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Observations on the Notion of Minimality in the Generalized State Space

by

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Abstract

Minimality is explored in the generalized state space. In the regular case two distinct notions are seen to accrue to the term, one which has a true dynamic meaning and one which does not. Implications for generalized state space systems are revealed.

1. Introduction.

The notion of minimality of state space representations of a strictly proper transfer function is important from a number of points of view. It variously implies that the representation is controllable and observable and that the state vector has least possible dimension over all such realisations, and often provides the conditions for certain design problems to be resolved. Similarly minimality is important in the generalized state space. For non-regular systems minimality summarises a number of properties [1], [5], among which are that the system is controllable and observable in a generalized sense and that the generalized state has least dimension. For regular generalized state space systems the situation is simpler in that these two properties are essentially all that it summarises.

This paper considers regular generalized state space systems and makes some observations which have relevance for non-regular systems but which will be explored in this context elsewhere. In particular the least possible dimension for regular generalized state space realisations of a given transfer function matrix is characterised and its minimality established. As with least order [10], minimality is a property of the particular realisation although the dimension of the generalized state vector in such a realisation is a property of the transfer function matrix. Section 4 provides a characterisation of the minimal generalized state dimension entirely in transfer function matrix terms. Section 5 examines the relationship between all minimal generalized state space realisations of the same transfer function matrix.

2. Preliminary Results.

Let $P(s) \in \mathbb{R}[s]^{p\times m}$ be a matrix polynomial with

$$P(s) = P_0 + P_1 s + \cdots + P_q s^q \tag{2.1}$$

and $P_q \neq 0$. The McMillan degree $\delta_M(P(s))$ is the total number of infinite poles of $P(s)$ [10] or equivalently the highest degree of minors of all orders of $P(s)$[7]. Another characterization of $\delta_M(P(s))$ is

Lemma 1. [2],[7]

$$\delta_M(P(s)) = \text{rank}_{\mathbf{R}} \begin{bmatrix} P_1 & P_2 & \cdots & P_q \\ P_2 & P_3 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ P_q & 0 & \cdots & 0 \end{bmatrix} \tag{2.2} \quad \square$$

Note that P_0 in (2.1) plays no role in the determination of $\delta_M(P(s))$. The relationship between $\delta_M(P(s))$ of a square invertible $P(s) \in \mathbb{R}[s]^{p\times p}$ and its infinite frequency structure is

Lemma 2. [11] Let $P(s) \in \mathbb{R}[s]^{p\times p}$, with $\text{rank}_{\mathbf{R}(s)} P(s) = p$ have Smith McMillan form at $s = \infty$, $S^{\infty}_{P(s)}$, given by

$$S^{\infty}_{P(s)} = \text{diag}\Big[s^{q_1}, s^{q_2}, \ldots, s^{q_v}, \underbrace{1, 1, \ldots, 1}_{k-v}, \frac{1}{s^{\hat{q}_{k+1}}}, \ldots, \frac{1}{s^{\hat{q}_p}}\Big] \tag{2.3}$$

where $q_1 \geq q_2 \geq \cdots \geq q_v \geq 1$; $\hat{q}_p \geq \hat{q}_{p-1} \geq \cdots \geq \hat{q}_{k+1} \geq 1$ are respectively the *degrees of the poles and zeros at* $s = \infty$ of $P(s)$ and v and $p-k$ are called the *multiplicities* of these poles and zeros. Then

$$\delta_M(P(s)) = \sum_{i=1}^{v} q_i = \deg|P(s)| + \sum_{i=k+1}^{p} \hat{q}_i \tag{2.4} \quad \square$$

