On the Solution of ARMA Representations.

by

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The objective of this paper is to determine a closed formula for the solution of a continuous time AutoRegressive Moving Average (ARMA) Representation. The importance of the above formula is that it gives rise to the solution of analysis, synthesis and design problems.

Keywords : ARMA representations, fundamental matrix, closed formula, solution.

1. Introduction.

Consider the nonhomogeneous system of linear algebraic and differential equations described in matrix form by :

$$A(\rho)\ y(t) = B(\rho)\ u(t) \quad (1.1)$$

where $\rho:=d/dt$ denotes the differential operator i.e. $\rho\ \ y(t):=dy(t)/dt$, $A(\rho)=A_0+A_1\rho+\cdots+A_q\rho^q \in\mathbb{R}[\rho]^{r\times r}$ with $\text{rank}_{\mathbb{R}(\rho)}A(\rho)=r$, $B(\rho)=B_0+B_1\rho+\cdots+B_s\rho^s\in\mathbb{R}[\rho]^{r\times m}$, $y(t):(0-,+\infty)\longrightarrow\mathbb{R}^r$ is the *output* of the system and $u(t):(0-,+\infty)\longrightarrow\mathbb{R}^m$ is the *input* of the system, a piecewise sufficiently differentiable function. Following the terminology of Willems (1991) we call the set of equations (1.1) an ARMA representation (*AutoRegressive Moving Average* representation) of $\mathcal{B}$ (behaviour), where $\mathcal{B}$ is the solution space of equations (1.1) defined by

$$\mathcal{B} = \pi_y(\mathcal{B}^f) \qquad (1.2a)$$

with

$$\mathcal{B}^f := \{(y(t),\ u(t)) : (0-,+\infty) \longrightarrow \mathbb{R}^r\times\mathbb{R}^m \mid \text{(ARMA) is satisfied } \forall t\in(0-,+\infty)\}$$

and

$$\pi_y : \mathbb{R}^r\times\mathbb{R}^m \longrightarrow \mathbb{R}^r \quad \pi_y(y(t),\ u(t)) = y(t) \qquad (1.2b)$$

In case where $A(\rho)=\rho E-A\in\mathbb{R}[\rho]^{r\times r}$ and $B(\rho)=B\in\mathbb{R}^{r\times m}$ then the ARMA representation (1.1) is the known generalized state space representation

$$E\frac{dx(t)}{dt} = A\ x(t) + B\ u(t) \quad (1.3)$$

while in case where $\det[E]\neq 0$, (1.3) is the known state space representation. For a survey of singular systems of the form (1.3) see Lewis (1986).

ARMA representations of the form (1.1) find numerous applications in analysis of circuits (Newcomb 1981), neural networks (DeClaris et al 1984), economics (the Leontieff model, see Luenberger 1977), power systems (Stott 1979) e.t.c. Analytic solutions of the ARMA representation (1.3) has been derived by many different techniques (Luenberger 1977, Campbell et al 1977, Campbell 1980, Yip & Sincovec 1981, Cobb 1980, 1982) while the numerical solution of the same system has been treated by Gear (1971), Brayton et al (1972), Sincovec et al (1981) and others. Following similar lines with Vardulakis (1991) we produce a closed formula for the solution of the general ARMA representation (1.1) in terms of the fundamental matrix H_k of the matrix $A(s)^{-1}$ (which can be easily calculated according to Fragulis et al (1991)), and the finite and infinite Jordan pairs of A(s). More specifically in section 3 and 4 we derive a closed formula for the homogeneous and forced response respectively of the ARMA representation (1.1) while in section 5 we derive a closed formula for the whole response of (1.1).

2. Problem formulation.

Consider the Autoregressive Moving Average Representation (ARMA—Representation) :

$$A(\rho)\ y(t) = B(\rho)\ u(t) \quad (2.1)$$

where ρ denotes the differential operator,

$$A(\rho)=A_0+A_1\rho+\cdots+A_q\rho^q\in\mathbb{R}[\rho]^{r\times r}, \text{ with } \det[A(\rho)]\neq 0 \quad (2.2a)$$

$$B(\rho)=B_0+B_1\rho+\cdots+B_s\rho^s\in\mathbb{R}[\rho]^{r\times m}, \quad (2.2b)$$

