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AN EXTENSION OF WOLOVICH'S DEFINITION OF EQUIVALENCE OF LINEAR SYSTEMS

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Abstract

Wolovich's classical definition of equivalence for linear systems is extended to the generalised study of linear systems. It is shown that this new kind of equivalence is an alternative characterisation of a notion of equivalence for general dynamical systems for which two other characterisations have recently been given.

Keywords : Linear systems, equivalence.

I Introduction

The conventional theory of linear systems deals with the finite frequency (exponential and sinusoidal) behaviour of such systems. In this theory the transformation of *Strict System Equivalence* (Rosenbrock 1970) plays a central role. This transformation does indeed possess the property of preserving the finite frequency structure of any polynomial matrix description to which it is applied. Another notion of "*equivalence*" between general dynamical systems was subsequently proposed by Wolovich (1974) and was based on the intuitive idea that two general linear systems should be deemed equivalent in case any state space reductions of them are related by the usual change of basis in the state space or in system matrix terms, system similarity (Rosenbrock, 1970). This notion of equivalence implied a number of desirable invariants and reduces to the standard definition of system similarity when both systems are in state space form. Pernebo (1977) has shown that strict system equivalence in Rosenbrock's sense is equivalent to the existence of a certain bijective mapping between the sets of solutions to the differential equations describing the system. A consequence of this proposition was that Wolovich's definition and strict system equivalence are seen as identical notions of equivalence.

The generalised theory of linear systems seeks a more complete study of linear system behaviour by considering additionally the possible significant impulsive motion. This necessitates treating the system's infinite frequency behaviour on an equal basis to its finite frequency behaviour and in this respect the above transformations do not permit the type of integrated study required since it is well known that they do not preserve the infinite frequency properties of the system. Within this spirit of an integrated study Hayton et.al. (1986) and Pugh et al. (1987) proposed the transformation of *Complete System Equivalence* for generalised state space systems, while Hayton et al. (1990) have proposed the transformation of *Full System Equivalence* for general systems described by mixed linear algebraic and high order differential equations. Recently a characterisation of full system equivalence has been given by Pugh et al. (1992) in the manner of Pernebo (1977) where the existence of a certain bijective mapping between the sets of *finite* and *infinite* solutions of the underlying differential equations is seen to specify the equivalence. These transformations do indeed have the property of simultaneously preserving the finite and infinite frequency structure of systems to which they are applied.

In this paper the view of Wolovich (1974) will be adopted and a notion of "*equivalence*" between two general system descriptions will be attributed on the basis of the equivalence of their underlying generalised state space models. This represents a natural extension of the Wolovich ideas since generalized state space systems are the most simple form of system equations which simultaneously exhibit finite and infinite frequency behaviour. It will be seen that this notion of equivalence again implies a number of desirable invariants and reduces to complete system equivalence when both systems are in generalized state space form. It will be seen that this new kind of equivalence is identical to full system equivalence and so preserves the system's finite and infinite frequency structure.

II Preliminary Results

Consider a linear time invariant multivariable system Σ described by a polynomial matrix description (PMD) :

$$A(\rho)\beta(t) = B(\rho)u(t) \tag{1a}$$

$$y(t) = C(\rho)\beta(t) + D(\rho)u(t) \tag{1b}$$

where $\rho = d/dt, A(\rho) \in \mathbf{R}[\rho]^{r\times r}$ with $|A(\rho)| \neq 0, B(\rho) \in \mathbf{R}[\rho]^{r\times m}, C(\rho) \in \mathbf{R}[\rho]^{p\times r}, D(\rho) \in \mathbf{R}[\rho]^{p\times m}, \beta(t) : (0-, \infty) \to \mathbf{R}^r$ the *pseudo state* of Σ, $u(t) : (0-, \infty) \to \mathbf{R}^m$ the *control input* and $y(t)$ the *output* of Σ, and let

$$P(s) = \begin{bmatrix} A(s) & B(s) \\ -C(s) & D(s) \end{bmatrix} \in \mathbf{R}[s]^{(r+p)\times(r+m)} \tag{2}$$

be its Rosenbrock system matrix. Σ may be written

$$\begin{bmatrix} A(\rho) & B(\rho) & 0 \\ -C(\rho) & D(\rho) & I_p \\ 0 & -I_m & 0 \end{bmatrix} \begin{bmatrix} \beta(t) \\ -u(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ I_m \end{bmatrix} u(t) \tag{3a}$$

$$y(t) = [0 \quad 0 \quad I_p] \begin{bmatrix} \beta(t) \\ -u(t) \\ y(t) \end{bmatrix} \tag{3b}$$

which is called the normalized form of Σ and denoted $\Sigma^{(N)}$.

$$\mathcal{P}(s) = \left[\begin{array}{ccc|c} A(s) & B(s) & 0 & 0 \\ -C(s) & D(s) & I_p & 0 \\ 0 & -I_m & 0 & I_m \\ \hline 0 & 0 & -I_p & 0 \end{array}\right] =: \begin{bmatrix} \mathcal{T}(s) & \mathcal{U} \\ -\mathcal{V} & 0 \end{bmatrix} \tag{4}$$

is then (Verghese 1978) the *normalized form* of $P(s)$.

