# On the Solution Space of Continuous Time AR Representations.

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**Abstract.**
The finite and infinite behaviour of autoregressive (AR) representations i.e. singular systems of linear homogeneous differential and algebraic equations is examined in relation with the structural invariants of the polynomial matrix A(s) which describes the autoregressive representation. It is shown that the finite and infinite zeros as well as the minimal bases of the right and left kernels of the polynomial matrix A(s) play a characteristic role in the evaluation of the vector space of solutions of an autoregressive representation.

**Keyword** : autoregressive representation, behaviour, structural invariants.

## 1. Introduction.

Consider a system of linear homogeneous differential and algebraic equations described in matrix form by:

$$A(\rho)\,\beta(t) = 0 \qquad (1.1)$$

where $\rho = d/dt$ is the differential operator, $A(\rho) = A_0 + A_1\rho + \cdots + A_k\rho^k \in \mathbb{R}[\rho]^{p\times m}$ with $\operatorname{rank}_{\mathbb{R}} A(\rho) = r$ and $\beta(t) : [0-,+\infty) \longrightarrow \mathbb{R}^m$. Following the terminology of Willems (1986) we call equation (1.1) an *AR representation* (*AutoRegressive representation*) of $\mathcal{B}$ (*behaviour*), where $\mathcal{B}$ is the distribution solution vector space of equations (1.1).

In the case when p=m=r i.e. $A(\rho)$ is a square and nonsingular polynomial matrix, the vector space of solutions of the homogeneous matrix differential equation (1.1) has been studied by many authors (Gantmacher 1971, Verghese 1978, Gohberg et.al. 1982, Anderson et al 1985, Vardulakis 1991). Through these studies it has been shown that the vector space of solutions of (1.1) is a direct sum of two vector spaces known as the *smooth* or finite and *impulsive* or infinite solution spaces of (1.1) each vector space related respectively with the finite and infinite frequency behaviour of $A(\rho)$. It has been shown also that (1.1) excibits either smooth (finite) or impulsive) (infinite) behaviour depending on the finite and infinite zero structure of $A(\rho)$ and whether a number of certain constraints related to the initial values of $\beta(t)$ and its derivatives at $t = 0-$ are satisfied. (Verghese et al 1979, Vardulakis 1991). An important result was that the dimension of the finite solution space of (1.1) is equal to the number of the finite zeros of $A(\rho)$ (Cohberg et.al. 1982) while the dimension of the infinite solution space of (1.1) is equal to the number of the infinite zeros of $A(\rho)$ (Verghese 1979, Anderson et al 1985).

In system theory however we need sometimes descriptions of dynamical systems where there is no distinction between inputs and outputs. In such cases the model (1.1) with $A(\rho)$ singular or non–square is very useful (Blomberg 1983, Willems 1986,1991, Kuijper 1992). However as noted by Willems (1991) the natural definition of the behaviour of the system (1.1) remains an open problem. In this note we give a solution of the above problem by investigating the distribution solution vector space of the system (1.1). More specifically in section 2 and 3 following similar lines with Vardulakis (1991) we investigate the solutions sets of the system (1.1) which are due to the finite and infinite zero structure of $A(\rho)$. Our assumption that $A(\rho)\in\mathbb{R}[\rho]^{p\times m}$ and $\operatorname{rank}_{\mathbb{R}} A(\rho) = r$ i.e. $A(\rho)$ is a *singular* polynomial matrix, (where not necessary p=m=r), provides the polynomial matrix $A(\rho)$ with extra structural invariants i.e. left and/or right kernels. As we show in Section 4 the role of the right kernel of $A(\rho)$ is to provide the system (1.1) with extra independent solutions while the left kernel (Section 5) gives rise to certain constraints between the initial conditions $\beta(0-),\beta^{(1)}(0-),..,\beta^{(k-1)}(0)$ which must be satisfied so that the system (1.1) has a solution. Finally in Section 6 we present the vector space of solutions of (1.1) and its relation with the structural invariants of the polynomial matrix $A(\rho)$. It is shown that in contrast to the case when $A(\rho)$ is nonsingular in which case each element of the solution vector space of (1.1) is a specific vector valued function, in the singular case the elements of its solution vector space are equivalence classes. We would like to note here that in the case when we are interested for $\mathcal{B}$={the set of infinitely continuously differentiable functions} we can always assume $A(\rho)\in\mathbb{R}[\rho]^{p\times m}$ with p>m and full row rank. However when we are interested for $\mathcal{B}$={the set of distribution vector functions} we have not the ability to reduce the AR–representation (1.1) via any left unimodular operations to a full row rank AR–representation with the same behaviour because we know (Verghese 1979) that the impulsive behaviour of a homogeneous system as in (1.1) doesn't remain invariant under unimodular transformations. This is the reason we use as $A(\rho)$ an arbitrary polynomial matrix.

