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Generalized State Space Representations for Linear Multivariable Systems

by

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Abstract –

In this paper we propose two different algorithms which reduce a general polynomial matrix model [A(s), B(s), C(s), D(s)] of a linear multivariable system Σ into a full system equivalent model [sE–A, B, C, D] in generalised state space form with the same properties i.e. the same (i) transfer function matrix, (ii) system poles and zeros in $\mathbb{C}\cup\{\infty\}$, (iii) decoupling zeros in $\mathbb{C}\cup\{\infty\}$, (iv) controllability and observability indices.

1. Introduction

Consider a linear multivariable system Σ described by a polynomial matrix model :

$$(\Sigma):\quad A(\rho)\,\beta(t) = B(\rho)\,u(t) \qquad (1.1a)$$
$$y(t) = C(\rho)\,\beta(t) + D(\rho)\,u(t) \qquad (1.1b)$$

where (ρ:=d/dt) $A(\rho)\in\mathbb{R}[\rho]^{r\times r}$ with $\text{rank}_{\mathbb{R}}A(\rho)=r$, $B(\rho)\in\mathbb{R}[\rho]^{r\times m}$, $C(\rho)\in\mathbb{R}[\rho]^{p\times r}$ and $D(\rho)\in\mathbb{R}[\rho]^{p\times m}$, $\beta(t):[0-,\infty)\longrightarrow\mathbb{R}^r$ the *pseudo state* of Σ, $u(t):[0-,\infty)\longrightarrow\mathbb{R}^m$ the input vector and $y(t): [0-,\infty)\longrightarrow\mathbb{R}^p$ the output vector. (1.1) may be rewritten as

$$(\Sigma^{(N)}):\ \mathcal{T}(\rho)\,\xi(t) = \mathcal{U}\,u(t) \qquad (1.2a)$$
$$y(t) = \mathcal{V}\,\xi(t) \qquad (1.2b)$$

where

$$\mathcal{T}(\rho)=\begin{bmatrix} A(\rho) & B(\rho) & 0\\ -C(\rho) & D(\rho) & I_p\\ 0 & -I_m & 0\end{bmatrix}\in\mathbb{R}[\rho]^{\bar{r}\times\bar{r}}$$

$$\mathcal{U}=\begin{bmatrix}0\\0\\I_m\end{bmatrix}\in\mathbb{R}^{\bar{r}\times\bar{m}}\ ;\ \xi(t)=\begin{bmatrix}\beta(t)\\-u(t)\\y(t)\end{bmatrix} \qquad (1.3)$$

and $\mathcal{V}=(0\ \ 0\ \ I_p)\in\mathbb{R}^{p\times\bar{r}}$ $\quad\bar{r}=r+p+m$

The system $\Sigma^{(N)}$ of (1.2) is called by Verghese (1978) the *normalized form* of the system Σ.

FORMULATION OF THE PROBLEM

Given the polynomial matrix description (PMD) [A(ρ), B(ρ), C(ρ), D(ρ)] of Σ determine :

(i) a positive integer $\lambda\in\mathbb{Z}^+$ and

(ii) a quintuple of matrices

$E, A\in\mathbb{R}^{\lambda\times\lambda}$, $B\in\mathbb{R}^{\lambda\times m}$, $C\in\mathbb{R}^{p\times\lambda}$ and $D\in\mathbb{R}^{p\times m}$

such that the system Σ_R

$$(\Sigma_R):\quad E\dot{x}(t) = Ax(t) + Bu(t) \qquad (1.4a)$$
$$y(t)= Cx(t) + D\dot{u}(t) \qquad (1.4b)$$

is "equivalent" with the system Σ i.e. the two systems exhibits the same finite and infinite system properties (the same (i) transfer function, (ii) system poles and zeros in $\mathbb{C}\cup\{\infty\}$, (iii) decoupling zeros in $\mathbb{C}\cup\{\infty\}$, (iv) controllability and observability indices). □

There are two different ways to solve the above problem. The first and the most common until now is to produce an algorithm, which reduces a general PMD of a linear multivariable system Σ to a system Σ_R in the generalized state space form, and in the sequel to show step by step that all the finite and infinite system properties of Σ remain invariant This is the way which Verghese 1978, Bosgra & Van Der Weiden 1981 and Vardulakis 1991 select to follow in the past. The second and more direct way according , Anderson *et al.* 1985 and the authors is to produce a reduction algorithm and in the sequel to show that this reduction algorithm is translated via a *system equivalence transformation*, which has the property to preserve the finite and infinite system properties, between the two systems . The benefits of the second choice are obvious. The system equivalence tools which we have to work is the *strong system equivalence* transformation (Anderson *et al* 1985) and the *full system equivalench* transformation (Hayton *et al* 1990). However the strong system equivalence transformation has the main disadvantage that is composed by two separate system transformations in addition to full system equivalence transformation which is composed by only one transformation and this is the reason which we shall use in the sequel the transformation of *full system equivalence*.