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Definition 1. If $T(s) \in \mathbb{R}(s)^{p\times m}$, a quintuple $[E \in \mathbb{R}^{r\times r}, A \in \mathbb{R}^{r\times r}, B \in \mathbb{R}^{r\times m}, C \in \mathbb{R}^{p\times r}, D \in \mathbb{R}^{p\times m}]$ for some $r \in \mathbf{Z}^+$ such that

$$T(s) = C(sE - A)^{-1}B + D \tag{2.5}$$

is called a *realization* of $T(s)$. □

When $T(s) \in \mathbb{R}_{spr}(s)^{p\times m}$, $\exists$ a realisation $[I_n, J, B, C, 0]$ for some $n \in \mathbf{Z}^+$ such that

$$T(s) = C(sI_n - J)^{-1}B. \tag{2.6}$$

This is the usual state space realization of $T(s)$. When $T(s) \in \mathbb{R}[s]^{p\times m}$ then $\exists$ a realisation $[-J_\infty, -I_\mu, B_\infty, C_\infty, D_\infty]$ such that

$$T(s) = C_\infty(I_\mu - sJ_\infty)^{-1}B_\infty + D_\infty \tag{2.7}$$

A realization $[E, A, B, C, D]$ of a rational matrix $T(s)$ corresponds to a regular system Σ in generalized state space form defined by

$$E\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t) \tag{2.8}$$

or analogously the polynomial system matrix

$$P(s) = \begin{pmatrix} sE - A & B \\ -C & D \end{pmatrix} \tag{2.9}$$

where $|sE - A| \neq 0$, with transfer function matrix

$$G(s) := C(sE - A)^{-1}B + D = T(s).$$

Minimality in the conventional state space gives rise to a number of distinct notions when considered in the generalized state space setting.

Definition 2. [10],[13] $P(s)$ of (2.9) having no finite (resp. infinite) decoupling zeros is *irreducible* (resp. *irreducible at infinity*). If $P(s)$ has both these properties it is termed *strongly irreducible*. □

Strong irreducibility is an extension of state space minimality in the sense that it characterises when the system (2.8) is controllable and observable in a generalized sense ([6], and references therein).

Lemma 3. [13] $P(s)$ of (2.9) is strongly irreducible then the pole structure in $\mathbf{C} \cup \{\infty\}$ of $G(s)$ is isomorphic to the zero structure in $\mathbf{C}\cup\{\infty\}$ of $sE - A$ and the zero structure of $G(s)$ in $\mathbf{C}\cup\{\infty\}$ is isomorphic to the zero structure of $P(s)$ in $\mathbf{C} \cup \{\infty\}$. □

Lemma 4. [12] Let $P(s) = P_0 + P_1 s + \cdots + P_q s^q$, $P_i \in \mathbb{R}^{(r+p)\times(r+m)}$, $i \in q$ be a general Rosenbrock system matrix written as a matrix polynomial, then

$$f = \delta_M(P(s)) = \operatorname{rank}_{\mathbb{R}} \begin{bmatrix} P_1 & P_2 & \dots & P_q \\ P_2 & P_3 & \dots & 0 \\ \vdots & & \ddots & \vdots \\ P_q & 0 & \dots & 0 \end{bmatrix} \geq \delta_M[G(s)] \tag{2.10}$$

with equality iff $P(s)$ is strongly irreducible. □

An obvious extension of minimality is,

Definition 3. A realization $[E, A, B, C, D]$ of $T(s) \in \mathbb{R}(s)^{p\times m}$ is *minimal realization* iff the corresponding generalized state space system (2.8) has the least possible number of states $x(t)$. □

Equivalently a regular realisation is minimal if the corresponding system (2.8) has the smallest possible number of system equations occurring in it. This notion of minimality arises when that introduced by [1, 3, 4] in the more general case of nonregular systems is restricted to the regular case.

In the special case when $T(s) \in \mathbb{R}_{spr}(s)^{p\times m}$ then the realization $[I_n, J, B, C, D]$ is minimal iff $n = \delta_m(T(s)) \equiv \nu(T(s))$ [10]. In the sequel we consider the values of $r \in \mathbf{Z}^+$ can a realization $[E, A, B, C, D]$ of the general rational matrix $T(s) \in \mathbb{R}(s)^{p\times m}$ be called minimal.