## 2. Finite elementary divisors and solutions of AR representations.

Consider the following AR representation

$$A(\rho)\beta(t) = 0 \qquad (2.1)$$

where $\rho = d/dt$ is a differential operator, $A(\rho)=A_k\rho^k+A_{k-1}\rho^{k-1}+\cdots+A_1\rho+A_0\in\mathbb{R}[\rho]^{p\times m}$ with $\mathrm{rank}_{\mathbb{R}}A(\rho)=r$, $\beta(0-),\beta^{(1)}(0-),\ldots,\beta^{(k-1)}(0-)$ be the "initial conditions" of $\beta(t)$ and its k−1 derivatives at t=0− and $\beta(t):(0-,+\infty)\longrightarrow\mathbb{R}^m$ i.e. an m–dimensional vector valued function which is to be found. Assume that $\beta(t)$ belongs to the space of infinitely continuously differentiable functions so that $\beta^{(q)}(0-)=\beta^{(q)}(0+)=:\beta^{(q)}(0)$, q=0,1,2,.. and where $\beta^{(q)}(t)$ denotes the qth derivative of $\beta(t)$ with respect to t.

Let us assume that $A(s)\in\mathbb{R}[s]^{p\times m}$ has $\ell$ distinct eigenvalues $\lambda_1, \lambda_2, \ldots,\lambda_\ell$ where for simplicity of notation we assume that $\lambda_i\in\mathbb{R}$, $i\in\ell$ and let

$$U_L(s)\, S^{\mathbb{C}}_{A(s)}\, U_R(s) =$$

$$\text{block diag}[1,1,..,1,f_\mu(s),f_{\mu+1}(s),\ldots,f_r(s),0_{p-r,m-r}] \qquad (2.2)$$

$1\le\mu\le r$ be the Smith form of A(s) (in $\mathbb{C}$) where $f_i(s)\in\mathbb{R}[s]$ are the invariant polynomials of A(s) and $f_i(s)/f_{i+1}(s)$ $i=\mu,\mu+1,\ldots,r-1$. Assume that the partial multiplicities of the eigenvalue $\lambda_i$, $i\in\ell$ are $0\le\sigma_{i\mu}\le\sigma_{i\mu+1}\le\cdots\le\sigma_{ir}$.

Let $u_j(s)\in\mathbb{R}[s]^{m\times 1}$ $j\in r$ be the columns of $U_R(s)$ and $u_j^{(q)}(s)=(d^q/ds^q)u_j(s)$, $q=0,1,\ldots,(\sigma_{ij}-1)$. Let also

$$\beta^i_{jq} := \frac{1}{q!}\, u_j^{(q)}(\lambda_i) \qquad i\in\ell \text{ and } j=\mu,\mu+1,\ldots,r$$

Define the vector valued functions

$$\beta^i_{jq}:=\left[\frac{t^{\sigma_{ij}-1-q}}{(\sigma_{ij}-1-q)!}\beta^i_{j0}+\frac{t^{\sigma_{ij}-2-q}}{(\sigma_{ij}-2-q)!}\beta^i_{j1}+\cdots+\right.$$

$$\left.+\frac{t}{1!}\beta^i_{j(\sigma_{ij}-2-q)}+\beta^i_{j(\sigma_{ij}-1-q)}\right]e^{\lambda_i t}$$

$$i\in\ell\,;\ j=\mu,\mu+1,\ldots,r\ ;\ q=0,1,\ldots,\sigma_{ij}-1$$

Let

$$\Psi_{ij}(t) := [\beta^i_{j(\sigma_{ij}-1)}(t), \beta^i_{j(\sigma_{ij}-2)}(t),\ldots,\beta^i_{j1}(t),\beta^i_{j0}(t)]$$

$$C_{ij} := [\beta^i_{j0}, \beta^i_{j1},\ldots, \beta^i_{j(\sigma_{ij}-2)}, \beta^i_{j(\sigma_{ij}-1)}]\in\mathbb{R}^{r\times\sigma_{ij}}$$