In the first part of this paper we present some preliminary results concerning the transformation of full system equivalence and its properties. In the second part we present two theoretical algorithms which reduce a general PMD of a linear multivariable system Σ to a full system equivalent system Σ_R in generalised state space system form.

The above algorithm becomes more realistic in the third part with the extension of the singular system realisation for arbitrary matrix fraction descriptions presented by Shaohua Tan and Joos Vandewalle (1988). We close this paper with an illustrative example.

2. Preliminary results.

Consider the set $P(p,m)$ of $(r+p)\times(r+m)$ polynomial matrices with $r\geq\max(-p,-m)$. A matrix transformation with many important systems theory implications is the following

Definition 1. (Hayton *et al* 1988) Let $T_1(s), T_2(s) \in P(p,m)$. Then $T_1(s)$, $T_2(s)$ are said to be fully equivalent (f.e.) if there exist polynomial matrices $M(s)$, $N(s)$ such that

$$[M(s)\ \ T_2(s)]\begin{bmatrix} T_1(s) \\ -N(s)\end{bmatrix} = 0 \qquad (2.1)$$

where the compound matrices

$$[M(s)\ \ T_2(s)] \quad ; \quad \begin{bmatrix} T_1(s) \\ -N(s)\end{bmatrix} \qquad (2.2)$$

satisfy the following

(i) they have full normal rank

(ii) they have no finite nor infinite zeros

(iii) the following McMillan degree conditions hold

$$\delta_M[M(s)\ \ T_2(s)]=\delta_M[T_2(s)]$$
$$\delta_M\begin{bmatrix} T_1(s) \\ -N(s)\end{bmatrix}=\delta_M[T_1(s)] \qquad (2.3)$$

□

A linear time invariant multivariable system Σ of the form (1.1) may be represented by an $(r+p)\times(r+m)$ (with $r>0$) polynomial system matrix

$$P(s) = \begin{bmatrix} A(s) & B(s) \\ -C(s) & D(s)\end{bmatrix} \qquad (2.4)$$

as has been described by Rosenbrock (1970) or by the *normalized form* of $P(s)$

$$\mathcal{P}(s) = \begin{bmatrix} \mathcal{T}(s) & \mathcal{U} \\ -\mathcal{V} & 0\end{bmatrix} \qquad (2.5)$$

as has been described by Verghese (1978). Let the set of all such matrices be denoted by $\mathcal{P}(p,m)$. Then we have

Definition 2. (Hayton *et al* 1990) $P_1(s)$, $P_2(s)\in\mathcal{P}(p,m)$ are said to be *full system equivalent* (f.s.e) if there exist polynomial matrices $M(s)$, $N(s)$, $X(s)$, $Y(s)$ such that

$$\begin{bmatrix} M(s) & 0 \\ X(s) & I\end{bmatrix}\begin{bmatrix} A(s) & B(s) \\ -C(s) & D(s)\end{bmatrix}=\begin{bmatrix} A(s) & B(s) \\ -C(s) & D(s)\end{bmatrix}\begin{bmatrix} N(s) & Y(s) \\ 0 & I\end{bmatrix} \qquad (2.6)$$

where (2.6) is a f.e. transformation. □

Some interesting results concerning the transformation of full system equivalence are included in the following

Theorem 1. (Walker 1988, Hayton et al 1990, Karampetakis & Vardulakis 1992)

(i) Full system equivalence is an equivalence relation.

(ii) Under full system equivalence the following are invariant

(a) the generalized order f and the Rosenbrock degree d_r

(b) the transfer function and thus the finite and infinite transmission poles and zeros

(c) the sets of finite and infinite system poles and zeros

(d) the sets of finite and infinite decoupling zeros

(e) the controllability and observability indices

(iii) Every system matrix $P(s)$ is full system equivalent with its normalized form $\mathcal{P}(s)$. □

Theorem 2. (Karampetakis & Vardulakis 1992) Let $P_1(s)$, $P_2(s)\in\mathcal{P}(p,m)$ be two strongly irreducible system matrices i.e. both systems do not have finite or infinite decoupling zeros. Then

$$P_1(s) \overset{f.s.e.}{\sim} P_2(s) \Leftrightarrow G_1(s) = G_2(s)$$

where $G_i(s)=C_i(s)A_i^{-1}(s)B_i(s)+D_i(s)$ are the transfer function matrices of $P_i(s)$ $i=1,2$. □

3. Generalized State Space Realizations for Linear Multivariable Systems.

The problem of the reduceness of a linear multivariable system to an "equivalent" generalised space system has been considered in the past by many authors, among them, Verghese (1978), Bosgra & Van Der Weiden (1981), Anderson *et al* (1985) and Vardulakis (1991). The solutions given by the above authors can be formulated by two different theoretical algorithms proposed in this section. We start with

Algorithm 1.