$y(t) : (0-,+\infty) \longrightarrow \mathbb{R}^r$ is the "output" of the ARMA representation and $u(t) : (0-,+\infty) \longrightarrow \mathbb{R}^m$ is the "input" of the ARMA representation, where $\{H_i\ i\in\mathbb{Z}\}$ be the fundamental matrix sequence at infinity of $A(s)^{-1}$ which is easily implemented (Fragulis at al 1991) and ($C\in\mathbb{R}^{r\times n}$, $J\in\mathbb{R}^{n\times n}$,

$B\in\mathbb{R}^{n\times r}$), ($C_\infty\in\mathbb{R}^{r\times\mu}$, $J_\infty\in\mathbb{R}^{\mu\times\mu}$, $B_\infty\in\mathbb{R}^{\mu\times r}$) be its finite and infinite Jordan pairs respectively :

$$A^{-1}(s)=H_{\hat{q}_r}s^{\hat{q}_r}+\cdots+H_1 s+H_0+H_{-1}\frac{1}{s}+$$

$$+H_{-2}\frac{1}{s^2}+\cdots=C_\infty(I_\mu-sJ_\infty)^{-1}B_\infty+C(sI_n-J)^{-1}B \tag{2.3}$$

PROBLEM Based on the equations (2.1)–(2.3) find out an easily implemented closed formula solution for the ARMA–representation (2.1).

3. Closed formula for the solution of the AR–representation $A(\rho)y_{hom}(t)=0$.

From Vardulakis (1991) we obtain $y_{hom}(t)$ to be the following

In case where $q<\hat{q}_r$

$$y_{hom}(t)=-\,[\delta^{(\hat{q}_r-1)}(t)I_r,\dots,\delta(t)I_r]\times$$

$$\times\begin{bmatrix} H_{\hat{q}_r} & 0 & & \\ H_{\hat{q}_r-1} & H_{\hat{q}_r} & & 0 \\ \vdots & \vdots & \ddots & \\ H_{s-\hat{q}_r} & H_{s-\hat{q}_r+1} & \cdots & H_{\hat{q}_r} \\ \vdots & \vdots & & \vdots \\ H_1 & H_2 & & H_q \end{bmatrix}\times$$

$$\times\begin{bmatrix} A_0 & A_1 & \cdots & A_{q-1} \\ 0 & A_0 & \cdots & A_{q-2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_0 \end{bmatrix}\begin{bmatrix} y(0-) \\ y^{(1)}(0-) \\ \\ y^{(q-1)}(0-) \end{bmatrix}+$$

$$+Ce^{Jt}[J^{q-1}B,J^{q-2}B,\dots,B]\times$$

$$\begin{bmatrix} A_q & 0 & \cdots & 0 \\ A_{q-1} & A_q & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_1 & A_2 & \cdots & A_q \end{bmatrix}\begin{bmatrix} y(0-) \\ y^{(1)}(0-) \\ \\ y^{(q-1)}(0-) \end{bmatrix} \tag{3.1a}$$

In case where $q>\hat{q}_r$

$$y_{hom}(t)=-[\delta^{(\hat{q}_r-1)}(t)I_r,\delta^{(\hat{q}_r-2)}(t)I_r,\dots,\delta(t)I_r]\times$$

$$\times\begin{bmatrix} H_{\hat{q}_r} & 0 & \cdots & 0 \\ H_{\hat{q}_r-1} & H_{\hat{q}_r} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ H_1 & H_2 & \cdots & H_{\hat{q}_r} \end{bmatrix}\times$$

$$\times\begin{bmatrix} A_0 & A_1 & \cdots & A_{\hat{q}_r-1} & \cdots & A_{q-1} \\ 0 & A_0 & \cdots & A_{\hat{q}_r-2} & \cdots & A_{q-2} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_0 & \cdots & A_{q-\hat{q}_r} \end{bmatrix}\begin{bmatrix} y(0-) \\ y^{(1)}(0-) \\ \\ y^{(q-1)}(0-) \end{bmatrix}+ \tag{3.1b}$$

$$+\,C\,e^{Jt}\,[J^{q-1}B,\,J^{q-2}B,\dots,B]\times$$

$$\times\begin{bmatrix} A_q & 0 & \cdots & 0 \\ A_{q-1} & A_q & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_1 & A_2 & \cdots & A_q \end{bmatrix}\begin{bmatrix} y(0-) \\ y^{(1)}(0-) \\ \\ y^{(q-1)}(0-) \end{bmatrix}$$