Consider the set $P(p,m)$ of $(r+p)\times(r+m)$ polynomial matrices where the integer $r \geq \max\{-p,-m\}$.

Definition 1. (Hayton et.al. 1988)

Two matrices $T_1(s), T_2(s) \in P(p,m)$ are said to be *fully equivalent* (f.e.) in case there exist polynomial matrices $M(s), N(s)$ of appropriate dimensions such that :

$$[M(s) \quad T_2(s)] \begin{bmatrix} T_1(s) \\ -N(s) \end{bmatrix} = 0 \tag{5}$$

where the compound matrices

$$L(s) \triangleq [M(s) \quad T_2(s)] \; ; \; R(s) \triangleq \begin{bmatrix} T_1(s) \\ -N(s) \end{bmatrix} \tag{6}$$

satisfy

i) they have full normal rank, (7a)

ii) they have no finite nor infinite zeros, (7b)

iii) the following McMillan degree conditions hold

$$(L(s)) = \delta_M(T_2(s)) \; ; \; \delta_M(R(s)) = \delta_M(T_1(s)) \quad (7c) \; \square$$

Let $P_0(p,m)$ be the set of $(r+p)\times(r+m)$ Rosenbrock system matrices then the following extension of definition 1 to the set $P_0(p,m)$ can be made.

Definition 2. (Hayton et.al. 1990)

Let $P_1(s), P_2(s) \in P_0(p,m)$ be representations of the general dynamical systems Σ_1 and Σ_2 respectively. $P_1(s)$ and $P_2(s)$ are said to be *full system equivalent* (f.s.e.) if there exist polynomial matrices $M(s), N(s)$, $X(s), Y(s)$ such that

$$\begin{bmatrix} M(s) & 0 \\ X(s) & I \end{bmatrix} \begin{bmatrix} A_1(s) & B_1(s) \\ -C_1(s) & D_1(s) \end{bmatrix} = \begin{bmatrix} A_2(s) & B_2(s) \\ -C_2(s) & D_2(s) \end{bmatrix} \begin{bmatrix} N(s) & Y(s) \\ 0 & I \end{bmatrix} \tag{8}$$

where (8) is a (f.e.) transformation. $\square$

It was shown in Hayton et al. (1990) that an analogous definition relating to the normalized forms of the system matrices leads to an identical notion of equivalence in the sense that two Rosenbrock system matrices are (f.s.e.) if and only if the corresponding normalized form system matrices are so related. In addition it can be shown that any Rosenbrock system matrix is (f.s.e.) with its normalized form. For ease of presentation this section will concentrate on Rosenbrock system matrices, but it should be borne in mind (for the sequel) that the following definitions and results apply equally to the normalized forms of the system matrices. It is also important to note in the sequel that (f.s.e.) defines an equivalence relation on $P_0(p,m)$.

In the case of definition 2 where $M(s), N(s), X(s), Y(s)$ are constant matrices and Σ_1, Σ_2 are in generalized state space form, then Σ_1 and Σ_2 are termed *completely system equivalent* (c.s.e.). An interpretation of (c.s.e.) in terms of bijective maps between the "finite" and "infinite" solution spaces of the two generalized state space systems was given by Hayton et.al. (1986) through the notion of *fundamental equivalence*. An extension of this interpretation to the general case of (f.s.e) and dynamical systems of the form (1) has been proposed by Pugh et.al. (1992). Specifically

Definition 3. (Pugh et.al. 1992)

Let Σ_1, Σ_2 be two systems of the form (1) with respective solution spaces $\mathcal{X}_u^1, \mathcal{X}_u^2$ for a given fixed u(t). Σ_1, Σ_2 are said to be *fundamentally equivalent* if and only if the following hold

(i) $\exists$ a bijective mapping between $\mathcal{X}_u^1$, $\mathcal{X}_u^2$

$$\begin{pmatrix} \beta_2(t) \\ u(t) \end{pmatrix} = \begin{pmatrix} L(\rho) & T(\rho) \\ 0 & I \end{pmatrix} \begin{pmatrix} \beta_1(t) \\ u(t) \end{pmatrix} \tag{9}$$

where $\beta_i(t)$ denotes the pseudostate of Σ_i, (i=1,2).