Step 1. DATA : $[A(s)\in\mathbb{R}[s]^{r\times r}$, $B(s)\in\mathbb{R}[s]^{r\times m}$, $C(s)\in\mathbb{R}[s]^{p\times r}$, $D(s)\in\mathbb{R}[s]^{p\times m}]$ be the PMD of Σ or equivalently $[\mathcal{T}(s)\in\mathbb{R}[s]^{\bar{r}\times\bar{r}}$, $\mathcal{U}\in\mathbb{R}^{\bar{r}\times m}$, $-\mathcal{V}\in\mathbb{R}^{p\times\bar{r}}]$, $\bar{r}=r+p+m$ be the PMD of its normalized form.

Step 2. Compute a strongly irreducible realization $[A_0(s)\in\mathbb{R}[s]^{q\times q}$, $B_0(s)\in\mathbb{R}[s]^{q\times\bar{r}}$, $C_0(s)\in\mathbb{R}[s]^{\bar{r}\times q}$,

$D_0(s)\in\mathbb{R}[s]^{\bar r\times\bar r}]$ of $\mathcal{T}(s)$ where the polynomial matrices $A_0(s)$, $B_0(s)$, $C_0(s)$, $D_0(s)$ are pencils.

Step 3. The Rosenbrok system matrix of the "equivalent" generalised state space realisation Σ_R of Σ will be the following

$$P_R(s)=\left[\begin{array}{cc|c} A_0(s) & B_0(s) & 0 \\ -C_0(s) & D_0(s) & \mathcal{U} \\ \hline 0 & -\mathcal{V} & 0 \end{array}\right] \quad (3.1)$$

□

Corollary 1. Let

$$sE-A:=\begin{bmatrix} A_0(s) & B_0(s) \\ -C_0(s) & D_0(s) \end{bmatrix}\ ;\ B:=\begin{bmatrix} 0 \\ \mathcal{U} \end{bmatrix}$$

$$\text{and } C:=(0\ \ \mathcal{V}) \quad (3.2)$$

Then the "equivalent" generalised state space system of (1.1) will be the following :

$$(\Sigma_R):\quad E\,\dot x(t)=A\,x(t)+B\,u(t) \quad (3.3a)$$

$$y(t)=C\,x(t) \quad (3.3b)$$ □

Corollary 2. In case where $C_0(s)\in\mathbb{R}^{p\times r}$, $B_0(s)\in\mathbb{R}^{r\times m}$ and $D_0(s)=0$ then the above algorithm is reduced to the one proposed by Verghese (1978) and Anderson *et al* (1985).

The Bosgra & Van Der Weiden reduction algorithm comes also from a strongly irreducible realization of $\mathcal{T}(s)$ in (1.2) (see Bosgra & Van Der Weiden (1981), Karampetakis & Vardulakis (1994)) whose construction is based on a selection procedure of linearly independent rows and columns of a Hankel matrix obtained from the matrices $\mathcal{T}_i$, $i\in t$ defined through the matrix

$\mathcal{T}(s)=\mathcal{T}_q s^q+\cdots+\mathcal{T}_1 s+\mathcal{T}_0$ in (1.3). □

We would like now to show that the generalised state space system proposed in (3.3) exhibits the same finite and infinite system properties with the system Σ defined in (1.1). It is easily to prove this, if we only show that the two systems are full system equivalent (see Theorem 1), and this is the point which we prove in the sequel.

Theorem 3. The linear multivariable systems (1.1) and (3.3) are full system equivalent.

