resp. infinite) transmission zeros of $\sum$. □

These results may then be extended to the system matrix representations of $\sum$ and $\sum'$ as follows

Theorem 4. The set of finite (resp. infinite) system zeros of $\sum'$ coincide with the set of finite (resp. infinite) system poles of $\sum$ and the set of finite (resp. infinite) system poles of $\sum'$ coincide with the set of finite (resp. infinite) system zeros of $\sum$.

Proof. We have that

$$\{\text{system zeros in } \mathbb{C}\cup\{\infty\} \text{ of } \textstyle\sum'\}$$

$$\equiv \{\text{zeros of } G^{-1}(s) \text{ in } \mathbb{C}\cup\{\infty\}\}$$

$$+ \{\text{decoupling zeros of } \textstyle\sum' \text{ in } \mathbb{C}\cup\{\infty\}\}$$

$$= \{\text{poles of } G(s) \text{ in } \mathbb{C}\cup\{\infty\}\}$$

$$+ \{\text{decoupling zeros in } \mathbb{C}\cup\{\infty\}\}$$

by Theorem 2 and Corollary 2. This latter is simply

$$\{\text{system poles in } \mathbb{C}\cup\{\infty\} \text{ of } \textstyle\sum\}$$

Under the similar arguments we obtain the second result. □

Lemma 1. (Varduiakis 1991) If $G(s)$ is square and nonsinguiar i.e. $G(s) \in \mathbb{R}[s]^{p\times p}$, $\text{rank}_{\mathbb{R}(s)}G(s) = p$, then

$$\{\text{total number of poles of } G(s) \text{ in } \mathbb{C}\cup\{\infty\}\}$$

$$(\text{orders accounted for})$$

$$=\{\text{total number of zeros of } G(s) \text{ in } \mathbb{C}\cup\{\infty\}\}$$

$$(\text{orders accounted for})$$

□

Theorem 5. The systems $\sum$ and $\sum'$ have the same generalized order f, complexity c and Rosenbrock degree d_R (Rosenbrock, 1974).

Proof. If we denote with $\#\{\cdot\}$ the total number of the quantities of the specific set, counted according to degree and multiplicity, then by Theorem 2, Corollary 2 and Lemma 1,

$$
\begin{aligned}
f_{\Sigma'} &= \#\{\text{zeros of } \mathcal{T}'(s) \text{ in } \mathbb{C}\cup\{\infty\}\} \\
&= \#\{\text{system poles of } \Sigma' \text{ in } \mathbb{C}\cup\{\infty\}\} \\
&= \#\{\text{poles of } G^{-1}(s) \text{ in } \mathbb{C}\cup\{\infty\}\} \\
&- \#\{\text{decoupling zeros of } \Sigma' \text{ in } \mathbb{C}\cup\{\infty\}\} \\
&= \quad \#\{\text{zeros of } G(s) \text{ in } \mathbb{C}\cup\{\infty\}\}- \\
&- \#\{\text{decoupling zeros of } \Sigma \text{ in } \mathbb{C}\cup\{\infty\}\} \\
&= \quad \#\{\text{poles of } G(s) \text{ in } \mathbb{C}\cup\{\infty\}\} \\
&- \#\{\text{decoupling zeros of } \Sigma \text{ in } \mathbb{C}\cup\{\infty\}\} \\
&= \#\{\text{system poles of } \Sigma \text{ in } \mathbb{C}\cup\{\infty\}\} \\
&= \#\{\text{zeros of } \mathcal{T}(s) \text{ in } \mathbb{C}\cup\{\infty\}\} \\
&= f_\Sigma
\end{aligned}
$$

Now c and the d_R of Σ and Σ' are the same according to Rosenbrock and Van Der Weiden (1977) and so the theorem is proved. □

Note that in contrast to the generalized order of Σ' which coincides with that of Σ, the order n of Σ' doesn't always coincide as we can see the the following:

Example 1. Let a Rosenbrock system matrix of a system Σ be:

$$P(s) = \left[\begin{array}{c|c} s^2+5s+6 & s+1 \\ \hline 2s-5 & 3s+2 \end{array}\right] \quad (\text{E.1})$$

with order $n_\Sigma = \deg|s^2+5s+6| = 2$. The Rosenbrock system matrix of the inverse system Σ' is

$$P'(s) = \left[\begin{array}{cc|c} s^2+5s+6 & s+1 & 0 \\ 2s-5 & 3s+2 & -1 \\ \hline 0 & 1 & 0 \end{array}\right] \quad (\text{E.2})$$

with order $n'_\Sigma = \deg|P(s)| = 3 \neq 2 = n_\Sigma$. However, one can easily see the $f'_\Sigma = 3 = f_\Sigma$ which verifies the theorem. □