3. Minimality in the Generalized State Space

Lemma 5. [11],[12] Let $T(s) = T_0 + T_1 s + \cdots + T_q s^q \in \mathbb{R}[s]^{p\times m}$, $\operatorname{rank}_{\mathbb{R}(s)} T(s) = r$ and $r \geq 1$ with Smith-McMillan form at infinity given by

$$S^{\infty}_{T(s)} = \operatorname{diag}[s^{q_1}, s^{q_2}, \dots, s^{q_v}, \underbrace{1, 1, \dots, 1}_{k-v}, \frac{1}{\hat{q}_{k+1}}, \dots, \frac{1}{\hat{q}_r}, 0_{p-rm-r}] \tag{3.1}$$

where $q_1 \geq q_2 \geq \cdots \geq q_v > 0$ and $\hat{q}_r \geq \hat{q}_{r-1} \geq \cdots \geq \hat{q}_{k+1} > 0$ are respectively the degrees of its poles and zeros at $s = \infty$ and v and $r - k$ their respective multiplicities. Then

(i) the McMillan degree of $\tilde{T}(w) := \frac{1}{w}T(\frac{1}{w}) \in \mathbb{R}^{p\times m}_{spr}(w)$ is given by

$$\mu = \delta_m(\tilde{T}(w)) = \Sigma_1^k(q_i + 1) = \Sigma_1^v(q_i + 1) + k - v$$
$$= q_1 + q_2 + \cdots + q_v + k \quad (q_i \geq 0) \tag{3.2}$$

(ii) if $[I_\mu, J_\infty, B_\infty, C_\infty, 0]$ is an irreducible realization of $\tilde{T}(w)$ with $J_\infty \in \mathbb{R}^{\mu\times\mu}$ in Jordan form then

$$J_\infty = \operatorname{blockdiag}[J_{\infty_1}, J_{\infty_2}, \dots, J_{\infty_v}, 0_{k-v,k-v}] \in \mathbb{R}^{\mu\times\mu} \tag{3.3}$$

where

$$J_{\infty i} = \begin{bmatrix} 0 & 1 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & & \ddots & \vdots \\ \vdots & & \ddots & 1 \\ 0 & 0 & \dots & 0 \end{bmatrix} \in \mathbb{R}^{(q_i+1)\times(q_i+1)}, i = 1, 2, \dots, v \tag{3.4}$$ □

Theorem 1. A realization $[-J_\infty, -I_\mu, B_\infty, C_\infty, D_\infty]$, $\mu \in \mathbf{Z}^+$ of $T(s) \in \mathbb{R}[s]^{p\times m}$ with J_∞ in Jordan normal form is minimal iff in (3.2)

$$\mu = \sum_1^v (q_i+1) = \nu + \sum_1^v q_i \quad (3.5)$$

Proof: We shall show that $\exists$ a realization of $T(s)$ with the dimension μ of (3.5) and that $\exists$ no other realization of $T(s)$ with dimension less than μ.

Consider $\tilde{T}(w) = \frac{1}{w}T(\frac{1}{w}) \in \mathbb{R}^{p\times m}_{pr}(w)$ and let $[I_\mu, J_\infty, B_\infty, C_\infty, 0]$ be an irreducible realization of $\tilde{T}(w)$ according to Lemma 5. Then

$$\tilde{T}(w) = \frac{1}{w}T\left(\frac{1}{w}\right) = C_\infty(wI_\mu - J_\infty)^{-1}B_\infty \overset{w=\frac{1}{s}}{\Longrightarrow}$$

$$sT(s) = C_\infty\left(\frac{1}{s}I_\mu - J_\infty\right)^{-1}B_\infty \Longrightarrow$$

$$T(s) = C_\infty(I_\mu - sJ_\infty)^{-1}B_\infty \quad (3.6)$$

where from Lemma 5 $J_\infty \in \mathbb{R}^{\mu\times\mu}$ is in Jordan form and so satisfies (3.3), (3.4).