The above solution will be free of impulsive terms in case where the coefficients of the impulse terms in (3.1) are equal to zero or equivalently when the initial conditions $\{y(0), y^{(1)}(0),\dots,y^{(q-1)}(0)\}$ of the system (2.1) belong to the following space H_{i0}

In case where $q<\hat{q}_r$

$$H_{i0}:=\left\{\begin{bmatrix} y(0-) \\ y^{(1)}(0-) \\ \\ y^{(q-1)}(0-) \end{bmatrix} : \begin{bmatrix} H_{\hat{q}_r} & 0 & & \\ H_{\hat{q}_r-1} & H_{\hat{q}_r} & & 0 \\ \vdots & \vdots & \ddots & \\ H_{s-\hat{q}_r} & H_{s-\hat{q}_r+1} & \cdots & H_{\hat{q}_r} \\ \vdots & \vdots & & \vdots \\ H_1 & H_2 & & H_q \end{bmatrix}\times\begin{bmatrix} A_0 & A_1 & \cdots & A_{q-1} \\ 0 & A_0 & \cdots & A_{q-2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_0 \end{bmatrix}\begin{bmatrix} y(0-) \\ y^{(1)}(0-) \\ \\ y^{(q-1)}(0-) \end{bmatrix}=0\right\} \tag{3.2a}$$

In case where $q>q_r$

$$H_{i0}:=\left\{\begin{bmatrix} y(0-) \\ y^{(1)}(0-) \\ \\ y^{(q-1)}(0-) \end{bmatrix} : \begin{bmatrix} H_{\hat{q}_r} & 0 & \cdots & 0 \\ H_{\hat{q}_r-1} & H_{\hat{q}_r} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ H_1 & H_2 & \cdots & H_{\hat{q}_r} \end{bmatrix}\times\right.$$

$$\times \begin{bmatrix} A_0 & A_1 & \cdots & A_{\hat{q}_r-1} & \cdots & A_{q-1} \\ 0 & A_0 & \cdots & A_{\hat{q}_r-2} & \cdots & A_{q-2} \\ \vdots & \vdots & \ddots & \vdots & & \vdots \\ 0 & 0 & \cdots & A_0 & \cdots & A_{q-\hat{q}_r} \end{bmatrix} \begin{bmatrix} y(0-) \\ y^{(1)}(0-) \\ \vdots \\ y^{(q-1)}(0-) \end{bmatrix} = 0 \Bigg\} \qquad (3.2b)$$

or equivalently

$$H_{i0} := \Bigg\{ y^{(i)}(0)\ (i=0,1,..,q-1) : \sum_{i=k}^{\hat{q}_r} \sum_{j=0}^{q-i+k-1} \Big[H_i A_j y^{(i+j-k)}(0) \Big] = 0, \quad k=1,2,..,\hat{q}_r \Bigg\} \qquad (3.3)$$

Definition 1. H_{i0} is called the admissible initial condition space of (2.1) under zero inputs. □

If $\{y(0), y^{(1)}(0),...,y^{(q-1)}(0)\}$ belongs to H_{i0} then the solution of the homogeneous system (2.1) will be, according to (3.1), the following

$$y_{hom}(t)=Ce^{Jt}[J^{q-1}B, J^{q-2}B,.., B] \times \begin{bmatrix} A_q & 0 & \cdots & 0 \\ A_{q-1} & A_q & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_1 & A_2 & \cdots & A_q \end{bmatrix} \begin{bmatrix} y(0-) \\ y^{(1)}(0-) \\ \vdots \\ y^{(q-1)}(0-) \end{bmatrix} \qquad (3.4)$$

4. Closed formula for the forced response of the ARMA–representation $A(\rho)y_{for}(t)= B(\rho)u(t)$.

In this section we examine the forced response of the system (2.1). Taking Laplace transforms in (2.1) and assuming zero initial conditions $\{y^{(i)}(0)=0_{r\times 1}\ i=0,1,..,q-1\}$ we obtain

$$A(s)\ \hat{y}_{for}(s) = B(s)\ \hat{u}(s) - [s^{s-1}I_r, s^{s-2}I_r, ..., I_r] \times \begin{bmatrix} B_s & 0 & \cdots & 0 \\ B_{s-1} & B_s & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ B_1 & B_2 & \cdots & B_s \end{bmatrix} \begin{bmatrix} u(0-) \\ u^{(1)}(0-) \\ \vdots \\ u^{(s-1)}(0-) \end{bmatrix} \Leftrightarrow$$