(ii) Σ_1 and Σ_2 have the same output. $\square$

The definition of fundamental equivalence for general dynamical systems coincides with the corresponding definition when the systems are in generalized state space form (Hayton et.al. 1986), where the only difference then is that in (9), $L(\rho)$ and $T(\rho)$ are constant matrices. Why the map (9) should be constant in this case will be addressed in the sequel.

The formal connection between (f.s.e.) (definition 2) and fundamental equivalence (definition 3) is established by

Theorem 1.

Let Σ_1 and Σ_2 be two general dynamical systems (generalized state space systems). Σ_1 and Σ_2 are *fully system equivalent* (*completely system equivalent*) if and only if they are *fundamentally equivalent*. Further if the (f.s.e.) transformation which relates the two systems is of the form (8), then the mapping

$$\begin{pmatrix} \beta_2(t) \\ u(t) \end{pmatrix} = \begin{pmatrix} N(\rho) & Y(\rho) \\ 0 & I \end{pmatrix} \begin{pmatrix} \beta_1(t) \\ u(t) \end{pmatrix} \tag{10}$$

which is constructed from the right transforming matrix in (8), is a bijective mapping.

Proof. See Pugh et al. (1992). $\square$

The following is noted in Pugh et al.(1992) is

Corollary 1.

The absence of finite zeros in the compound matrix

$$\begin{pmatrix} A(\rho) & B(\rho) \\ -C(\rho) & D(\rho) \\ N(\rho) & Y(\rho) \\ 0 & I \end{pmatrix} \tag{11}$$

is a necessary and sufficient condition for the part of the inverse of (10) relating to the finite parts of the solutions of Σ_1, Σ_2 to be uniquely determined. The absence of infinite zeros in (11) together with the McMillan degree condition

$$\delta_M \begin{pmatrix} A(\rho) & B(\rho) \\ -C(\rho) & D(\rho) \\ N(\rho) & Y(\rho) \\ 0 & I \end{pmatrix} = \delta_M \begin{pmatrix} A(\rho) & B(\rho) \\ -C(\rho) & D(\rho) \end{pmatrix} \tag{12}$$

is a necessary and sufficient condition for that part of (10) relating the impulsive parts of the solutions of Σ_1, Σ_2 to be uniquely determined. $\square$

Important properties of the two identical transformations proposed in definitions 2 and 3, are the following

Lemma 1. (Hayton et.al. 1990, Karampetakis and Vardulakis 1992)

Under *full system equivalence* the following are invariant

(i) the generalized order f and the Rosenbrock degree d_R,

(ii) the transfer function and hence the sets of finite and infinite transmission poles and zeros,

(iii) the sets of finite and infinite system poles and zeros,

(iv) the sets of finite and infinite input (output, input-output) decoupling zeros,

(v) the sets of input (output) dynamical indices. $\square$

For the precise definitions of the terms used in Lemma 1 the reader is referred to Rosenbrock (1970) and Verghese (1978). We present next an extension of the well known Wolovich definition of equivalence (Wolovich 1974). This extension relates to the complete solution space of the system (1), not simply its finite solution space, and its connection with the ideas noted in this section will be established.

III An Extension of the Wolovich Definition Of Equivalence

The notion of normalised form of the system equations, or what is the same thing, the associated system matrix permits consistent definitions of finite and infinite frequency system properties to be given (Verghese 1978). In that sense it therefore facilitates the integrated study of the finite frequency and impulsive behaviours of the system. The initial definitions given here therefore relate to normalized forms.

Consider the *normalized form* Σ^N of the general dynamical system Σ of (1) i.e.

$$\mathcal{T}(\rho)\xi(t) = \mathcal{U}u(t) \tag{13a}$$

$$y(t) = \mathcal{V}\xi(t) \tag{13b}$$

where

$$\mathcal{T}(\rho) = \begin{bmatrix} A(\rho) & B(\rho) & 0 \\ -C(\rho) & D(\rho) & -I_m \\ 0 & I_p & 0 \end{bmatrix}, \mathcal{U} = \begin{bmatrix} 0 \\ 0 \\ I_p \end{bmatrix} \tag{14}$$

$$\mathcal{V} = [0 \quad 0 \quad I_m], \xi(t) = [\beta(t)^T, -u(t)^T, y(t)^T]^T$$

Consider the generalized state space system

$$E\dot{x}(t) = Ax(t) + Bu(t) \tag{15a}$$

$$y(t) = Cx(t) + Du(t) \tag{15b}$$

Following Wolovich (1974) the equivalence of two general dynamical systems will be defined in two parts. The first step is to establish the notion of the equivalence of Σ to a generalized state space form, while the second step will involve the notion of equivalence of two such generalized state space forms. With regard to the first step the following is proposed.