Any result concerning the order and the generalized order of the square inverse system Σ', will carry over to the *least order* $v(G^{-1}(s))$ and the *generalized least order* $\delta_M(G^{-1}(s))$ of Σ'. Thus we have:

Theorem 6. (Rosenbrock & Van Der Weiden 1977) $G(s)$ and $G^{-1}(s)$ have the same McMillan degree and not necessary the same least order. □

4. Equivalence of Inverse Systems.

Morf (1975), Levy et.al. (1977) proposed a transformation, called Morf System Equivalence, based on certain relations between the generalized transfer function matrices of (2.7) and is given by the following:

Definition 4. (*Morf System Equivalence*) Let Σ_1, Σ_2 be two generalized dynamical systems of the form (2.1). Σ_1 and Σ_2 are said to be *Morf System Equivalent* if there exist polynomial matrices $X(s), Y(s), K(S), L(s)$ such that

$$\begin{bmatrix} K(s) & 0 \\ X(s) & I \end{bmatrix} P_{M_1} = P_{M_2} \begin{bmatrix} I & 0 \\ Y(s) & L(s) \end{bmatrix} \qquad (4.1)$$

where P_{M_i} is defined in (2.7) and

$$\{K(s), P_2(s)\} \text{are left coprime} \qquad (4.2a)$$

$$\{P_1(s), L(s)\} \text{are right coprime} \qquad (4.2b)$$

□

It can easily be shown that the coprimeness conditions (4.2) may be equivalently applied to the complete matrices occurring in (4.1). Thus for example $K(s)$, $P_2(s)$ are left coprime iff the matrices

$$\begin{bmatrix} K(s) & 0 \\ X(s) & I \end{bmatrix}, \ P_{M_2}$$

are left coprime.

An important result of Morf System Equivalence is that it gives rise to the same equivalence classes as that given by Fuhrmann's (1977) definition of system equivalence and so preserves the finite zero structure of equivalent systems. However, if we are interested in the preservation of the infinite zero structure of the system also, we need an extension of the above transformation. A transformation which has the specific requirements, as we shall see in the sequel, is the following

Definition 5. (*Generalized Morf System Equivalence*) Let Σ_1, Σ_2 be two generalized dynamical systems of the form (2.1). Σ_1 and Σ_2 are said to be *Generalized Morf System Equivalent* (GMSE) if there exist polynomial matrices $X(s), Y(s), K(s), L(s)$ such that

$$\begin{bmatrix} K(s) & 0 \\ X(s) & I \end{bmatrix} P_{M_1} = P_{M_2} \begin{bmatrix} I & 0 \\ Y(s) & L(s) \end{bmatrix} \qquad (4.3)$$

where the compound matrices

$$\left[\begin{array}{cc|c} K(s) & 0 & \\ & & P_{M_2} \\ X(s) & I & \end{array} \right] \text{ and } \left[\begin{array}{cc} \multicolumn{2}{c}{P_{M_1}} \\ \hline -I & 0 \\ -Y(s) & -L(s) \end{array} \right] \qquad (4.4)$$

satisfy the conditions of FSE. □

It is proved (Walker 1988) that definition 3 defines an equivalence relation in the set of polynomial system matrices P_0. If $(P_1(s), P_2(s)) \in P_0 \times P_0$ are FSE then we write $(P_1(s), P_2(s)) \in \Gamma_{FSE}$. Let Γ_{GMSE} be the subset of $P_0 \times P_0$ such that if $(P_1(s), P_2(s)) \in \Gamma_{GMSE}$ then $P_1(s)$ and $P_2(s)$ are Generalized Morf System Equivalent. An important theorem, which gives the coincidence of the two equivalence classes of *Full System Equivalence* and *Generalized Morf System Equivalence* respectively is the following

Theorem 7. Let Σ_1 and Σ_2 be respectively two linear time invariant multivariable systems. Then

$$\Sigma_1 \overset{(FSE)}{\sim} \Sigma_2 \Leftrightarrow \Sigma_1 \overset{(GMSE)}{\sim} \Sigma_2$$

Proof.