$i\in\ell$, $j=\mu,\mu+1,\ldots,r$ and

$$J_{ij} := \begin{bmatrix} \lambda_i & 1 & \cdots & 0 \\ 0 & \lambda_i & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ \vdots & & & 1 \\ 0 & 0 & \cdots & \lambda_i \end{bmatrix}\in\mathbb{R}^{\sigma_{ij}\times\sigma_{ij}} \qquad \begin{matrix} i\in\ell \\ j=\mu,\mu+1,..,r \end{matrix}$$

$$\Psi_i(t) := [\underbrace{\Psi_{ik}(t)}_{\sigma_{ik}},\ \underbrace{\Psi_{i,k+1}(t)}_{\sigma_{i,k+1}},\ldots,\ \underbrace{\Psi_{ir}(t)}_{\sigma_{ir}}] \qquad i\in\ell$$

$$C_i := [C_{ik}, C_{i,k+1},\ldots,C_{ir}]\in\mathbb{R}^{r\times m_i} \qquad i\in\ell$$

$$J_i := \text{block diag}[J_{ik}, J_{ik+1}, \ldots, J_{ir}]\in\mathbb{R}^{m_i\times m_i} \qquad i\in\ell$$

where $m_i=\sigma_{i\mu}+\sigma_{i\mu+1}+\cdots+\sigma_{ir}$, $i\in\ell$.

Finally let

$$\Psi(t) := [\Psi_1(t), \Psi_2(t),\ldots,\Psi_l(t)] \qquad i\in\ell$$

$$C := [C_1, C_2,\ldots,C_l]\in\mathbb{R}^{r\times n}$$

$$J := \text{block diag}[J_1, J_2, \ldots, J_\ell]\in\mathbb{R}^{n\times n}$$

where

$$n := m_1+m_2+\cdots+m_l = \deg\left[\prod_{j=\mu}^{r} f_j(s)\right]$$

Then we have the following

**Theorem 1.** The columns of the following matrix

$$\Psi(t) := [\Psi_1(t), \Psi_2(t),\ldots,\Psi_l(t)] = C\,e^{Jt}$$

construct a solution space $X\subseteq\mathcal{B}$ of (2.1) with dimension

| dimX=n:=total number of finite zeros of A(s) |
|---|

□

An interesting result concerning the solutions of (2.1) which belongs to X and thus are due to the finite zero structure of A(s) is given by the following :

**Proposition 2.** Let $\Psi(t)$ be a basis matrix as above which constructs a solution space $X\subseteq\mathcal{B}$ of (2.1). If $p\ge m$, $\mathrm{rank}_{\mathbb{R}}A_k = m$ and Forney minimal order = 0 then for *every* set of initial conditions $\beta^{(q)}(0-)\in\mathbb{R}^m$, q=0,1,..,k−1 there exist a *unique* $x(0-)=[x_1(0-), x_2(0-), \ldots,x_n(0-)]^T\in\mathbb{R}^n$, such that

$$\beta(t) = \Psi(t)\,x(0-)$$

is a solution of (2.1) satisfying the given initial conditions. □

## 3. Infinite elementary divisors and solutions of AR representations.

Consider (2.1) and define the "*dual*" polynomial matrix $\tilde{A}(w)$ of A(s) as

$$\tilde{A}(w):=A_k+A_{k-1}w+\cdots+A_1w^{k-1}+A_0w^k:=w^k\,A\!\left(\frac{1}{w}\right)$$

Let $\tilde{U}_L(w) \in \mathbb{R}(w)^{p \times p}$, $\tilde{U}_R(w) \in \mathbb{R}(w)^{m \times m}$ be rational matrices having no poles and zeros at w=0 and such that

$$\tilde{U}_L(w)\, \tilde{A}(w)\, \tilde{U}_R(w) = S^{0}_{\tilde{A}(w)}(w)$$

where $S^{0}_{\tilde{A}(w)}(w)$ is the Smith Form of $\tilde{A}(w)$ at w=0. Let now $\tilde{U}_R(w) = [\tilde{u}_1(w), \tilde{u}_2(w), \ldots, \tilde{u}_m(w)]$ where $\tilde{u}_j(w) \in \mathbb{R}(w)^{m \times 1}$ and $\tilde{u}_j^{(q)}(w)$, $\tilde{A}^{(q)}(w)$ be the qth derivatives of $\tilde{u}_j(w)$ and $\tilde{A}(w)$ with respect to w for $q = 0, 1, .., \mu_j - 1$ i.e. $\mu_j = q_1 + \hat{q}_j$ where $\mathbf{q_1 = k}$ and $\hat{q}_r \geq \hat{q}_{r-1} \geq \cdots \geq \hat{q}_{\kappa+1}$ are the orders of the zeros of A(s) at $s = \infty$, and $j = \kappa+1, .., r$. Define

$$x_{jq} := \frac{1}{q!}\, \tilde{u}_j^{(q)}(0)$$

for $q = 0, 1, .., \mu_j - 1$ and $j = \kappa+1, .., r$. Then for initial conditions