The above realization corresponds to

$$P(s) = \left[\begin{array}{cc|c} I_{\bar\mu} - sJ'_\infty & 0 & B^1_\infty \\ 0 & I_{k-v} & B^2_\infty \\ \hline -C^1_\infty & -C^2_\infty & 0 \end{array}\right] \quad (3.7)$$

where $\bar\mu = \mu - (k-v) =_. \Sigma_1^v(q_i+1)$ and $J'_\infty :=$ blockdiag$(J_{\infty_1}, \ldots, J_{\infty_v})$. In (3.1) some orders of poles at $s=\infty$ may be equal to zero giving rise to zero Jordan blocks i.e. $J_{\infty_i} = [0]$ (see (3.3). A consequence of these zero Jordan blocks is the existence of some nondynamic variables which play no role (see [13]) in the realization of $T(s)$ and which may be absorbed via an additive constant term to yield the following realization

$$T(s) = C^1_\infty(I_{\bar\mu} - sJ'_\infty)^{-1}B^1_\infty + C^2_\infty . B^2_\infty \quad (3.8)$$

Then $[-J'_\infty, -I_{\bar\mu}, B^1{}_\infty, C^1{}_\infty, C^2{}_\infty B^2{}_\infty]$ where $\bar\mu =_. \Sigma_1^v(q_i+1)$, is a realization of $T(s)$.

Suppose now that there exists another realization of $T(s)$, which without loss of generality may be taken to be in its Jordan canonical form $[-\bar{J}_\infty, -I_{\hat\mu}, \bar{B}_\infty, \bar{C}_\infty, \bar{D}_\infty]$ with $\hat\mu < \bar\mu$ or equivalently

$$\sum_{i=1}^h(\bar{q}_i+1) < \sum_{i=1}^v(q_i+1) \quad (3.9)$$

where $\bar{q}_i$, $i = 1,2,..,h$ are the degrees of the zeros of $I_{\hat\mu} - s\bar{J}_\infty$ at $s=\infty$. From Lemma 4 we have

$$f := \delta_M(I_{\hat\mu} - s\bar{J}_\infty) \overset{\text{Lemma2}}{=} \sum_{i=1}^h \bar{q}_i \geq \delta_M(T(s))$$

$$= \sum_{i=1}^v q_i \Longrightarrow \sum_{i=1}^v (q_i+1) \geq \sum_{i=1}^h \bar{q}_i + v \quad (3.10)$$

Relating (3.9) with (3.10) it follows that

$$\sum_{i=1}^h(\bar{q}_i+1) < \sum_{i=1}^h \bar{q}_i + v \Longrightarrow h < v \quad (3.11)$$

However from [13]

$$\begin{aligned} &\{\text{system poles at } s=\infty\} \\ &= \{\text{transmission poles at } s=\infty\} \\ &+ \{\text{decoupling zeros at } s=\infty\} \end{aligned} \quad (3.12)$$

which means that

$$\begin{aligned} &\sharp\{\text{system poles at } s=\infty\} \\ &= \sharp\{\text{transmission poles at } s=\infty\} \\ &+ \sharp\{\text{decoupling zeros at } s=\infty\} \end{aligned} \quad (3.13)$$

where $\sharp$ denotes the total number elements in the indicated set. In multiplicity terms this means

$$h = v + \alpha \quad (3.14)$$

where $\alpha > 0$ is determined by the occurrence of the decoupling zeros at infinity. Now (3.14) contradicts (3.12) and so $[-\bar{J}_\infty, -I_{\hat\mu}, \bar{B}_\infty, \bar{C}_\infty, \bar{D}_\infty]$ has $\hat\mu \geq \bar\mu$. Thus $[-J'_\infty, -I_{\bar\mu}, B^1{}_\infty, C^1{}_\infty, C^2{}_\infty B^2{}_\infty]$ with $\bar\mu =_. \Sigma_1^v(q_i+1)$ is such that $\exists$ no other realization $[-\bar{J}_\infty, -I_{\hat\mu}, \bar{B}_\infty, \bar{C}_\infty, \bar{D}_\infty]$ of $T(s)$ with $\hat\mu < \bar\mu$. □