Proof. Consider the PMD (1.1) with its normalised form described by (1.2). Let also a strongly irreducible realization $[A_0(s)\in\mathbb{R}[s]^{q\times q}$, $B_0(s)\in\mathbb{R}[s]^{q\times\bar r}$, $C_0(s)\in\mathbb{R}[s]^{\bar r\times q}$, $D_0(s)\in\mathbb{R}[s]^{\bar r\times\bar r}]$ of the polynomial matrix $\mathcal{T}(s)$ ($\mathcal{T}(s)=C_0(s)A_0(s)^{-1}B_0(s)+D_0(s)$) where the polynomial matrices $A_0(s)$, $B_0(s)$, $C_0(s)$, $D_0(s)$ are pencils. Then the Rosenbrock system matrices

$$P_1(s)=\begin{bmatrix} I_q & 0 \\ 0 & \mathcal{T}(s)\end{bmatrix} \text{ and } P_2(s)=\begin{bmatrix} A_0(s) & B_0(s) \\ -C_0(s) & D_0(s)\end{bmatrix} \quad (3.4)$$

are strongly irreducible and have the same transfer function matrix ($\mathcal{T}(s)$). Thus from Theorem 2 Σ_1 and Σ_2 are full system equivalent and therefore there exist polynomial matrices $M(s)$, $N(s)\in\mathbb{R}[s]^{q\times q}$, $X(s)\in\mathbb{R}[s]^{\bar r\times q}$ and $Y(s)\in\mathbb{R}[s]^{q\times\bar r}$ such that

$$\begin{bmatrix} M(s) & 0 \\ X(s) & I_{\bar r}\end{bmatrix}\begin{bmatrix} I_q & 0 \\ 0 & \mathcal{T}(s)\end{bmatrix}=\begin{bmatrix} A_0(s) & B_0(s) \\ -C_0(s) & D_0(s)\end{bmatrix}\begin{bmatrix} N(s) & Y(s) \\ 0 & I_{\bar r}\end{bmatrix} \quad (3.5)$$

where (3.5) is a full equivalent transformation. Thus the following conditions will be satisfied

(a) According to the McMillan degree conditions of full equivalence transformation (3.5), we have that

$$\delta_M\left[\begin{array}{cc|cc} M(s) & 0 & A_0(s) & B_0(s) \\ X(s) & I_{\bar r} & -C_0(s) & D_0(s)\end{array}\right]=\delta_M\begin{bmatrix} A_0(s) & B_0(s) \\ -C_0(s) & D_0(s)\end{bmatrix} \quad (3.6a)$$

$$\text{and } \delta_M\begin{bmatrix} I_q & 0 \\ 0 & \mathcal{T}(s) \\ -N(s) & -Y(s) \\ 0 & -I_{\bar r}\end{bmatrix}=\delta_M\begin{bmatrix} I_q & 0 \\ 0 & \mathcal{T}(s)\end{bmatrix} \quad (3.6b)$$

which implies (see Hayton *et al* 1990) that the polynomial matrix $N(s)$ is constant i.e. $N(s)=N\in\mathbb{R}^{q\times q}$. From (3.6b), the conclusion that $N(s)$ is constant and the fact that the McMillan degree of a polynomial matrix is independent of its constant terms we conclude that

$$\delta_M\begin{bmatrix} \mathcal{T}(s) \\ -Y(s)\end{bmatrix}=\delta_M(\mathcal{T}(s)) \quad (3.7)$$

(b) The compound matrices

$$Q=\left[\begin{array}{cc|cc} M(s) & 0 & A_0(s) & B_0(s) \\ X(s) & I_{\bar r} & -C_0(s) & D_0(s)\end{array}\right]$$

$$\text{and } R=\begin{bmatrix} I_q & 0 \\ 0 & \mathcal{T}(s) \\ -N(s) & -Y(s) \\ 0 & -I_{\bar r}\end{bmatrix} \quad (3.8)$$

have no zeros in $\mathbb{C}\cup\{\infty\}$. Thus (3.5) may be rewritten under internal recoordination (which doesn't change the structure of the involved matrices (Hayton *et al.* 1990)) as

$$\left[\begin{array}{cc|cc} M(s) & 0 & A_0(s) & B_0(s) \\ X(s) & I_{\bar r} & -C_0(s) & D_0(s) \end{array}\right]\begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ -N & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix} \times$$

$$\times \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ N & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix}\left[\begin{array}{cc} I_q & 0 \\ 0 & \mathcal{T}(s) \\ \hline -N & -Y(s) \\ 0 & -I_{\bar r} \end{array}\right] = 0 \Leftrightarrow$$

$$\begin{bmatrix} M(s)-A_0(s)N & 0 \\ X(s)+C_0(s)N & I_{\bar r} \end{bmatrix}\begin{bmatrix} I_q & 0 \\ 0 & \mathcal{T}(s) \end{bmatrix} =$$

$$= \begin{bmatrix} A_0(s) & B_0(s) \\ -C_0(s) & D_0(s) \end{bmatrix}\begin{bmatrix} 0 & Y(s) \\ 0 & I_{\bar r} \end{bmatrix} \quad (3.9)$$