$$\begin{bmatrix} \beta(0-) \\ \beta^{(1)}(0-) \\ \vdots \\ \beta^{(k-1)}(0-) \end{bmatrix} = - \begin{bmatrix} x_{jq+1} \\ x_{jq+2} \\ \vdots \\ x_{jq+q_1} \end{bmatrix} \qquad \begin{matrix} q = 0, 1, \ldots, \hat{q}_j - 1 \\ j = \kappa+1, \ldots, r \end{matrix}$$

we obtain respectively the linearly independent impulsive solutions

$$\begin{aligned} \beta^{\infty}_{j0}(t) &:= x_{j0}\delta(t) \\ \beta^{\infty}_{j1}(t) &:= x_{j0}\delta^{(1)}(t) + x_{j1}\delta(t) \\ &\cdots\cdots\cdots\cdots\cdots\cdots \qquad (3.1) \\ \beta^{\infty}_{j\hat{q}_j-1}(t) &:= x_{j0}\delta^{(\hat{q}_j-1)}(t) + \cdots + x_{j\hat{q}_j-1}\delta(t) \end{aligned}$$

**Theorem 3.** The columns of matrix

$\Psi(t) := [\Psi_{\kappa+1}(t), \Psi_{\kappa+2}(t), .., \Psi_r(t)] = [\psi^{\infty}_1(t), \psi^{\infty}_2(t), .., \psi^{\infty}_{\hat{q}}(t)]$

where

$$\Psi_j(t) := [\beta^{\infty}_{j0}(t), \beta^{\infty}_{j1}(t), \ldots, \beta^{\infty}_{j\hat{q}_j-1}(t)] \in \mathbb{R}[s]^{m \times \hat{q}_j} \qquad j = \kappa+1, .., r$$

construct a solution space $X^{\infty} \subseteq \mathcal{B}$ with dimension

$$\boxed{\dim X^{\infty} := \hat{q} = \hat{q}_{\kappa+1} + \hat{q}_{\kappa+2} + \cdots \hat{q}_r} \qquad \square$$

## 4. Right minimal indices and solutions of AR representations.

$A(s) \in \mathbb{R}[s]^{p \times m}$ according to our assumption has rank r and therefore the dimension of the right null space of A(s) is equal to **m−r**. Consider a minimal polynomial basis of the right null space of A(s), let

$$[\bar{u}_{r+1}(s), \bar{u}_{r+2}(s), \ldots, \bar{u}_m(s)]$$

The greatest degrees of the columns $\bar{u}_i(s)$, $i = r+1, .., m$ let $\{\ell_{r+1}, \ell_{r+2}, .., \ell_m\}$ are called *right minimal indices* of A(s). Let also $\tilde{A}(w)$ and $\tilde{u}_i(w)$ be the "dual" polynomials of A(s) and $\bar{u}_i(s)$. Denote $\tilde{u}_j^{(q)}(w)$ and $\tilde{A}^{(q)}(w)$ the qth order derivative of $\tilde{u}_j(w)$ and $\tilde{A}(w)$ with respect to w for $q = 0, 1, .., q_1 + \ell_j - 1$ and $j = r+1, .., m$ and define

$$y_{jq} := \frac{1}{q!}\, \tilde{u}_j^{(q)}(0)$$

for $q = 0, 1, .., q_1 + \ell_j - 1$ and $j = r+1, .., m$. Then for initial conditions

$$\begin{bmatrix} \beta(0-) \\ \beta^{(1)}(0-) \\ \vdots \\ \beta^{(k-1)}(0-) \end{bmatrix} = - \begin{bmatrix} y_{jq+1} \\ y_{jq+2} \\ \vdots \\ y_{jq+q_1} \end{bmatrix} \qquad \begin{matrix} q = 0, 1, \ldots, \ell_j - 1 \\ j = r+1, \ldots, m \end{matrix}$$

we obtain the linearly independent impulsive solutions

$$\begin{aligned} \beta^{0}_{j0}(t) &:= y_{j0}\delta(t) \\ \beta^{0}_{j1}(t) &:= y_{j0}\delta^{(1)}(t) + y_{j1}\delta(t) \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ \beta^{0}_{j\ell_j-1}(t) &:= y_{j0}\delta^{(\ell_j-1)}(t) + \cdots + y_{j\ell_j-1}\delta(t) \end{aligned}$$