An important result arises when the realization $[-\bar{J}_\infty, -I_{\hat\mu}, \bar{B}_\infty, \bar{C}_\infty, \bar{D}_\infty]$ has no infinite decoupling zeros (it cannot possess finite decoupling zeros) and has no nondynamic variables. Then $\hat\mu = \bar\mu =_. \Sigma_1^v(q_i+1)$ since the polar structure of $T(s)$ is isomorphic to the zero structure of $(I_{\hat\mu} - s\bar{J}_\infty)$ (Lemma 3). Thus

Corollary 2. A necessary and sufficient condition for a realization $[-\bar{J}_\infty, -I_{\hat\mu}, \bar{B}_\infty, \bar{C}_\infty, \bar{D}_\infty]$ of $T(s) \in \mathbb{R}[s]^{p\times m}$ to be minimal is that it be irreducible at infinity and have no nondynamic variables. □

The extension to realisations of a general rational matrix follows, which generalises [4] for $[E, A, B, C, 0]$ realisations.

Theorem 2. A realization $[E, A, B, C, D]$, $r \in \mathbf{Z}^+$ of $T(s) \in \mathbb{R}(s)^{p\times m}$ is minimal iff it is strongly irreducible and has no nondynamic variables. Then

$$r = \nu(T(s)) + \sum_{i=1}^v(q_i+1) \quad (3.15)$$

is called the *generalized least dimension* of $T(s)$.

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Corollary 2. A necessary and sufficient condition for a realization $[-\bar{J}_\infty, -I_{\hat{\mu}}, \bar{B}_\infty, \bar{C}_\infty, \bar{D}_\infty]$ of $T(s) \in \mathbb{R}[s]^{p\times m}$ to be minimal is that it be irreducible at infinity and have no nondynamic variables. □

The extension to realisations of a general rational matrix follows, which generalises [4] for $[E, A, B, C, 0]$ realisations.

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$$r = \nu(T(s)) + \sum_{i=1}^v (q_i + 1) \qquad (3.15)$$

is called the *generalized least dimension* of $T(s)$.

Proof: Using Lemma 1 and that the pole structure of the polynomial part of $T(s)$ is isomorphic with the pole structure of $T(s)$ at infinity [11] gives

$$\sum_{i=1}^{k} q_i = \operatorname{rank}_{\mathbb{R}} \begin{bmatrix} T_1 & T_2 & \dots & T_q \\ T_2 & T_3 & \dots & 0 \\ \vdots & & \ddots & \vdots \\ T_q & 0 & \dots & 0 \end{bmatrix} \tag{4.5}$$

Using also (4.5) we obtain that

$$\operatorname{rank}_{\mathbb{R}} \begin{bmatrix} T_0 & T_1 & \dots & T_q \\ T_1 & T_2 & \dots & 0 \\ \vdots & & \ddots & \vdots \\ T_q & 0 & \dots & 0 \end{bmatrix} \tag{4.6}$$

$$= \delta_M(sT_0 + s^2T_1 + \dots + s^{q+1}T_q)$$

$$= \sum_{i=1}^{k}(q_i + 1) = \sum_{i=1}^{v}(q_i + 1) + (k - v)$$

and

$$\operatorname{rank}_{\mathbb{R}} \begin{bmatrix} T_2 & T_3 & \dots & T_q \\ T_3 & T_4 & \dots & 0 \\ \vdots & & \ddots & \vdots \\ T_q & 0 & \dots & 0 \end{bmatrix}$$

$$= \delta_M(T_2s + T_3s^2 + \dots + T_qs^{q-1})$$

$$= \delta_M\left(\frac{T_1s + T_2s^2 + \dots + T_qs^q}{s}\right) = \sum_{i=1}^{v}(q_i - 1) \tag{4.7}$$