which implies that

$$M(s)-A_0(s)N = 0 \quad \text{and} \quad X(s)+C_0(s)N = 0 \quad (3.10)$$

$$A_0(s)Y(s)+B_0(s) = 0 \Leftrightarrow Y(s) = -A_0(s)^{-1}B_0(s)$$

and thus relation (3.9) may be rewritten as

$$\begin{bmatrix} 0 & 0 \\ 0 & I_{\bar r} \end{bmatrix}\begin{bmatrix} I_q & 0 \\ 0 & \mathcal{T}(s) \end{bmatrix} =$$

$$= \begin{bmatrix} A_0(s) & B_0(s) \\ -C_0(s) & D_0(s) \end{bmatrix}\begin{bmatrix} 0 & -A_0(s)^{-1}B_0(s) \\ 0 & I_{\bar r} \end{bmatrix} \quad (3.11)$$

Based on relation (3.11) we may now construct the following transformation

$$\left[\begin{array}{c|c} 0 & 0 \\ I_{\bar r} & 0 \\ \hline 0 & I_p \end{array}\right]\left[\begin{array}{c|c} \mathcal{T}(s) & \mathcal{U} \\ \hline -\mathcal{V} & 0 \end{array}\right] =$$

$$= \left[\begin{array}{cc|c} A_0(s) & B_0(s) & 0 \\ -C_0(s) & D_0(s) & \mathcal{U} \\ \hline 0 & -\mathcal{V} & 0 \end{array}\right]\left[\begin{array}{c|c} -A_0(s)^{-1}B_0(s) & 0 \\ I_{\bar r} & 0 \\ \hline 0 & I_m \end{array}\right] \quad (3.12)$$

Taking into account the conditions of full equivalence of the full equivalent transformation (3.11), it is easy to prove that the transformation (3.12) is a full system equivalence transformation. It can be easily seen that the compound matrices of the transformations (3.11) and (3.12) are related each other with the following strict equivalence transformations (which have the property to preserve the finite and infinite zero structure)

$$\left[\begin{array}{c|cccc} 0 & 0 & A_0(s) & B_0(s) & 0 \\ 0 & I_{\bar r} & -C_0(s) & D_0(s) & 0 \\ \hline I_p & 0 & 0 & 0 & 0 \end{array}\right] =$$

$$\left[\begin{array}{cc|ccc} 0 & 0 & A_0(s) & B_0(s) & 0 \\ I_{\bar r} & 0 & -C_0(s) & D_0(s) & \mathcal{U} \\ 0 & I_p & 0 & -\mathcal{V} & 0 \end{array}\right]\begin{bmatrix} 0 & I_{\bar r} & 0 & 0 & -\mathcal{U} \\ I_p & 0 & 0 & \mathcal{V} & 0 \\ 0 & 0 & I_q & 0 & 0 \\ 0 & 0 & 0 & I_{\bar r} & 0 \\ 0 & 0 & 0 & 0 & I_m \end{bmatrix} \quad (3.13)$$

and

$$\left[\begin{array}{c|c} 0 & 0 \\ \mathcal{T}(s) & 0 \\ A_0(s)^{-1}B_0(s) & 0 \\ -I_{\bar r} & 0 \\ \hline 0 & I_m \end{array}\right] = \begin{bmatrix} 0 & I_p & 0 & -\mathcal{V} & 0 \\ I_{\bar r} & 0 & 0 & 0 & \mathcal{U} \\ 0 & 0 & I_q & 0 & 0 \\ 0 & 0 & 0 & I_{\bar r} & 0 \\ 0 & 0 & 0 & 0 & -I_m \end{bmatrix} \times$$

$$\times \left[\begin{array}{cc} \mathcal{T}(s) & \mathcal{U} \\ -\mathcal{V} & 0 \\ \hline A_0(s)^{-1}B_0(s) & 0 \\ -I_{\bar r} & 0 \\ 0 & -I_m \end{array}\right] \quad (3.14)$$

a fact which proves that the compound matrices of the transformation (3.12) satisfy the same conditions with the compound matrices of the full equivalent transformation (3.11). Thus the transformation (3.12) satisfy all the conditions of full equivalence. We can also easily see using the symmetric transformation of (3.5) that the symmetric full system equivalence transformation of (3.12) is the following

$$\left[\begin{array}{cc|c} C_0(s)A_0(s)^{-1} & I_{\bar r} & 0 \\ \hline 0 & 0 & I_p \end{array}\right]\left[\begin{array}{cc|c} A_0(s) & B_0(s) & 0 \\ -C_0(s) & D_0(s) & \mathcal{U} \\ \hline 0 & -\mathcal{V} & 0 \end{array}\right] =$$

$$= \left[\begin{array}{c|c} \mathcal{T}(s) & \mathcal{U} \\ \hline -\mathcal{V} & 0 \end{array}\right]\left[\begin{array}{cc|c} 0 & I_{\bar r} & 0 \\ \hline 0 & 0 & I_m \end{array}\right] \quad (3.15)$$