**Theorem 4.** The columns of the matrix

$\Psi(t) := [\Psi_{r+1}(t), \Psi_{r+2}(t), \ldots, \Psi_m(t)] = [\psi^0_1(t), \psi^0_2(t), .., \psi^0_{\hat{\ell}}(t)]$

where

$$\Psi_j(t) := [\beta^{\infty}_{j0}(t), \beta^{\infty}_{j1}(t), \ldots, \beta^{\infty}_{j\ell_j-1}(t)] \in \mathbb{R}[s]^{m \times \hat{q}_j} \qquad j = r+1, .., m$$

construct a solution space $X^0 \subseteq \mathcal{B}$ with dimension

$$\boxed{\dim X^0 := \hat{\ell} = \ell_{r+1} + \ell_{r+2} + \cdots + \ell_m}$$

with $X^{\infty} \cap X^0 = \emptyset$. $\square$

## 5. Left minimal indices and solutions of AR representations.

Contrary to regular AR representations i.e. $A(s)\in\mathbb{R}[s]^{p\times p}$ and $\mathrm{rank}_{\mathbb{R}(s)}A(s) = p$, a singular AR representations does not always have a solution or more specifically there exist some initial conditions $\beta^{(q)}(0-)$ q=0,1,..,k which do not gives rise to a solution (the system is incompatible). As we shall see in the sequel this strange behavior is due to the left kernel of A(s). More analytically let a left minimal polynomial basis of the left kernel of A(s) be

$$\{v_{r+1}(s),\ v_{r+2}(s),\dots,v_p(s)\}$$

where

$$v_i(s) = v_{i0} + v_{i1}s + \cdots + v_{i\eta_i}s^{\eta_i} \in\mathbb{R}[s]^{1\times p} \qquad i=r+1,..,p$$

and $\{\eta_{r+1},\ \eta_{r+2},\dots,\ \eta_p\}$ be the *left minimal indices* of A(s). Then a necessary and sufficient condition for the existence of a solution is given by the following

**Theorem 5.** The AR representation (2.1) has a solution if and only if the following $\hat{\eta} := \eta_{r+1} + \eta_{r+2} + \cdots + \eta_p$ conditions between $\beta^{(q)}(0-)$, $q=0,1,..,q_1-1$ are satisfied

$$\begin{bmatrix} v_{i\eta_i} & 0 & \cdots & 0 \\ v_{i\eta_i-1} & v_{i\eta_i} & \cdots & 0 \\ \vdots & \vdots & \ddots & v_{i\eta_i} \\ v_{i1} & v_{i2} & \cdots & v_{iq_1} \end{bmatrix} \begin{bmatrix} A_0 & A_1 & \cdots & A_{q_1-1} \\ 0 & A_0 & \cdots & A_{q_1-2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_0 \end{bmatrix} \times$$

$$\times \begin{bmatrix} \beta(0-) \\ \beta^{(1)}(0-) \\ \vdots \\ \beta^{(q_1-1)}(0-) \end{bmatrix} = 0 \qquad i=r+1,..,p$$

□

## 6. The whole solution space of an AR representation.

In this section we show that the right null space of A(s) determines the uniqueness of the solutions of (4.1). Consider the minimal basis of the right kernel of A(s) as in section 4

$$\{\bar{u}_{r+1}(s),\ \bar{u}_{r+2}(s),\dots,\bar{u}_m(s)\}$$

Then one can prove the following

**Proposition 6.** The AR representation (2.1) with initial conditions

$$\begin{bmatrix} \beta(0-) \\ \beta^{(1)}(0-) \\ \vdots \\ \beta^{(q_1-1)}(0-) \end{bmatrix} \in \mathrm{Kernel} \begin{bmatrix} A_{q_1} & 0 & \cdots & 0 \\ A_{q_1-1} & A_{q_1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \\ A_1 & A_2 & \cdots & A_{q_1} \end{bmatrix} \tag{6.1}$$

has the solution

$$\beta(t) = \int_0^t \bar{u}_{r+1}(t)z_1(t-\tau)d\tau + \int_0^t \bar{u}_{r+2}(t)z_2(t-\tau)d\tau + \cdots + \int_0^t \bar{u}_m(t)z_{m-r}(t-\tau)d\tau$$

where $z_1(t)$, $z_2(t),\cdots$, $z_{m-r}(t)$ are arbitrary functions and $\bar{u}_{r+1}(s):=\mathcal{L}[\bar{u}_{r+1}(t)],..,\ \bar{u}_m(s):=\mathcal{L}[\bar{u}_m(t)]$. □

In the regular case i.e. $A(s)\in\mathbb{R}[s]^{p\times p}$ and $\mathrm{rank}_{\mathbb{R}(s)}A(s)=p$, and if condition (6.1) is satisfied then the only solution is $\beta(t)=0$. A consequence of Theorem 5 and Proposition 6 is the following theorem of uniqueness of solution.