From (4.5), (4.7) the number v of infinite poles is

$$v = \operatorname{rank}_{\mathbb{R}} \begin{bmatrix} T_1 & T_2 & \dots & T_q \\ T_2 & T_3 & \dots & 0 \\ \vdots & & \ddots & \vdots \\ T_q & 0 & \dots & 0 \end{bmatrix} - \operatorname{rank}_{\mathbb{R}} \begin{bmatrix} T_2 & T_3 & \dots & T_q \\ T_3 & T_4 & \dots & 0 \\ \vdots & & \ddots & \vdots \\ T_q & 0 & \dots & 0 \end{bmatrix} \tag{4.8}$$

and so

$$r = \nu(T(s)) + \sum_{i=1}^{v}(q_i + 1)$$

$$\overset{(4.3)}{=} \operatorname{rank}_{\mathbb{R}} \begin{bmatrix} T_{-1} & T_{-2} & \dots & T_{-\sigma} \\ T_{-2} & T_{-3} & \dots & T_{-\sigma-1} \\ \vdots & & \ddots & \vdots \\ T_{-\sigma} & T_{-\sigma-1} & \dots & T_{-2\sigma+1} \end{bmatrix} \tag{4.9}$$

$$+ \sum_{i=1}^{v} q_i + v$$

(4.4) then follows by using (4.5) and (4.8). □

The above result gives a method of calculating the generalized least dimension of $T(s)$ without the necessity of computing the degrees of the infinite poles, as (3.15) otherwise indicates, in order to ascertain the multiplicity of the infinite poles.

Example 1: Consider the transfer function matrices

$$G_1(s) = \begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix}; \quad G_2(s) = \begin{pmatrix} s & s^2 \\ 0 & s \end{pmatrix} \tag{4.10}$$

which both have McMillan degree equal to 2. The Laurent expansion of these matrices are

$$G_1(s) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} s + \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$G_2(s) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} s^2 + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

It follows from (4.4) that the generalized least dimensions of these two transfer function matrices are

$$\mu_{G_1(s)} = 4; \quad \mu_{G_2(s)} = 3$$

and it may be readily verified that

$$P_1(s) = \left[\begin{array}{cccc|cc} 1 & -s & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -s & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ \hline -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \end{array}\right]$$

$$P_2(s) = \left[\begin{array}{ccc|cc} 1 & -s & 0 & 0 & 0 \\ 0 & 1 & -s & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ \hline -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 & 0 \end{array}\right]$$

are minimal realisations of $G_1(s)$, $G_2(s)$ respectively. It is thus noted that the McMillan degree gives no particular indication of the magnitude of the generalized least order. □

5. Properties of Strongly Irreducible and Minimal Realisations

In conventional state space theory all least order state space realisations of a given transfer function matrix are system similar [10]. We enquire as to the nature of the relationship which exists between strongly irreducible or indeed minimal generalized state space realisations.

Definition 4. [8] Two Rosenbrock system matrices $P_1(s)$, $P_2(s)$ in generalized state space form are *completely system equivalent* (C.S.E.) if $\exists$ constant matrices M, N, X, Y such that

$$\begin{bmatrix} M & 0 \\ X & I \end{bmatrix} \begin{bmatrix} sE_1 - A_1 & B_1 \\ -C_1 & D_1 \end{bmatrix} = \begin{bmatrix} sE_2 - A_2 & B_2 \\ -C_2 & D_2 \end{bmatrix} \begin{bmatrix} N & Y \\ 0 & I \end{bmatrix} \qquad (5.1)$$

where the compound matrices

$$[M \;\; sE_2 - A_2]; \qquad \begin{bmatrix} sE_1 - A_1 \\ -N \end{bmatrix} \qquad (5.2)$$

are such that

(i) they have full normal rank (5.3a)