We can also observe that : The realisation $\{A_0(s)\in\mathbb{R}[s]^{q\times q},\ B_0(s)\in\mathbb{R}[s]^{q\times \bar r},\ C_0(s)\in\mathbb{R}[s]^{\bar r\times q},\ D_0(s)\in\mathbb{R}[s]^{\bar r\times \bar r}\}$ is a strongly irreducible realization and thus the finite system poles (finite zeros of $A_0(s)$) coincide with the finite transmission poles (finite poles of $\mathcal{T}(s)$) (Verghese (1978)). However $\mathcal{T}(s)$ is a polynomial matrix and therefore $A_0(s)$ must include no finite zeros or equivalently $A_0(s)$ must be unimodular. Thus the matrices

$A_0(s)^{-1}B_0(s)$ and $C_0(s)A_0(s)^{-1}$ presented in (3.14) and (3.15) respectively are polynomial matrices and therefore the transformations (3.12) and (3.15) are full system equivalence transformations.

Finally we have from Theorem 1 that the linear multivariable system (1.1) is full system equivalent with its normalised form (1.2), which is according to the above full system equivalent to the generalised state space system (3.3). Thus according to the transitivity property of full system equivalence (see Theorem 1) we obtain that the systems (1.1) and (3.3) are full system equivalent.□

While the first algorithm is based on the realisation of $\mathcal{T}(s)$ defined in (1.2), the next algorithm is based on the realization of $\mathcal{T}(s)^{-1}$ and give us a theoretical extension of the known algorithm presented by Vardulakis (1991).

Algorithm 2.

Step 1. DATA : $[A(s)\in\mathbb{R}[s]^{r\times r},\ B(s)\in\mathbb{R}[s]^{r\times m},\ C(s)\in\mathbb{R}[s]^{p\times r},\ D(s)\in\mathbb{R}[s]^{p\times m}]$ be the PMD of Σ or equivalently $[\mathcal{T}(s)\in\mathbb{R}[s]^{\bar{r}\times\bar{r}},\ \mathcal{U}\in\mathbb{R}^{\bar{r}\times m},\ -\mathcal{V}\in\mathbb{R}^{p\times\bar{r}}]$, $\bar{r}=r+p+m$ be the PMD of its normalized form.

Step 2. Compute a strongly irreducible realization $[sE-A\in\mathbb{R}[s]^{\lambda_{av}\times\lambda_{av}},\quad B\in\mathbb{R}^{\lambda_{av}\times\bar{r}},\quad C\in\mathbb{R}^{\bar{r}\times\lambda_{av}},\ D\in\mathbb{R}^{\bar{r}\times\bar{r}}]$ of $\mathcal{T}(s)^{-1}$.

Step 3. The "equivalent" generalized state space system Σ_R of Σ will be the following

$$(\Sigma_R):\ E\dot{x}(t) = Ax(t) + B\mathcal{U}u(t) \qquad (3.16a)$$
$$y(t) = \mathcal{V}Cx(t) + \mathcal{V}D\mathcal{U}u(t) \qquad (3.16b)\ □$$

Theorem 4. The generalised state space system (3.16) is full system equivalent with the linear multivariable system Σ defined in (1.1).

Proof. This follow in a similar way to the result in Theorem 3 (see Karampetakis (1993)). The full system equivalence transformations which relate the systems (1.2) and (3.16) (comes from the proof of this theorem) are the following

$$\begin{bmatrix} \mathcal{T}(s)C(sE-A)^{-1} & 0 \\ 0 & I_p \end{bmatrix}\begin{bmatrix} sE-A & B\mathcal{U} \\ -\mathcal{V}C & \mathcal{V}D\mathcal{U} \end{bmatrix} = \begin{bmatrix} \mathcal{T}(s) & \mathcal{U} \\ -\mathcal{V} & 0 \end{bmatrix}\begin{bmatrix} C & -D\mathcal{U} \\ 0 & I_m \end{bmatrix} \qquad (3.17)$$

and

$$\begin{bmatrix} B & 0 \\ \mathcal{V}D & I_p \end{bmatrix}\begin{bmatrix} \mathcal{T}(s) & \mathcal{U} \\ -\mathcal{V} & 0 \end{bmatrix} = \begin{bmatrix} sE-A & B\mathcal{U} \\ -\mathcal{V}C & \mathcal{V}D\mathcal{U} \end{bmatrix}\begin{bmatrix} (sE-A)^{-1}B\mathcal{T}(s) & 0 \\ 0 & I_m \end{bmatrix} \qquad (3.18)$$