Definition 4.

The systems (1) and (15) are **"equivalent"** if and only if the two following conditions hold

(i) there is a constant bijective mapping

$$\begin{bmatrix} \xi(t) \\ u(t) \end{bmatrix} = \begin{bmatrix} C_0 & D_0 \\ 0 & I \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \tag{16}$$

between the set of solutions $\mathcal{X}_u^1$, $\mathcal{X}_u$ of the normalized form (13) and the generalized state space system (15), for each $u(t)$.

(ii) the systems (13) and (15) have the same output for the given u(t). □

Note that the equivalence is defined in terms of Σ^N the normalized form of Σ, and not directly in terms of Σ itself. Inserting (16) in (13b) we obtain that

$$\begin{aligned} y(t) &= \mathcal{V}(C_0x(t) + D_0u(t)) \\ &= \mathcal{V}C_0x(t) + \mathcal{V}D_0u(t) \\ &\equiv Cx(t) + Du(t) \end{aligned} \tag{17}$$

The condition (ii) of the definition 4 means that $C \equiv \mathcal{V}C_0$ and $D \equiv \mathcal{V}D_0$ which indicates, on taking into account condition (i), that the following diagram commutes

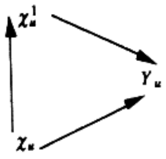

Diagram 1 Y_u is the set of outputs corresponding to u.

The above is a natural extension of the Wolovich definition of equivalence to the generalized state space setting. Notice that in the Definition 4, as in the original Wolovich definition, the map (16) is taken to be constant without any seeming justification. An explanation however can be given, of why this map should be constant. A necessary tool for proving this is the following

Lemma 2. (Pugh et.al. 1992)

Consider the general dynamical system (1) and the relation

$$\begin{pmatrix} \xi_1(t) \\ u(t) \end{pmatrix} = \begin{pmatrix} N(\rho) & Y(\rho) \\ 0 & I \end{pmatrix} \begin{pmatrix} \xi(t) \\ u(t) \end{pmatrix} \tag{18}$$

A necessary and sufficient condition for relation (18) to be a map in the formal sense (of being a many-one relation) is

$$\delta_M \begin{pmatrix} A(\rho) & B(\rho) \\ -C(\rho) & D(\rho) \\ N(\rho) & Y(\rho) \\ 0 & I \end{pmatrix} = \delta_M \begin{pmatrix} A(\rho) & B(\rho) \\ -C(\rho) & D(\rho) \end{pmatrix} \tag{19}$$ □

Based on Lemma 4 we can now prove the following

Theorem 2.

Consider the two dynamical systems (13) and (15). Let

$$\begin{pmatrix} \xi(t) \\ u(t) \end{pmatrix} = \begin{pmatrix} N(\rho) & Y(\rho) \\ 0 & I \end{pmatrix} \begin{pmatrix} x(t) \\ u(t) \end{pmatrix} \tag{20}$$

be a relation between the solution-input spaces $(\xi(t)^\top \quad u(t)^\top)^\top$ of (13) and $(x(t)^\top \quad u(t)^\top)^\top$ of (15), where $N(\rho) = N_q\rho^q + \ldots + N_1\rho + N_0$ and $Y(\rho) = Y_q\rho^q + \ldots + Y_1\rho + Y_0$ (where at least one of N_q, Y_q is nonzero). Then (20) is a map if and only if it is constant.

Proof.

By Lemma 2 a necessary and sufficient condition for (20) to be a map is

$$\delta_M \begin{pmatrix} \rho E - A & B \\ -C & 0 \\ N(\rho) & Y(\rho) \\ 0 & I \end{pmatrix} = \delta_M \begin{pmatrix} \rho E - A & B \\ -C & 0 \end{pmatrix} \overset{[6]}{\Leftrightarrow}$$

$$\text{rank}_{\mathbf{R}} \begin{pmatrix} E & \ldots & 0 & 0 & \ldots & 0 \\ 0 & \ldots & 0 & 0 & \ldots & 0 \\ N_q & \ldots & 0 & Y_q & \ldots & 0 \\ N_{q-1} & \ldots & 0 & Y_{q-1} & \ldots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ N_1 & \ldots & N_q & Y_1 & \ldots & Y_q \end{pmatrix} = \text{rank}_R \begin{pmatrix} E & 0 \\ 0 & 0 \end{pmatrix}$$