$$\hat{y}_{for}(s) := \hat{y}^1_{for}(s) - \hat{y}^2_{for}(s) := \qquad (4.1)$$

$$=[A^{-1}(s)B(s)]\ \hat{u}(s) - A^{-1}(s)[s^{s-1}I_r, s^{s-2}I_r,..., I_r] \times \begin{bmatrix} B_s & 0 & \cdots & 0 \\ B_{s-1} & B_s & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ B_1 & B_2 & \cdots & B_s \end{bmatrix} \begin{bmatrix} u(0-) \\ u^{(1)}(0-) \\ \vdots \\ u^{(s-1)}(0-) \end{bmatrix}$$

where $\hat{y}_{for}(s):=\mathcal{L}[y_{for}(t)]$ and $\hat{u}(s):=\mathcal{L}[u(t)]$ ($\mathcal{L}$ denotes the Laplace transform). It is known from Vardulakis (1991) that

$$\hat{y}^1_{for}(s) = A^{-1}(s)\ B(s)\ \hat{u}(s) = \qquad (4.2)$$

$$=[C \quad C_\infty]\Psi\Phi \begin{bmatrix} I_m \\ sI_m \\ \vdots \\ s^{\hat{q}_r+s}I_m \end{bmatrix} \hat{u}(s) + C(sI_n-J)^{-1}\Omega\hat{u}(s)$$

where

$$\Psi := \left[\begin{array}{c|c} J^{s-1}B\ J^{s-2}B\ \cdots B & 0_{n,(\hat{q}_r+1)r} \\ \hline 0_{\mu . sr} & B_\infty\ \ J_\infty B_\infty\ \cdots\ J_\infty^{\hat{q}_r}B_\infty \end{array} \right] \in \mathbb{R}^{(n+\mu)\times(\hat{q}_r+1+s)r} \qquad (4.3)$$

$$\Phi := \begin{bmatrix} B_s & 0 & & & & & \\ B_{s-1} & B_s & & & & 0 & \\ \vdots & \vdots & \ddots & & & & \\ B_1 & B_2 & \cdots & & & & \\ B_0 & B_1 & & & & & \\ 0 & B_0 & & & & & \\ \vdots & \vdots & & & & & \\ 0 & 0 & \cdots & B_0 & B_1 & \cdots & B_{s-1}\ B_s \end{bmatrix} \in \mathbb{R}^{(\hat{q}_r+1+s)r\times(\hat{q}_r+1+s)m} \qquad (4.4)$$

$$\Omega:=[J^sBB_s+J^{s-1}BB_{s-1}+\cdots+JBB_1+BB_0] \in \mathbb{R}^{r\times m} \qquad (4.5)$$

and ($C\in\mathbb{R}^{r\times n}, J\in\mathbb{R}^{n\times n}, B\in\mathbb{R}^{n\times r}$), ($C_\infty^{\ r\times\mu}$, $J_\infty^{\ \mu\times\mu}$, $B_\infty^{\ \mu\times r}$) be the finite and infinite Jordan pair of $A(s)$. Consider now the second term $\hat{y}^2_{for}(s)$ of (4.1)

$$\hat{y}^2_{for}(s) = A^{-1}(s)[s^{s-1}I_r, s^{s-2}I_r, ..., I_r]\times \qquad (4.6a)$$

$$\times \begin{bmatrix} B_s & 0 & \cdots & 0 \\ B_{s-1} & B_s & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ B_1 & B_2 & \cdots & B_s \end{bmatrix} \begin{bmatrix} u(0-) \\ u^{(1)}(0-) \\ \vdots \\ u^{(s-1)}(0-) \end{bmatrix}$$