□

Corollary 3. The linear multivariable systems Σ in (1.1) and Σ_R in (3.16) (or in (3.3)) are full system equivalent and thus will possess the same finite and infinite system properties according to Theorem 1. □

As we can see from Algorithms 1 and 2 the construction of an "equivalent" generalized state space realization of the system Σ in (1.1) is concentrated on the computation of a strongly irreducible realization either of $\mathcal{T}(s)$ or $\mathcal{T}(s)^{-1}$. Verghese (1978) and Anderson *et al* (1985) (Vardulakis 1991) gave a solution to this problem with the construction of a strongly irreducible realization of $\mathcal{T}(s)$ ($\mathcal{T}(s)^{-1}$) in terms of finite and infinite Jordan pairs of $\mathcal{T}(s)$ ($\mathcal{T}(s)^{-1}$). A consequence of this kind of solution is, that their algorithms are not easily implemented. In contrast to the above authors, Bosgra and Van Der Weiden (1981) proposed a more practical solution to the above problem. A different approach to the implementation of the above algorithms is given in the next section by an extension of the known generalized state space realisation method for MFDs (matrix fraction descriptions) presented by Shaohua Tan & Van Dewalle (1988).

4. An extension of Tan & Van Dewall's model.

We present some lemmas which will be useful in the sequel

Lemma 1. (Vardulakis 1991, pp.144) Let $T(s)\in\mathbb{R}(s)^{p\times m}$ with $\text{rank}_{\mathbb{R}(s)}T(s)=m(=p)$. Then $T(s)$ is column (row) reduced at $s=\infty$ iff the pole–zero structure at $s=\infty$ of $T(s)$ is given by the pole–zero structure of its columns (rows) taken separately. □

Lemma 2. (Janssen 1988) Let $T(s)\in\mathbb{R}(s)^{p\times m}$ and $T(s)=Q(s)^{-1}R(s)$ $(=R(s)Q(s)^{-1})$, $R(s)\in\mathbb{R}[s]^{p\times m}$, $Q(s)\in\mathbb{R}[s]^{p\times p}$ $(Q(s)\in\mathbb{R}[s]^{m\times m})$ is a left (right) MFD of $T(s)$. If the compound matrix $[Q(s)\ \ R(s)]$ $([Q(s)^T\ \ R(s)^T]^T)$ possess no zeros in $\mathbb{C}\cup\{\infty\}$ then

$$\delta_M(Q(s)^{-1}R(s))=\sum_{i=1}^{k} q_i[Q\ \ R]=\delta_M(Q(s)\ \ R(s))$$

$$\left[\delta_M(R(s)Q(s)^{-1})=\sum_{i=1}^{k} q_i[Q^T\ \ R^T]^T= = \delta_M(Q(s)^T\ \ R(s)^T)^T\right] \qquad (4.1)$$

where $q_i[Q\ \ R]\geq 0$ ($q_i[Q^T\ \ R^T]^T\geq 0$) are the orders of the infinite poles of $[Q(s)\ \ R(s)]$ ($[Q(s)^T\ \ R(s)^T]^T$).□

Consider now the system Σ in (1.1) with its normalized form in (1.2) and let $S^{\infty}_{\mathcal{T}(s)}(s)$ be the Smith McMillan form at $s=\infty$ of $\mathcal{T}(s)$

$$S^{\infty}_{\mathcal{T}(s)}(s) := \text{diag}[s^{q_1}, s^{q_2}, ..., s^{q_k}, \frac{1}{s^{\hat{q}_{k+1}}}, ..., \frac{1}{s^{\hat{q}_{\bar{r}}}}] \qquad (4.2)$$

Then we propose the following algorithm for the construction of an "equivalent" generalized state space system of Σ in (1.1).

Algorithm 3.

Step 1. [Computation of U(s), Q(s), R(s)]

Compute a unimodular matrix $U(s)\in\mathbb{R}[s]^{\bar{r}\times\bar{r}}$ such that

$$\mathcal{T}(s)^{-1} = I_{\bar{r}} \times \mathcal{T}(s)^{-1} = [U(s)] \times [\mathcal{T}(s)U(s)]^{-1} \qquad (4.3)$$

where the compound matrix

$$\begin{bmatrix} Q(s) \\ R(s) \end{bmatrix} := \begin{bmatrix} \mathcal{T}(s)U(s) \\ U(s) \end{bmatrix} \qquad (4.4)$$

is column reduced.