$$\Leftrightarrow \quad N_i = 0, \quad ; \quad Y_i = 0 \quad i = 2, \ldots, q$$

$$\text{and} \quad N_1 = HE \quad ; \quad Y_1 = 0 \tag{21}$$

for some constant matrix H. Thus

$$\begin{pmatrix} \xi(t) \\ u(t) \end{pmatrix} = \begin{pmatrix} N_0 + HE\rho & Y_0 \\ 0 & I \end{pmatrix} \begin{pmatrix} x(t) \\ u(t) \end{pmatrix} =$$

$$\begin{pmatrix} N_0 + H(Ax(t) + Bu(t)) + Y_0u(t) \\ u(t) \end{pmatrix}$$

$$= \begin{pmatrix} N_0 + HA & Y_0 + HB \\ 0 & I \end{pmatrix} \begin{pmatrix} x(t) \\ u(t) \end{pmatrix} \tag{22}$$

and so the thorem is proved. □

Theorem 2 confirms our intuition that the physical system variables $\xi(t)$ are constant linear combinations of the generalised state variables $x(t)$. It also justifies why the definition of fundamental equivalence for generalised state space systems (Hayton et al. 1986) should be based on constant maps as follows.

Corollary 2.

The definition 3 of fundamental equivalence for general dynamical systems coincides with the definition of fundamental equivalence for generalized state space systems (Hayton et.al. 1986), when (1) is in generalized state-space form.

Proof.

If there exists a bijective map of the form (20) between two generalized state space systems then according to Theorem 2 this map is a constant map which verifies the corollary. □

(16) is an interesting map. The reason for this is that the solution sets of $\mathcal{X}_0^1$ and $\mathcal{X}_0$ of the homogeneous systems (13a) and (15a) form vector spaces with dimension equal to the generalized 'er $f_1 = \delta_M(\mathcal{T}(s))$ and $f = \delta_M(\rho E - A)$ (Vardulakis 1991) respectively. (16) is then a vector space isomorphism between $\mathcal{X}_0^1$ and $\mathcal{X}_0$, and so in particular it will preserve this generalized order i.e. $\delta_M(\mathcal{T}(s)) = \delta_M(\rho E - A)$. The bijective map (16) has also the property of preserving the controllability subspaces of since it is a bijection between the solution/input pairs of these systems. Less obviously (16) also preserves the observability subspaces of these systems (13) and (15) (Pugh et.al. 1992). It is then reasonable to call (13) and (15) "equivalent". Some additional properties which arise from the above definition and provide points of comparison with Wolovich (1974), are included in the following results. The first of these establishes invariants of an external nature.

Theorem 3.

The "equivalent" systems (1) and (15) are

(i) Partial state/input transfer matrix equivalent,

(ii) Input/output transfer matrix equivalent

Proof.

(i) Laplace transforming (13a) and (15a) and ignoring the ini- ' conditions gives

$$\mathcal{T}(s)\bar{\xi}(s) = \mathcal{U}\bar{u}(s)$$

$$(sE - A)\bar{x}(s) = B\bar{u}(s)$$

Thus from condition (i) of definition 4 it follows that

$$\bar{\xi}(s) = C_0\bar{x}(s) + D_0\bar{u}(s) = C_0(sE - A)^{-1}B\bar{u}(s) + D_0\bar{u}(s)$$

but

$$\bar{\xi}(s) = \mathcal{T}^{-1}(s)\mathcal{U}\bar{u}(s)$$

and so necessarily

$$C_0(sE - A)^{-1}B + D_0 \equiv \mathcal{T}^{-1}(s)\mathcal{U} \tag{23}$$

(ii) From (13)

$$\bar{y}(s) = \mathcal{V}(s)\mathcal{T}^{-1}(s)\mathcal{U}(s)\bar{u}(s)$$

while from (15)

$$\bar{y}(s) = (C(sE - A)^{-1}B + D)\bar{u}(s)$$

By condition (ii) of Definition 4 these outputs are the same for any given input $\bar{u}(t)$ and so necessarily

$$C(sE - A)^{-1}B + D = \mathcal{V}\mathcal{T}^{-1}(s)\mathcal{U}$$
$$(= C(s)A^{-1}(s)B(s) + D(s)) \tag{24}$$

□

From an internal point of view we have the following results from the definition 4.

Theorem 4.

(i) Definition 4 reduces to the definition of complete system equivalence when the general dynamical system (1) which underlies (13) is in generalized state space form.