**Theorem 7.** The AR representation (2.1) *has a unique* solution if and only if the following two conditions are satisfied

i) $p\geq m$ and $\mathrm{rank}_{\mathbb{R}(s)}A(s)=m$ where $A(s)\in\mathbb{R}[s]^{p\times m}$

and

ii) $$\begin{bmatrix} v_{i\eta_i} & 0 & \cdots & 0 \\ v_{i\eta_i-1} & v_{i\eta_i} & \cdots & 0 \\ \vdots & \vdots & \ddots & v_{i\eta_i} \\ v_{i1} & v_{i2} & \cdots & v_{iq_1} \end{bmatrix} \begin{bmatrix} A_0 & A_1 & \cdots & A_{q_1-1} \\ 0 & A_0 & \cdots & A_{q_1-2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_0 \end{bmatrix} \times$$

$$\times \begin{bmatrix} \beta(0-) \\ \beta^{(1)}(0-) \\ \vdots \\ \beta^{(q_1-1)}(0-) \end{bmatrix} = 0 \qquad i=r+1,..,p$$

□

It follows from Theorem 7 that in the case where the right kernel of A(s) is not the null space and $\beta_0(t)$ is a solution of (2.1) then

$$\beta(t) = \beta_0(t) + \int_0^t \bar{u}_{r+1}(t)z_1(t-\tau)d\tau + \int_0^t \bar{u}_{r+2}(t)z_2(t-\tau)d\tau + \cdots + \int_0^t \bar{u}_m(t)z_{m-r}(t-\tau)d\tau$$

is also a solution of (2.1), where $z_1(t)$, $z_2(t)$, $\cdots$, $z_{m-r}(t)$ are arbitrary real functions. □

Let

$$\mathcal{X} = \{x(t) \mid x(t) := \int_0^t \bar{u}_{r+1}(t) z_1(t-\tau)d\tau + $$
$$+ \int_0^t \bar{u}_{r+2}(t) z_2(t-\tau)d\tau + \cdots + \int_0^t \bar{u}_m(t) z_{m-r}(t-\tau)d\tau\}$$

Define the following sets of solutions

$$\tilde{X} = X \oplus \mathcal{X} = \{\beta(t) \mid \beta(t) := x_1(t)+x_2(t) \text{ where } x_1(t)\in X \text{ and } x_2(t)\in\mathcal{X}\}$$

$$\tilde{X}^{\infty} = X^{\infty} \oplus \mathcal{X} = \{\beta(t) \mid \beta(t) := x_1(t)+x_2(t) \text{ where } x_1(t)\in X^{\infty} \text{ and } x_2(t)\in\mathcal{X}\}$$

$$\tilde{X}^{0} = X^{0} \oplus \mathcal{X} = \{\beta(t) \mid \beta(t) := x_1(t)+x_2(t) \text{ where } x_1(t)\in X^{0} \text{ and } x_2(t)\in\mathcal{X}\}$$

It is obvious that **the spaces $\tilde{X}$, $\tilde{X}^{\infty}$ and $\tilde{X}^0$ are the distribution solution vector spaces of (2.1) which are due to the finite and infinite elementary divisors of A(s) and the right kernel of A(s) respectively.**

An important question which arises from the above is if the solution sets of the AR representation (2.1) constituted a vector space and if the answer is yes, what is the dimension of this vector space.

Let $\mathcal{B} = \{\beta(t) \mid$ solution space of the AR representation (2.1)$\}$. Define the following relation between the solutions of (2.1) $R \subset \mathcal{B} \times \mathcal{B}$

$$R(\beta_1(t), \beta_2(t)) = \{ (\beta_1(t), \beta_2(t)) \mid \beta_1(t) - \beta_2(t) \in \mathcal{X} \text{ where } \beta_1(t), \beta_2(t) \in \mathcal{B} \} \qquad (6.2)$$

**Proposition 8.** The relation (6.2) is an equivalence relation. □

We call an **equivalence class** of the element $\beta(t) \in \mathcal{B}$, and we denote this with $[\beta(t)]$, the set of all the elements of $\mathcal{B}$ which are equivalent to $\beta(t)$ or equivalently

$$[\beta(t)] := \{\beta_1(t) \in \mathcal{B} \mid (\beta(t), \beta_1(t)) \in R\} = \beta(t) \oplus \mathcal{X} =$$
$$= \{ \beta(t) + x(t) \text{ where } \beta(t) \in \mathcal{B} \text{ and } x(t) \in \mathcal{X} \} \qquad (6.3)$$