Step 2. [Computation of Q_c, R_c]

Let

$$S^{\infty}_{[Q(s)^T\ \ R(s)^T]^T}(s) = \begin{bmatrix} \text{diag}[s^{\bar{q}_1}, s^{\bar{q}_2}, ..., s^{\bar{q}_{\bar{r}}}] \\ 0_{\bar{r}\times\bar{r}} \end{bmatrix} \qquad (4.5)$$

Define the matrix

$$S^T(s) = \begin{bmatrix} s^{\bar{q}_1}\ s^{\bar{q}_1-1} \cdots s\ 1 & \cdots & 0 \\ 0 & \cdots & 0 \\ \vdots & \ddots & \\ 0\ \ - & \cdots\ s^{\bar{q}_{\bar{r}}}\ s^{\bar{q}_{\bar{r}}-1} & \cdots s\ 1 \end{bmatrix} \qquad (4.6)$$

and write the polynomial matrices Q(s) and R(s) as follows

$$Q(s) = Q_cS(s)$$
$$R(s) = R_cS(s) \qquad (4.7)$$

where Q_c and R_c are constant matrices.

Step 3. [Computation of the "equivalent" generalised state space realisation]

Construct the following core realisation

$$E_cs-A_c = \text{block diag}\{E_{c1}s-A_{c1}, ..., E_{c\bar{r}}s-A_{c\bar{r}}\} \qquad (4.8)$$

where $E_{ci}s-A_{ci}\in\mathbb{R}[s]^{(\bar{q}_i+1)\times(\bar{q}_i+1)}$:

$$E_{ci}s-A_{ci} = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ -1 & s & \cdots & 0 & 0 \\ \vdots & & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -1 & s \end{bmatrix} \qquad (4.9)$$

and

$$B_c^T=\text{block diag}[B_1, B_2, ..., B_{\bar{r}}]$$

$$\text{with } B_i=[1\ 0\cdots\ 0]\in\mathbb{R}^{1\times(\bar{q}_i+1)} \qquad (4.10)$$

$$C_c = I_n, \quad n = \sum_{i=1}^{\bar{r}} \bar{q}_i + \bar{r} \qquad (4.11)$$

The "equivalent" generalised state space model of the system Σ in (1.1) will be the following

$$(\Sigma_R): \quad E\dot{x}(t) = Ax(t) + Bu(t) \qquad (4.12a)$$
$$y(t) = Cx(t) + Du(t) \qquad (4.12b)$$

where

$$E=E_c,\ A=A_c-B_cQ_c,\ B=B_c\mathcal{U},\ C=\mathcal{V}R_cC_c=\mathcal{V}R_c \qquad (4.13)\ \square$$

Theorem 5. The generalised state space system Σ_R in (4.12) is full system equivalent with the system Σ in (1.1).

Proof. It is easily checked (Tan & Van Dewalle 1988) that the PMD $[sE_c-A_c+B_cQ_c, B_c, R_c]$

(a) is strongly irreducible

(b) realises $\mathcal{T}(s)^{-1}$ i.e.

$$\mathcal{T}(s)^{-1}=R_c(sE_c-A_c+B_cQ_c)^{-1}B_c$$

Thus from Theorem 4 the generalised state space system (4.12) will be full system equivalent with the system $\Sigma^{(N)}$ in (1.2) (and thus with the system Σ in (1.1)) under the full system equivalence transformations (3.17) and (3.18). □

Remark 1. The dimension of the pseudostate x(t) of the generalised state space system Σ_R in (4.12) is equal to

$$\lambda_{av} = \bar{r} + \delta_M(\mathcal{T}(s)) \qquad (4.14)$$

Proof. We can easily see from (4.8), (4.9) and (4.13) that

$$\lambda_{av} = \sum_{i=1}^{\bar{r}}(\bar{q}_i+1) \qquad (4.15)$$

The compound matrix $[Q(s)^T\ \ R(s)^T]^T$ has

(1991) has now been formulated in two different theoretical reduction algorithms. It was shown that these reduction algorithms gives rise to full system equivalence generalised state space models. The advantages of this formulation are to show that : a) the "equivalent" models proposed by the above authors are fully system equivalent to the original one and thus gain the same finite and infinite system properties, b) the tide connection of the algorithms which produce strongly irreducible realisations and those which produce "equivalent" generalised state space models, c) to give the relation between the solution–input vector spaces of the equivalent systems, through the transforming matrices of the full system equivalence transformation (3.12, 3.15, 3.17, 3.18) (see Karampetakis & Vardulakis 1994). An implementation of the proposed theoretical algorithms has been elucidated through an extension of the Wolovich's (1973, 1974) and Tan & Vandewalle's (1988) reduction algorithms.The above extension has been illustrated by an example presented in section 6.

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