(ii) the generalised state space system formed from(15a) and (16) i.e.

$$E\dot{x}(t) = Ax(t) + Bu(t) \tag{25a}$$

$$\xi(t) = C_0x(t) + D_0u(t) \tag{25b}$$

is strongly observable i.e. $((\rho E - A)^\top \quad C_0^\top)^\top$ has no finite nor infinite zeros.

Proof.

(i) Consider the case where (1) is in generalized state space form *i.e.* $A(\rho) = E_1\rho - A_1$, $B(\rho) = B_1$, $C(\rho) = C_1$ and $D(\rho) = D_1$ and so (11) is also in generalized state space form

$$\bar{E}\dot{\mathbf{x}}(t) = \bar{A}\mathbf{x}(t) + \bar{B}u(t) \tag{26a}$$

$$y(t) = \bar{C}\mathbf{x}(t) \tag{26b}$$

where

$$\bar{E} = \begin{bmatrix} E_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \bar{A} = \begin{bmatrix} A_1 & -B_1 & 0 \\ C_1 & -D_1 & -I \\ 0 & I & 0 \end{bmatrix},$$

$$\bar{B} = \begin{bmatrix} 0 \\ 0 \\ I \end{bmatrix}, \quad \bar{C} = [0 \quad 0 \quad I] \text{ and } \mathbf{x}(t) = [x(t)^T, -u(t)^T, y(t)^T]^T \tag{27}$$

If the two generalized state space systems (26) and (15) are "equivalent" in the sense of definition 4 then the two systems are fundamentally equivalent under a constant map and so by Theorem 1, (26) and (15) are (c.s.e.). Additionally note that

$$\left[\begin{array}{c|c} I & 0 \\ 0 & 0 \\ 0 & 0 \\ \hline 0 & I \end{array}\right] \begin{bmatrix} sE_1 - A_1 & B_1 \\ -C_1 & D_1 \end{bmatrix}$$
$$= \begin{bmatrix} s\bar{E} - \bar{A} & \bar{B} \\ -\bar{C} & 0 \end{bmatrix} \left[\begin{array}{c|c} I & 0 \\ 0 & 0 \\ C_1 & D_1 \\ \hline 0 & I \end{array}\right] \tag{28}$$

is a transformation of complete system equivalence, and so from the transivity property of (c.s.e.), the two generalized state space systems (1) and (15) will be (c.s.e.).

(ii) The mapping (16) is a bijective mapping, and so the generalized state space system (25) is strongly observable, (Verghese 1978). □

To complete the definition of equivalence in the Wolovich manner it is necessary for every general dynamical system in normalized form to possess an equivalent (in the sense of definition 4) generalized state space representation. It is always possible to construct an "equivalent" generalized state space system as can be seen in the following

Theorem 5.

Every general dynamical system of the form (1) has an equivalent (in the sense of definition 4) generalized state-space system representation.

...e (1978) proposed a reduction method which takes a ... irreducible realization $\{C_\infty, J_\infty, B_\infty\}$ of the denomi... atrix of the normalized system matrix (4) such that

$$\mathcal{T}(s) = C_\infty(I_\mu - sJ_\infty)^{-1}B_\infty \tag{29}$$

...r the generalized state-space system

$$\left[\begin{array}{cc|c} -\rho J_\infty & -B_\infty & 0 \\ C_\infty & 0 & \mathcal{U} \\ \hline 0 & \mathcal{V} & 0 \end{array}\right]\left[\begin{array}{c} z(t) \\ \xi(t) \\ \hline -u(t) \end{array}\right] = \begin{bmatrix} 0 \\ 0 \\ y(t) \end{bmatrix} \tag{30}$$

...ill show that this generalized state-space model is an ...ent model of the general dynamical system (1). In this ...note that from (30)

$$\xi(t) = [0 \quad I]\begin{bmatrix} z(t) \\ \xi(t) \end{bmatrix} := C_0\begin{bmatrix} z(t) \\ \xi(t) \end{bmatrix} \tag{31}$$

...apping between the solution sets of the normalized system ...and the generalized state-space system (30). However the ...ound matrix

$$\begin{bmatrix} \ldots E - A \\ C_0 \end{bmatrix} := \left[\begin{array}{cc} I_\mu - \rho J_\infty & -B_\infty \\ C_\infty & 0 \\ \hline 0 & I \end{array}\right] \overset{\text{F.E.}}{\sim} \begin{bmatrix} I_\mu - \rho J_\infty \\ C_\infty \end{bmatrix} \tag{32}$$