After some manipulations of (4.6a) we obtain that

$$\hat{y}^2_{for}(s) = [s^{\hat{q}_r+s-1} I_r,\ s^{\hat{q}_r+s-2} I_r, \ldots, I_r] \times$$

$$\times \begin{bmatrix} H_{\hat{q}_r} & 0 & & \\ H_{\hat{q}_r-1} & H_{\hat{q}_r} & & 0 \\ \vdots & \vdots & \ddots & \\ H_{s-\hat{q}_r} & H_{s-\hat{q}_r+1} & \cdots & H_{\hat{q}_r} \\ \vdots & \vdots & & \vdots \\ H_{-s+1} & H_{-s} & & H_0 \end{bmatrix} \begin{bmatrix} B_s & 0 & \cdots & 0 \\ B_{s-1} & B_s & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ B_1 & B_2 & \cdots & B_s \end{bmatrix} \times$$

$$\times \begin{bmatrix} u(0-) \\ u^{(1)}(0-) \\ \vdots \\ u^{(s-1)}(0-) \end{bmatrix} + [s^{-1} I_r,\ s^{-2} I_r, \ldots] \times$$

$$\times \begin{bmatrix} H_{-s} & H_{-s+1} & \cdots & H_{-1} \\ H_{-s-1} & H_{-s} & \cdots & H_{-2} \\ \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \cdots & \vdots \end{bmatrix} \times$$

$$\times \begin{bmatrix} B_s & 0 & \cdots & 0 \\ B_{s-1} & B_s & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ B_1 & B_2 & \cdots & B_s \end{bmatrix} \begin{bmatrix} u(0-) \\ u^{(1)}(0-) \\ \vdots \\ u^{(s-1)}(0-) \end{bmatrix} =$$

$$:= \hat{y}^{21}_{for}(s) + \hat{y}^{22}_{for}(s) \quad (4.6b)$$

We have also that

$$\hat{y}^{22}_{for}(s) = [s^{-1} I_r, s^{-2} I_r, \ldots] \begin{bmatrix} H_{-s} & H_{-s+1} & \cdots & H_{-1} \\ H_{-s-1} & H_{-s} & \cdots & H_{-2} \\ \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \cdots & \vdots \end{bmatrix}$$

$$\times \begin{bmatrix} B_s & 0 & \cdots & 0 \\ B_{s-1} & B_s & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ B_1 & B_2 & \cdots & B_s \end{bmatrix} \begin{bmatrix} u(0-) \\ u^{(1)}(0-) \\ \vdots \\ u^{(s-1)}(0-) \end{bmatrix} = \cdots =$$

$$= C\,[sI_n - J]^{-1}\, u_s(0-) \quad (4.7a)$$

where

$$u_s(0-) = [J^{s-1}B,\ J^{s-2}B, \ldots, B] \times$$

$$\times \begin{bmatrix} B_s & 0 & \cdots & 0 \\ B_{s-1} & B_s & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ B_1 & B_2 & \cdots & B_s \end{bmatrix} \begin{bmatrix} u(0-) \\ u^{(1)}(0-) \\ \vdots \\ u^{(s-1)}(0-) \end{bmatrix} \quad (4.7b)$$

Combining now (4.1), (4.2), (4.6) and (4.7) and after some manipulations, we obtain the Laplace transform of the forced response of the ARMA–representation (2.1)

$$\hat{y}_{for}(s) = [C \ \ C_\infty]\, \Psi\, \Phi \left\{ \begin{bmatrix} \hat{u}(s) \\ s\hat{u}(s) \\ \vdots \\ s^{\hat{q}_r+s}\hat{u}(s) \end{bmatrix} - \begin{bmatrix} 0 \\ u(0) \\ \vdots \\ s^{\hat{q}_r+s-1}u(0) + \cdots + u^{(\hat{q}_r+s-1)}(0-) \end{bmatrix} \right\} +$$

$$+ C(sI_n - J)^{-1}\, \Omega\, \hat{u}(s) - C\,[sI_n - J]^{-1}\, u_s(0-) +$$

$$+ [s^{\hat{q}_r-1} I_r,\ s^{\hat{q}_r-2} I_r, \ldots, I_r] \begin{bmatrix} H_{\hat{q}_r} & 0 & \cdots & 0 \\ H_{\hat{q}_r-1} & H_{\hat{q}_r} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ H_1 & H_2 & \cdots & H_{\hat{q}_r} \end{bmatrix} \times$$

$$\begin{bmatrix} B_0 & B_1 & \cdots & B_s & & \\ 0 & B_0 & & & \ddots & \\ \vdots & \vdots & \ddots & & & \\ 0 & 0 & \cdots & B_0 & B_1 \cdots B_{s-1} & B_s \end{bmatrix} \begin{bmatrix} u(0-) \\ u^{(1)}(0-) \\ \vdots \\ u^{(s-1)}(0-) \\ \vdots \\ u^{(\hat{q}_r+s-1)}(0-) \end{bmatrix}$$