(ii) they have no finite nor infinite zeros (5.3b) □

For strongly irreducible realisations we have

Theorem 4. Let $P_1(s), P_2(s)$ be two strongly irreducible Rosenbrock system matrices in generalized state space form. Then

$$P_1(s) \overset{\text{C.S.E.}}{\sim} P_2(s) \Longleftrightarrow G_1(s) = G_2(s) \qquad (5.4)$$

where

$$G_i(s) = C_i(sE_i - A_i)^{-1}B_i + D_i \;\; (i = 1, 2) \qquad (5.5)$$

Proof: [13] established that all strongly irreducible realisations give rise to the same transfer function matrix iff they are related by what was termed strong equivalence. [8] subsequently proved that the closed form matrix characterisation of strong equivalence is (C.S.E.). □

(C.S.E.) thus forms the basic connection between all strongly irreducible generalized state space realisations of a given transfer function matrix. Whether this equivalence is any further refined for minimal generalized state space realisations is indicated by the following.

Example 2: Consider the system matrices

$$P_1(s) = \left[\begin{array}{ccc|c} s-1 & 0 & 0 & 1 \\ 0 & 1 & -s & 0 \\ 0 & 0 & 1 & 1 \\ \hline -1 & -1 & 0 & 1 \end{array}\right],$$

$$P_2(s) = \left[\begin{array}{ccc|c} s-1 & 0 & 0 & 1 \\ 1 & 1 & -s & 0 \\ 0 & 0 & 1 & 1 \\ \hline -2 & -1 & 0 & 1 \end{array}\right]$$

which can be readily verified as being minimal realisations of the transfer function matrix

$$G(s) = \frac{s^2 - s + 1}{s - 1}$$

Now $P_1(s)$, $P_2(s)$ can be seen to be related by the obvious (C.S.E.) transformation

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] P_1(s) = P_2(s) \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \qquad (5.6)$$

while it can be easily verified that there exists no transformation of the form

$$\left[\begin{array}{ccc|c} h_{11} & h_{12} & h_{13} & 0 \\ h_{21} & h_{22} & h_{23} & 0 \\ h_{31} & h_{32} & h_{33} & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] P_1(s)$$

$$= P_2(s)\left[\begin{array}{ccc|c} h_{11} & h_{12} & h_{13} & 0 \\ h_{21} & h_{22} & h_{23} & 0 \\ h_{31} & h_{32} & h_{33} & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \qquad (5.7)$$

with the matrix

$$\begin{pmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{pmatrix}$$

invertible, as required by system similarity. For note that the (1,2) and (1,3) equations yield $h_{12} = h_{13} = 0$ and it follows from the (1,4) equation that $h_{11} = 0$. However solving the (1,2), (1,3), (4,3), (2,4) and (4,1) equations produces the solution $h_{11} = \frac{1}{2}$. Thus equations (5.7) are inconsistent. □

This example together with theorem 4 establishes that the basic relationship between all strongly irreducible realisations $[E, A, B, C, D,]$, $r \in \mathbf{Z}^+$ of a given transfer function matrix is (C.S.E.) and that this equivalence cannot be refined to system similarity (S.S.) even in the case of minimal generalized state space realisations. The example also demonstrates that (C.S.E.) cannot be refined to (S.S) even if the plant matrices $sE_i - A_i$, $i = 1, 2$ are taken to be in the Kronecker form of (3.18). The best that can be said as regards any refinement of (C.S.E.) for minimal realisations is the following.

Theorem 5. Let $P_1(s)$, $P_2(s)$ be the system matrices of two minimal generalized state space systems. Then $P_1(s)$, $P_2(s)$ give rise to the same transfer function matrix iff they are related by (5.1) with the transforming matrices M, N square and nonsingular.
Proof: If $P_1(s)$, $P_2(s)$ are related in the manner of (5.1) then they give rise to the same transfer function matrix [9]. Conversely if $P_1(s)$, $P_2(s)$ give rise to the same $G(s)$ then by Theorem 4 they are (C.S.E.). Since $P_1(s)$, $P_2(s)$ are minimal, they have the same dimensions and so [7, theorem 7] the statement of (C.S.E.) reduces to one in which the matrices M, N in (5.1) are square and nonsingular. □

The refinement of (C.S.E.) indicated in the above theorem is called *operations of strong equivalence* [13]. Thus operations of strong equivalence form the basic transformation connecting minimal generalized state space realisations of a given transfer function matrix, as is readily confirmed by Example 2. Theorem 5 is entirely consistent with the corresponding result [5] for the case of nonregular generalized state space systems with no direct feedthrough term. Example 2 also reveals the main undesirable feature of the notion of minimality as follows.