We can see that any equivalence class of an element $\beta(t)$ gives the solution of (2.1) under some specific initial conditions. In case where A(s) has not a right kernel then every equivalence class is composed by a unique element contrary to the singular case where to each equivalence class corresponds an arbitrary number of elements of $\mathcal{B}$. We conclude therefore that the whole solution space of the AR representation $\mathcal{B}$ is divided into equivalence classes which are defined by (6.3). Define now the following "*sum*" between equivalence classes of the form (6.3)

$$[\beta_1(t)] + [\beta_2(t)] := (\beta_1(t) + \beta_2(t)) \oplus \mathcal{X} =$$
$$= \{\beta_1(t)+\beta_2(t)+x(t) \text{ where } \beta_1(t),\beta_2(t)\in\mathcal{B} \text{ and } x(t)\in\mathcal{X}\}$$

and the "*product*"

$$\lambda\, [\beta(t)] := \lambda\beta(t) \oplus \mathcal{X} =$$
$$= \{ \lambda\beta(t) + x(t) \text{ where } \lambda\in\mathbb{R},\ \beta(t) \in \mathcal{B} \text{ and } x(t) \in \mathcal{X} \}$$

It is now easy to show the following

**Proposition 9.** The space which is spanned by the equivalence classes which are defined in (6.3) is a vector space $\hat{\mathcal{B}}$ and this is the solution vector space of the AR representation (2.1). □

Consider now the following spaces

$$\hat{X} = \{[\beta(t)] \mid \beta(t) \in \tilde{X}\} = \tilde{X}/R$$
$$\hat{X}^{\infty} = \{[\beta(t)] \mid \beta(t) \in \tilde{X}^{\infty}\} = \tilde{X}^{\infty}/R$$
$$\hat{X}^{0} = \{[\beta(t)] \mid \beta(t) \in \tilde{X}^{0}\} = \tilde{X}^{0}/R$$

It is obvious that the above spaces partition the sets $\tilde{X}$, $\tilde{X}^{\infty}$ and $\tilde{X}^0$ and are vector spaces.

**Theorem 10.** The vector space $\hat{\mathcal{B}} = \mathcal{B}/R = \{ [\beta(t)] \mid \beta(t) \in \tilde{X} \cup \tilde{X}^{\infty} \cup \tilde{X}^0\}$ has dimension

$$\boxed{\dim \hat{\mathcal{B}} = f := n + \hat{q} + \hat{\ell}}$$

$f = \dim\hat{\mathcal{B}}$ is called *generalized order* of AR representation (2.1). □

**Remark 11.** Note that if A(s) is regular then $\hat{\ell}=0$ and f coincides with what Verghese defined as generalized order of (2.1). □

We can see through the proof of Theorem 10 that $\dim \hat{X} = n$, $\dim \hat{X}^{\infty} = \hat{q}$ and $\dim \hat{X}^0 = \hat{\ell}$.

**Example 12.** Consider the AR representation

$$A(\rho)\,\beta(t) = 0 \Longrightarrow$$

$$\begin{bmatrix} \rho^3+\rho^2 & \rho^3+\rho^2-1 & \rho^3-\rho \\ -\rho^2-\rho & -\rho^2-\rho & -\rho^2+1 \end{bmatrix} \begin{bmatrix} \beta_1(t) \\ \beta_2(t) \\ \beta_3(t) \end{bmatrix} = 0_{2\times1} \qquad (E.1)$$

with

$$S^{\mathbb{C}}_{A(s)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & s+1 & 0 \end{bmatrix} \quad ; \quad S^{\infty}_{A(s)} = \begin{bmatrix} s^3 & 0 & 0 \\ 0 & 1/s & 0 \end{bmatrix}$$

and

$$\begin{bmatrix} s^3+s^2 & s^3+s^2-1 & s^3-s \\ -s^2-s & -s^2-s & -s^2+1 \end{bmatrix} \begin{bmatrix} -s+1 \\ 0 \\ s \end{bmatrix} = 0_{2\times1} \qquad (E.2)$$