$(\mathcal{L}^{-1})$

$$\Rightarrow \ y_{for}(t) = [H_{-s}\ H_{-s+1} \ldots H_0 \ldots H_{\hat{q}_r}] \times$$

$$\begin{bmatrix} B_s & 0 & & \\ B_{s-1} & B_s & & 0 \\ \vdots & \vdots & \ddots & \\ B_1 & B_2 & \cdots & \\ B_0 & B_1 & & \\ 0 & B_0 & & \\ \vdots & \vdots & \ddots & \\ 0 & 0 & \cdots B_0\ B_1 \cdots B_{s-1} & B_s \end{bmatrix} \times \begin{bmatrix} u(t) \\ u^{(1)}(t) \\ \vdots \\ u^{(s-1)}(t) \\ \vdots \\ u^{(\hat{q}_r+s)}(t) \end{bmatrix} +$$

$$+\int_0^t Ce^{J(t-\tau)}\Omega\, u(\tau)d\tau - Ce^{Jt}u_s(0-)+$$

$$+[\delta^{(\hat q_r-1)}(t)I_r,\ \delta^{(\hat q_r-2)}(t)I_r,..,\ \delta(t)I_r]\times$$

$$\times\begin{bmatrix} H_{\hat q_r} & 0 & \cdots & 0 \\ H_{\hat q_r-1} & H_{\hat q_r} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ H_1 & H_2 & \cdots & H_{\hat q_r}\end{bmatrix}\times$$

$$\begin{bmatrix} B_0 & B_1 & \cdots & B_s & & \\ 0 & B_0 & & & \ddots & \\ \vdots & \vdots & \ddots & & & \\ 0 & 0 & \cdots & B_0\ B_1 & \cdots B_{s-1} & B_s\end{bmatrix}\begin{bmatrix} u(0-) \\ u^{(1)}(0-) \\ \vdots \\ u^{(s-1)}(0-) \\ \vdots \\ u^{(\hat q_r+s-1)}(0-)\end{bmatrix} \quad (4.8)$$

where $u^{(i)}(t)$, $i=0,1,..,\hat q_r+s$, denotes the i–th ordinary derivative of u(t) and Ω, $u_s(0-)$ defined in (4.5) and (4.7) respectively, or equivalently

$$y_{for}(t) = \sum_{i=0}^{s}\sum_{j=i}^{s} H_{-i}B_j u^{(j-i)}(t) +$$

$$+\sum_{i=1}^{\hat q_r}\sum_{j=0}^{s} H_iB_ju^{(j+i)}(t) + \int_0^t Ce^{J(t-\tau)}\Omega u(\tau)d\tau -$$

$$+Ce^{Jt}u_s(0-)+\sum_{i=0}^{\hat q_r-1}\sum_{l=i+1}^{\hat q_r}\sum_{j=0}^{s}\Big[H_lB_j\, u^{(j+l-i-1)}(0-)\Big]\delta^{(i)}(t) \quad (4.9)$$

It can be easily seen from the above that, the forced response of the ARMA–representation (2.1) will be free of impulses iff the input initial conditions $\{u(0-),u^{(1)}(0-),...,\ u^{(\hat q_r+s-1)}(0-)\}$ satisfy the following constraints :

$$\begin{bmatrix} H_{\hat q_r} & 0 & \cdots & 0 \\ H_{\hat q_r-1} & H_{\hat q_r} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ H_1 & H_2 & \cdots & H_{\hat q_r}\end{bmatrix}\times$$

$$\begin{bmatrix} B_0 & B_1 & \cdots & B_s & & \\ 0 & B_0 & & & \ddots & \\ \vdots & \vdots & \ddots & & & \\ 0 & 0 & \cdots & B_0\ B_1 & \cdots B_{s-1} & B_s\end{bmatrix}\begin{bmatrix} u(0-) \\ u^{(1)}(0-) \\ \vdots \\ u^{(s-1)}(0-) \\ \vdots \\ u^{(\hat q_r+s-1)}(0-)\end{bmatrix}$$

$$= 0 \quad (4.10)$$

5. Closed formula for the solution of the ARMA–representation $A(\rho)y(t)=B(\rho)u(t)$.

In this section we examine the whole response of the system (2.1). Taking Laplace transforms in (2.1) we obtain