($\Rightarrow$) Let Σ_1 and Σ_2 be (FSE), then there exist polynomial matrices $M(s), N(s), X(s)$ and $Y(s)$ such that

$$\begin{bmatrix} M(s) & 0 \\ X(s) & I \end{bmatrix} \begin{bmatrix} A_1(s) & B_1(s) \\ -C_1(s) & D_1(s) \end{bmatrix} = \begin{bmatrix} A_2(s) & B_2(s) \\ -C_2(s) & D_2(s) \end{bmatrix} \begin{bmatrix} N(s) & Y(s) \\ 0 & I \end{bmatrix} \qquad (4.5)$$

where (4.5) satisfies the conditions (2.12), or equivalently that

$$\left[\begin{array}{cc|c} M(s) & 0 & 0 \\ X(s) & I & 0 \\ \hline 0 & 0 & I \end{array} \right] \begin{bmatrix} A_1(s) & B_1(s) & 0 \\ -C_1(s) & D_1(s) & I \\ 0 & I & 0 \end{bmatrix} = \begin{bmatrix} A_2(s) & B_2(s) & 0 \\ -C_2(s) & D_2(s) & I \\ 0 & I & 0 \end{bmatrix} \left[\begin{array}{cc|c} N(s) & Y(s) & 0 \\ 0 & I & 0 \\ \hline 0 & 0 & I \end{array} \right] \qquad (4.6)$$

where (4.6) now satisfies the conditions (2.12). However, it can be easily seen that

$$\begin{bmatrix} A_1(s) & B_1(s) & 0 \\ -C_1(s) & D_1(s) & I \\ 0 & I & 0 \end{bmatrix} = \begin{bmatrix} 0 & I & 0 \\ I & 0 & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} I & -C_1(s) & -D(s) \\ 0 & A_1(s) & -B_1(s) \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} 0 & 0 & I \\ I & 0 & 0 \\ 0 & -I & 0 \end{bmatrix} \qquad (4.7)$$

satisfies the conditions (2.12) because row or column permutations are operations which do not alter any of these conditions. In view of (4.6) and (4.7) we have that

$$\begin{bmatrix} I & X(s) & 0 \\ 0 & M(s) & 0 \\ \hline 0 & 0 & I \end{bmatrix} \begin{bmatrix} I & -C_1(s) & -D(s) \\ 0 & A_1(s) & -B_1(s) \\ 0 & 0 & I \end{bmatrix} = \begin{bmatrix} I & C_s(s) & -D_2(s) \\ 0 & A_2(s) & -B_2(s) \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} I & 0 & 0 \\ \hline 0 & N(s) & -Y(s) \\ 0 & 0 & I \end{bmatrix} \tag{4.8}$$

is a (FSE) transformation and so the two systems Σ_1 and Σ_2 are (GMSE).

($\Leftarrow$) Let Σ_1 and Σ_2 be (GMSE), then there exist polynomial matrices $M(s), N(s), X(s)$ and $Y(s)$ such that

$$\begin{bmatrix} M(s) & 0 \\ X(s) & I \end{bmatrix} P_{M_1} = P_{M_2} \begin{bmatrix} N(s) & Y(s) \\ 0 & I \end{bmatrix} \Rightarrow$$

$$\begin{bmatrix} M_{11} & M_{12} & 0 & I & -C_2 & -D_2 \\ M_{21} & M_{22} & 0 & 0 & A_2 & -B_2 \\ X_1 & X_2 & I & 0 & 0 & I \end{bmatrix} \begin{bmatrix} I & -C_1 & -D_1 \\ 0 & A_1 & -B_1 \\ 0 & 0 & I \\ -I & 0 & 0 \\ -Y_1 & -N_{11} & -N_{12} \\ -Y_2 & -N_{21} & -N_{22} \end{bmatrix} = 0 \tag{4.9}$$

where (4.9) satisfies the conditions (2.12 a. b. c) of FSE. From the McMillan degree conditions (2.12c) it follows (since the McMillan degree of a polynomial matrix is the highest degree of minors of all orders of that matrix) that the matrices X_1, X_2 and Y_1, Y_2 are at most constant matrices. It follows that the matrices

$$\begin{bmatrix} I & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & I & 0 & 0 \\ -Y_1 & 0 & 0 & 0 & I & 0 \\ -X_1 & -X_2 & 0 & 0 & 0 & -I \end{bmatrix} \text{ and } \begin{bmatrix} I & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & I & 0 & 0 \\ -Y_1 & 0 & 0 & 0 & I & 0 \\ X_1 & -X_2 & 0 & 0 & 0 & -I \end{bmatrix} \tag{4.10}$$

are such that the one is the inverse of the other. Consider the following equations taken from (4.9)

$$\begin{aligned} (1,1):&\quad M_{11} - I + C_2Y_1 + D_2Y_2 = 0 \\ (2,1):&\quad M_{21} - A_2Y_1 + B_2Y_2 = 0 \\ (3,1):&\quad X_1 - Y_2 = 0 \\ (3,2):&\quad -X_1C_1 + X_2A_1 - N_{21} = 0 \\ (3,3):&\quad -X_1D_1 - X_2B_1 + I - N_{22} = 0 \end{aligned} \tag{4.11}$$