In this case n=1, $\hat{q}=1$ $\hat{\ell}=1$ and $\hat{\eta}=0$ and (E.1) has generalized order $f = \dim \hat{\mathcal{B}} = n+\hat{q}+\hat{\ell} = 1+1+1 = 3$ and $\hat{\mathcal{B}}$ is given by :

$$\hat{B}=B/\mathrm{R}=\{\lambda_1[\begin{bmatrix}-1\\0\\1\end{bmatrix}e^{-t}]+\lambda_2[\begin{bmatrix}1\\-1\\0\end{bmatrix}\delta(t)]+\lambda_3[\begin{bmatrix}-1\\0\\1\end{bmatrix}\delta(t)]\}$$

with

$$B=\{\lambda_1\begin{bmatrix}-1\\0\\1\end{bmatrix}e^{-t}+\lambda_2\begin{bmatrix}1\\-1\\0\end{bmatrix}\delta(t)+\lambda_3\begin{bmatrix}-1\\0\\1\end{bmatrix}\delta(t)+$$

$$+\begin{bmatrix}-u^{(1)}(t)+u(t)\\0\\u(t)\end{bmatrix}\} \quad (E.6)$$

where u(t) is an arbitrary real function and $\lambda_1,\lambda_2,\lambda_3\in\mathbb{R}$.

□

**Example 13** Consider the AR representation

$$A(\rho)\,\beta(t)=0 \Longrightarrow$$

$$\begin{bmatrix}\rho^3+\rho^2 & -\rho^2-\rho\\ \rho^3+\rho^2-1 & -\rho^2-\rho\\ \rho^3-\rho & -\rho^2+1\end{bmatrix}\begin{bmatrix}\beta_1(t)\\ \beta_2(t)\end{bmatrix}=0_{3\times 2} \quad (E.1)$$

with

$$S^{\mathbb{C}}_{A(s)}=\begin{bmatrix}1&0\\0&s+1\\0&0\end{bmatrix} \quad ; \quad S^{\infty}_{A(s)}=\begin{bmatrix}s^3&0\\0&1/s\\0&0\end{bmatrix}$$

and

$$[-s+1\ \ 0\ \ s]\begin{bmatrix}s^3+s^2 & -s^2-s\\ s^3+s^2-1 & -s^2-s\\ s^3-s & -s^2+1\end{bmatrix}=0_{1\times 3} \quad (E.2)$$

In this case n=1, $\hat{q}=1$ $\hat{\ell}=0$ and $\hat{\eta}=1$. Note that A(s) has no right kernel and thus $\mathcal{X}=\{0\}$. The left kernel of A(s) gives rise to the constraints between the initial conditions which must be satisfied so that (E.1) has a solution and are the following

$$\beta_2(0-)-\beta_1^{(1)}(0-)+\beta_2^{(1)}(0-)-\beta_1^{(2)}(0-)=0 \quad (E.5)$$

Thus under the assumption that conditions (E.5) are satisfied, the generalized order of (E.1) is $f=\dim\hat{B}=n+\hat{q}+\hat{\ell}=1+1+0=2$ and $\hat{B}$ is given by

$$\hat{B}=B/\mathrm{R}=\{\lambda_1[\begin{bmatrix}0\\1\end{bmatrix}e^{-t}]+\lambda_2[\begin{bmatrix}0\\1\end{bmatrix}\delta(t)]\}$$

with

$$B=\{\lambda_1\begin{bmatrix}0\\1\end{bmatrix}e^{-t}+\lambda_2\begin{bmatrix}0\\1\end{bmatrix}\delta(t)\}\equiv\hat{B} \quad (E.6)$$

$$\lambda_1,\lambda_2\in\mathbb{R}$$

□

## 7. Conclusions.

A number of authors Verghese (1978), and Vardulakis (1991) have studied the distribution solution vector space of *regular* AR representations i.e. $A(\rho)\beta(t)=0$ where $A(\rho)\in\mathbb{R}[\rho]^{r\times r}$ and $\mathrm{rank}_{\mathbb{R}(s)}A(s)=r$. In this paper we extend these results to the case of *singular* AR representations i.e. $A(\rho)\beta(t)=0$ where $A(\rho)\in\mathbb{R}[\rho]^{p\times m}$ and $\mathrm{rank}_{\mathbb{R}(s)}A(s)=r$. Some differences distinguish the singular from the regular AR representations i.e. the infinite number of solutions or the absence of solutions. Summarizing, in sections 2,3 and 4 we obtain vector spaces which are due to the finite and infinite elementary divisors and the right kernel of a polynomial matrix while in section 5 we give the role of the left kernel in the existence of solutions of an AR representation. In section 6 we conclude that the uniqueness of a solution is dependent on the right kernel of A(s). We have shown also that the solution vector space (behaviour $\hat{B}$) of (2.1) is composed of equivalence classes and its dimension is equal to $f=n+\hat{q}+\hat{\ell}$ i.e. is equal to the total number of zeros at $\mathbb{C}\cup\{\infty\}$ and the sum of the right minimal indices of A(s) (order accounted for).

The meaning of the algebraic structure of a polynomial matrix in relation to the solution vector spaces of singular AR representations has thus been elucidated.

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