...es the McMillan degree condition in (12), has no finite nor ...e zeros because the realization $\{C_\infty, J_\infty, B_\infty\}$ is strongly ...cible and so the mapping (31) is an injective mapping. ...an see also from the form of C_0 that the mapping (31) is ...y a surjective mapping. Hence (31) is a bijection and so the ...alized state space model (30) satisfies the first condition ...e definition 4 of equivalence. We have also that

$$y(t) = \mathcal{V}\xi(t) \overset{(31)}{=} \mathcal{V}C_0\begin{bmatrix} z(t) \\ \xi(t) \end{bmatrix} = [0 \quad \mathcal{V}]\begin{bmatrix} z(t) \\ \xi(t) \end{bmatrix} \tag{33}$$

...h is the output from the generalized state space represen... ...n in (30). Thus the second condition of the equivalence in ...ition 4 is also fulfilled and so the theorem is proved. □

...defi... ...n of equivalence given in Definition 4 is a special ...of the definition of fundamental equivalence of Definition ... a formulation as an ...d as such will possess, by Theorem 1, a formulation as an ...e.) transformation. In fact it is an alternative characteriza... ...of (f.s.e.) in the context of general dynamical systems (1) ...generalised state-space systems of the form (15).

...eorem 6.

...e general dynamical system (1) and the generalized state-...ce system (15) are equivalent in the sense of definition 4 if ...l only if they are (f.s.e.).

...oof

...) Suppose (1) and (15) are (f.s.e.). It has already been ...ed that any system Σ is (f.s.e.) to its normalised form ...$^{V)}$, and so from the transitivity of (f.s.e.) it follows that ...) and (15) are (f.s.e.). Hence there exist polynomial matri... ...$M(\rho), N(\rho), X(\rho), Y(\rho)$ such that

$$\left[\begin{array}{cc|cc} M(\rho) & 0 & \mathcal{T}(\rho) & \mathcal{U} \\ X(\rho) & I & -\mathcal{V} & 0 \end{array}\right]\left[\begin{array}{cc} \rho E - A & B \\ -C & D \\ \hline -N(\rho) & -Y(\rho) \\ 0 & -I \end{array}\right] = 0 \tag{34}$$

...ere (34) is a (f.e.) transformation. According to the McMil...n degree conditions on the compound polynomial matrices ...(34) we obtain that $Y(\rho) = Y_0$ is a constant matrix and $N(\rho) = N_0 + HE\rho$ and so (34) may be rewritten as

$$\left[\begin{array}{cc|cc} M(\rho) & 0 & \mathcal{T}(\rho) & \mathcal{U} \\ X(\rho) & I & -\mathcal{V} & 0 \end{array}\right]\left[\begin{array}{cc} \rho E - A & B \\ -C & D \\ \hline -N_0 - HE\rho & -Y_0 \\ 0 & -I \end{array}\right] = 0 \tag{35}$$

Postmultipling and premultipling respectively the first and the second compound matrix in (35) with

$$Q = \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ -H & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix} \quad \text{and} \quad Q^{-1} = \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ H & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix} \tag{36}$$

we obtain that

$$\left[\begin{array}{cc|cc} M(\rho) - \mathcal{T}(\rho)H & 0 & \mathcal{T}(\rho) & \mathcal{U} \\ X(\rho) + \mathcal{V}H & I & -\mathcal{V} & 0 \end{array}\right] \times \left[\begin{array}{cc} \rho E - A & B \\ -C & D \\ \hline -N_0 - HA & -Y_0 + HB \\ 0 & -I \end{array}\right] = 0 \tag{37}$$

which gives according to Theorem 1 that

$$\begin{bmatrix} \xi(t) \\ u(t) \end{bmatrix} = \begin{bmatrix} N_0 + HA & Y_0 - HB \\ 0 & I \end{bmatrix}\begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \tag{38}$$

is a bijective mapping. (F.s.e.) has also the property to leave invariant the transfer function matrix and so the two systems will have the same output and so the systems (1) and (15) are equivalent in the sense of definition 4.

(⇒) If the two systems (13) and (15) are equivalent in the sense of definition 4 then it is obvious that they will be fundamental equivalent according to definition 3. Hence according to Theorem 1, they are (f.s.e.). Again since any system is (f.s.e.) to its normalised form it follows from the transitivity property of (f.s.e.) that (1) and (15) will be so related. □

Corollary 3.

The general dynamical system (1) and generalized state space representation (30) are (f.s.e.).

Proof.

From Theorem 5 the general dynamical system (1) and the generalized state space system (30) are equivalent in the sense of definition 4, and so from Theorem 6, they are (f.s.e.). □

Theorem 7.

Two generalized state space systems are (f.s.e.) iff they are (c.s.e.).

Proof.