$$A(s)\,\hat y(s) = [s^{q-1}I_r,\ s^{q-2}I_r,\ ...,\ I_r]\times$$

$$\times\begin{bmatrix} A_q & 0 & \cdots & 0 \\ A_{q-1} & A_q & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_1 & A_2 & \cdots & A_q\end{bmatrix}\begin{bmatrix} y(0-) \\ y^{(1)}(0-) \\ \vdots \\ y^{(q-1)}(0-)\end{bmatrix}$$

$$+ B(s)\,\hat u(s) - [s^{s-1}I_r,\ s^{s-2}I_r,\ ...,\ I_r]$$

$$\times\begin{bmatrix} B_s & 0 & \cdots & 0 \\ B_{s-1} & B_s & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ B_1 & B_2 & \cdots & B_s\end{bmatrix}\begin{bmatrix} u(0-) \\ u^{(1)}(0-) \\ \vdots \\ u^{(s-1)}(0-)\end{bmatrix} \Leftrightarrow$$

$$\hat y(s) := \underbrace{A^{-1}(s)\,[s^{q-1}I_r,\ s^{q-2}I_r,\ ...,\ I_r]\times\begin{bmatrix} A_q & 0 & \cdots & 0 \\ A_{q-1} & A_q & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_1 & A_2 & \cdots & A_q\end{bmatrix}\begin{bmatrix} y(0-) \\ y^{(1)}(0-) \\ \vdots \\ y^{(q-1)}(0-)\end{bmatrix}}_{\hat y_{hom}(s)} +$$

$$+ \underbrace{[A^{-1}(s)B(s)]\,\hat u(s) - A^{-1}(s)[s^{s-1}I_r,\ s^{s-2}I_r,\ ...,I_r]\begin{bmatrix} B_s & 0 & \cdots & 0 \\ B_{s-1} & B_s & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ B_1 & B_2 & \cdots & B_s\end{bmatrix}\begin{bmatrix} u(0-) \\ u^{(1)}(0-) \\ \vdots \\ u^{(s-1)}(0-)\end{bmatrix}}_{\hat y_{for}(s)} \quad (5.1)$$

where $\hat y(s):=\mathcal{L}[y(t)]$ and $\hat u(s):=\mathcal{L}[u(t)]$. It is easily seen from (5.1) and sections 3, 4 that the whole response of the system will be equal to the sum of the free response and the forced response of the ARMA—representation (2.1). Thus the solution of the ARMA—representation (2.1) will be equal to :

<u>In case where $q<\hat{q}_r$</u>

$$y(t) = -\left[\delta^{(\hat{q}_r-1)}(t)I_r,\ \delta^{(\hat{q}_r-2)}I_r, .., \delta(t)I_r\right]\times$$

$$\begin{bmatrix} H_{\hat{q}_r} & 0 & & \\ H_{\hat{q}_r-1} & H_{\hat{q}_r} & 0 & \\ \vdots & \vdots & \ddots & \\ H_{s-\hat{q}_r} & H_{s-\hat{q}_r+1} & \cdots & H_{\hat{q}_r} \\ \vdots & \vdots & & \vdots \\ H_1 & H_2 & & H_q \end{bmatrix} \times \qquad (5.2a)$$

$$\times \begin{bmatrix} A_0 & A_1 & \cdots & A_{q-1} \\ 0 & A_0 & \cdots & A_{q-2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_0 \end{bmatrix} \begin{bmatrix} y(0-) \\ y^{(1)}(0-) \\ \vdots \\ y^{(q-1)}(0-) \end{bmatrix} +$$

$$+Ce^{Jt}[y_s(0-)-u_s(0-)]+\int_0^t Ce^{J(t-\tau)}\Omega\, u(\tau)d\tau +$$

$$+ [H_{-s}\ H_{-s+1}...H_0...H_{\hat{q}_r}] \times$$

$$\begin{bmatrix} B_s & 0 & & & \\ B_{s-1} & B_s & & 0 & \\ \vdots & \vdots & \ddots & & \\ B_1 & B_2 & \cdots & & \\ B_0 & B_1 & & \ddots & \\ 0 & B_0 & & & \\ \vdots & \vdots & \ddots & & \\ 0 & 0 & \cdots\ B_0\ B_1 & \cdots & B_{s-1}\ B_s \end{bmatrix} \begin{bmatrix} u(t) \\ u^{(1)}(t) \\ \vdots \\ u^{(s-1)}(t) \\ \vdots \\ u^{(\hat{q}_r+s)}(t) \end{bmatrix} +$$