In view of (4.11), internally recoordinating (Hayton et.al. 1990) the transformation (4.9) by means of the matrices (4.10) yields

$$\begin{bmatrix} I & M_{12} + D_2X_2 & 0 & I & -C_2 & -D_2 \\ 0 & M_{22} + B_2X_2 & 0 & 0 & A_2 & -B_2 \\ 0 & 0 & I & 0 & 0 & I \end{bmatrix} \begin{bmatrix} I & -C_1 & -D_1 \\ 0 & A_1 & -B_1 \\ 0 & 0 & I \\ -I & 0 & 0 \\ 0 & -Y_1C_1 - N_{11} & Y_1D_1 + N_{12} \\ 0 & 0 & -I \end{bmatrix} = 0 \tag{4.12}$$

In addition since the matrices (4.10) are constant and nonsingular, the compound matrices in (4.12) still satisfy all the requirements of Definition 5. Hence

$$\begin{bmatrix} I & M_{12} - D_2X_2 \\ 0 & M_{22} + B_2X_2 \end{bmatrix} \begin{bmatrix} -C_1 & D_1 \\ A_1 & B_1 \end{bmatrix} = \begin{bmatrix} -C_2 & D_2 \\ A_2 & B_2 \end{bmatrix} \begin{bmatrix} Y_1C_1 + N_{11} & -Y_1D_1 - N_{12} \\ 0 & I \end{bmatrix} \tag{4.13}$$

or equivalently under column or row permutations we obtain that

$$\begin{bmatrix} M_{22} - B_2X_2 & 0 \\ M_{12} - D_2X_2 & I \end{bmatrix} \begin{bmatrix} A_1 & B_1 \\ -C_1 & D_1 \end{bmatrix} = \begin{bmatrix} A_2 & B_2 \\ -C_2 & D_2 \end{bmatrix} \begin{bmatrix} Y_1C_1 + N_{11} & -Y_1D_1 - N_{12} \\ 0 & I \end{bmatrix} \tag{4.14}$$

is a transformation which satisfies the conditions (2.12 a, b, c) and so Σ_1 and Σ_2 are (FSE). □

We have thus proved that $\Gamma_{FSE} \equiv \Gamma_{GMSE}$ i.e that *Full System Equivalence* (FSE) is an equivalence relation which coincides with *Generalized Morf system Equivalence.* In view of this Theorem 1 yields that Generalized Morf System Equivalence has the property to leave invariant

(i) generalized order f, the order n and the Rosenbrock degree d_R,
(ii) transfer function and so the finite and infinite transmission poles and zeros,
(iii) finite and infinite system poles and zeros,
(iv) finite and infinite invariant zeros,
(v) sets of finite and infinite input (output) decoupling zeros,

vi) set of input (output) dynamical indices.

An important theorem which connects square inverse systems and the (GMSE) transformation is the following.

Theorem 8. Let Σ_1 and Σ_2 be two linear time invariant multivariable systems and Σ_1', Σ_2' are respectively the inverse system of Σ_1 and Σ_2. Then

$$\Sigma_1 \overset{(FSE)}{\sim} \Sigma_2 \Leftrightarrow \Sigma_1' \overset{(FSE)}{\sim} \Sigma_2'$$

Proof. The proof is based on the fact that P_M and P' are related by certain row and column operations (see (4.6), (4.7) and (4.9)) and that (FSE) defines the same equivalence classes as (GMSE). □

5. Conclusions.

It has been shown that the results of Roenbrock and Van Der Weiden (1977) concerning the relationship between certain invariants of a square invertible system and its inverse system can be extended to include the infinite frequency aspects. Thus for example the infinite decoupling zero structure of the inverse system is isomorphic to that of the original system, while the infinite transmission and system zeros of the inverse system are identical to the transfer function and system poles at infinity of the original system, and vice versa.

Kailath (1980) has indicated that, in the case of Fuhrmann's notion of strict system equivalence, the same form of equivalence which relates two polynomial realisations of a given $G(s)$ is induced between the derived polynomial realisations of the inverse of $G(s)$. It is established here that this relationship extends to the notion of FSE (Hayton et al., 1990) which has been established (Pugh et al., 1993) as the basic underlying notion of equivalence for the generalised study of well formed linear systems. Thus a more complete explanation of the relationship between the invariants of a square invertible system and those of its corresponding inverse emerges.

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