Example 3: Consider again

$$G(s) = \frac{s^2 - s + 1}{s - 1}$$

which has generalized least dimension 3. If unity output feedback is applied to $G(s)$ then the closed loop transfer function matrix

$$G_1(s) = \frac{s^2 - s + 1}{s^2}$$

results. Using (4.4) it can be readily ascertained that $G_1(s)$ has generalized least dimension 2. Thus the generalized least dimension is not invariant under constant output feedback. □

Example 3 demonstrates that the generalized least dimension of $G(s)$ is not a proportional feedback invariant, although the total number of poles of the system, as represented by the McMillan degree $\delta(G)$, is such an invariant. Thus since the generalized least dimension is the sum of $\delta(G)$ and the multiplicity of the infinite poles, it is this multiplicity which changes under such feedback. Thus $G(s)$ of the example has generalized least dimension 3 since it has McMillan degree 2 (one finite and one infinite pole) and infinite pole multiplicity of 1. By contrast $G_1(s)$ has generalized least order 2 since it has just two finite poles (giving McMillan degree 2).
Finally it should be noted that if

$$P(s) = \begin{pmatrix} sE - A & B \\ -C & D \end{pmatrix}$$

is minimal then the natural description of the closed loop system formed from $P(s)$ by constant output feedback as described by the matrix F, which is [10]

$$P_F(s) = \left[\begin{array}{ccc|c} sE - A & B & 0 & 0 \\ -C & D & I & 0 \\ 0 & -I & F & I \\ \hline 0 & 0 & -I & 0 \end{array}\right] \qquad (5.8)$$

will be strongly irreducible, although not necessarily minimal. Thus although a strongly irreducible realisation of the closed loop system may be immediately written down, a minimal realisation will have to be generated from this either by (C.S.E.) on $P_F(s)$, or by directly minimally realising $G_F(s)$.

6. Conclusions.

The notion of minimality of a realisation of a given transfer function matrix $G(s)$ has been considered in a generalized state space setting. It has been noted that the single notion of minimality in the conventional state space gives rise to two distinct notions in the generalized state space case, namely strong irreducibility and what has been termed here "minimality", the least possible dimension for the generalized state vector over all the generalized state space realisation of $G(s)$. Both strong irreducibility and minimality are terms which apply to a generalized state space realisation of a given transfer function matrix $G(s)$, although the actual value of the dimension of the generalized state vector of any minimal realisation of $G(s)$ is an invariant of $G(s)$ and has here been referred to as the generalized least dimension of $G(s)$. The characterisation of this least dimension is given in section 3 while an algorithm for its computation directly from $G(s)$ is provided in section 4. Section 5 indicates that the basic transformation connecting all irreducible realisations of a given transfer function matrix is (C.S.E.) and that a slight refinement of this equivalence is obtained in the case of minimal realisations.

The notion of strong irreducibility has its attractions in that it represents algebraically the controllability and observability prospects of the generalized state vector, as [6] has indicated. The notion of minimality however is attractive (presumably from a large-scale point of view) because it additionally represents a realisation with the least possible number of state variables, and (since the realisation is regular) the least possible number of equations. Thus while strong irreducibility has an exact dynamic interpretation, the notion of "minimality" does not. This distinction is revealed clearly in the results and observations of section 5 where it is seen that the desirable (dynamic) properties, such as invariance under constant output feedback, are possessed by strongly irreducible realisations. This is not the case with minimal realisations, where it is seen that under such feedback neither will the generalized least dimension itself be invariant, nor will the natural system matrix representation (5.8) of the feedback system be minimal even if the open loop system matrix has this property.

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