(⇒) In case where the general dynamical system (13) is in generalized state space form we obtain from Theorem 6 that if the two generalized state-space systems are (f.s.e.) then there exist a constant bijective mapping between the solution sets of the two systems of the form

$$\begin{bmatrix} x_2(t) \\ u(t) \end{bmatrix} = \begin{bmatrix} N & Y \\ 0 & I \end{bmatrix}\begin{bmatrix} x_1(t) \\ u(t) \end{bmatrix} \tag{39}$$

and so the two generalized state space systems are *fundamen-*

tally equivalent or according Theorem 1 (c.s.e.).

($\Leftarrow$) It is obvious that the conditions of (c.s.e.) coincide with the conditions of (f.s.e.) in the special form of (c.s.e.) and so the converse of the theorem is proved. □

As proposed by Wolovich (1974), it follows in view of definition 4, and the previous results that it is now possible to complete the definition of equivalence by defining equivalence between two general dynamical systems of the form (1). Whereas in the original Wolovich definition the transformation of system similarity (Rosenbrock 1970) plays a key role, in this study it will be the transformation of (c.s.e.).

Definition 5.

Two general dynamical systems Σ_1 and Σ_2 of the form (1) are **equivalent** if and only if their equivalent generalized state-space systems are (c.s.e.). □

Under this definition, Theorem 6 may be extended to the case where both systems are in the general form (1) as follows.

Theorem 8.

Two general dynamical systems Σ_1 and Σ_2 are equivalent in the sense of definition 5 if and only if they are (f.s.e.).

Proof

($\Rightarrow$) Consider two equivalent general dynamical systems Σ_1 and Σ_2. This means that their equivalent (in the sense of definition 4) generalized state-space representations S_1 and S_2 of the form (30) are (c.s.e.). From Theorem 6 it follows that Σ_1 and S_1 (resp. Σ_2 and S_2) are (f.s.e.). Further since (c.s.e.) is a special case of (f.s.e.) we have the following relation

$$\Sigma_1 \overset{\text{f.s.e.}}{\sim} S_1 \overset{\text{f.s.e.}}{\sim} S_2 \overset{\text{f.s.e.}}{\sim} \Sigma_2 \tag{40}$$

Using the transitivity property of (f.s.e.) we obtain that the two systems Σ_1 and Σ_2 are (f.s.e.).

($\Leftarrow$) Consider two (f.s.e.) general dynamical system Σ_1 and Σ_2 and let S_1 and S_2 respectively be their equivalent (in the sense of definition 4) generalised state-space systems. By Theorem 6 Σ_1 and S_1 (resp. Σ_2 and S_2) are (f.s.e.) and so

$$S_1 \overset{\text{f.s.e.}}{\sim} \Sigma_1 \overset{\text{f.s.e.}}{\sim} \Sigma_2 \overset{\text{f.s.e.}}{\sim} S_2 \tag{41}$$

Thus by the transitivity property of (f.s.e.), S_1 and S_2 are (f.s.e.) or further by (Theorem 7) S_1 and S_2 are (c.s.e.). Thus Σ_1, Σ_2 are equivalent in the sense of definition 5. □

An interesting property of this new kind of equivalence is

Corollary 4.

If two general dynamical systems Σ_1 and Σ_2 are equivalent in the sense of definition 5 then they have the same

(i) generalized order f and Rosenbrock degree d_R,

(ii) transfer function and so the sets of finite and infinite transmission poles and zeros,

(iii) sets of finite and infinite system poles and zeros,

(iv) sets of finite and infinite input (output) decoupling zeros,

(v) sets of input (output) dynamical indices, .

Proof.

From Theorem 8 we have that this new kind of equivalence defines the same equivalence classes as (f.s.e.) and so shares the same properties (Lemma 1). □

IV Conclusions

An extension of the Wolovich definition of equivalence, to encompass the generalised theory of linear systems, has been given. The extension is based on the notion that a general dynamical systems has an equivalent generalised state space reduction. In fact several reductions are available but the one selected here is that proposed by Verghese (1978). The basis of the definition is then that two general dynamical systems are equivalent in case their generalised state space reductions are completely system equivalent. Thus in the generalised study of linear systems the generalised state space system is appropriately seen to play the same role as the state space model in the conventional study, while complete system equivalence is seen to be as important in the generalised context as system similarity in the conventional. Overall the definition is seen to coincide with the previously defined transformation of full system equivalence and so has the property of simultaneously preserving the the system's *finite* and *infinite* frequency behaviour *i.e.* the behaviour as summarized by the generalized order, the sets of finite and infinite system zeros and poles, the transfer function and the sets of finite and infinite decoupling zeros. As such the proposed notion of equivalence provides some neat explanations of certain features of the transformation of full system equivalence and underlines its important role in the generalised study of linear systems

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