$$+[\delta^{(\hat{q}_r-1)}(t)I_r, .., \delta(t)I_r] \begin{bmatrix} H_{\hat{q}_r} & 0 & \cdots & 0 \\ H_{\hat{q}_r-1} & H_{\hat{q}_r} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ H_1 & H_2 & \cdots & H_{\hat{q}_r} \end{bmatrix} \times$$

$$\begin{bmatrix} B_0 & B_1 & \cdot & B_s \\ 0 & B_0 & & \\ \vdots & \vdots & \ddots & \\ 0 & 0 & \cdots\ B_0\ B_1 \cdots B_{s-1} & B_s \end{bmatrix} \begin{bmatrix} u(0-) \\ u^{(1)}(0-) \\ \vdots \\ u^{(s-1)}(0-) \\ \vdots \\ u^{(\hat{q}_r+s-1)}(0-) \end{bmatrix}$$

<u>In case where $q>\hat{q}_r$</u>

$$y(t)=-[\delta^{(\hat{q}_r-1)}(t)I_r,\ \delta^{(\hat{q}_r-2)}(t)I_r, .., \delta(t)I_r] \times$$

$$\begin{bmatrix} H_{\hat{q}_r} & 0 & \cdots & 0 \\ H_{\hat{q}_r-1} & H_{\hat{q}_r} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ H_1 & H_2 & \cdots & H_{\hat{q}_r} \end{bmatrix} \times$$

$$\times \begin{bmatrix} A_0 & A_1 & \cdots & A_{\hat{q}_r-1} & \cdots & A_{q-1} \\ 0 & A_0 & \cdots & A_{\hat{q}_r-2} & \cdots & A_{q-2} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_0 & \cdots & A_{q-\hat{q}_r} \end{bmatrix} \begin{bmatrix} y(0-) \\ y^{(1)}(0-) \\ \vdots \\ y^{(q-1)}(0-) \end{bmatrix} +$$

$$+Ce^{Jt}[y_s(0-)-u_s(0-)]+\int_0^t Ce^{J(t-\tau)}\Omega\, u(\tau)d\tau +$$

$$+ [H_{-s}\ H_{-s+1}...H_0...H_{\hat{q}_r}] \times$$

$$\begin{bmatrix} B_s & 0 & & & \\ B_{s-1} & B_s & & 0 & \\ \vdots & \vdots & \ddots & & \\ B_1 & B_2 & \cdots & & \\ B_0 & B_1 & & \ddots & \\ 0 & B_0 & & & \\ \vdots & \vdots & \ddots & & \\ 0 & 0 & \cdots\ B_0\ B_1 & \cdots & B_{s-1}\ B_s \end{bmatrix} \begin{bmatrix} u(t) \\ u^{(1)}(t) \\ \vdots \\ u^{(s-1)}(t) \\ \vdots \\ u^{(\hat{q}_r+s)}(t) \end{bmatrix} +$$

$$+ [\delta^{(\hat{q}_r-1)}(t)I_r,\ \delta^{(\hat{q}_r-2)}(t)I_r, .., \delta(t)I_r]\times$$

$$\begin{bmatrix} H_{\hat{q}_r} & 0 & \cdots & 0 \\ H_{\hat{q}_r-1} & H_{\hat{q}_r} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ H_1 & H_2 & \cdots & H_{\hat{q}_r} \end{bmatrix} \times \qquad (5.2b)$$

$$\times \begin{bmatrix} B_0 & B_1 & \cdots & B_s \\ 0 & B_0 & & \\ \vdots & \vdots & \ddots & \\ 0 & 0 & \cdots\ B_0\ B_1 \cdots B_{s-1} & B_s \end{bmatrix} \begin{bmatrix} u(0-) \\ u^{(1)}(0-) \\ \vdots \\ u^{(s-1)}(0-) \\ \vdots \\ u^{(\hat{q}_r+s-1)}(0-) \end{bmatrix}$$

where

$$y_s(0-) = [J^{q-1}B,\ J^{q-2}B, .., B]\times$$

$$\times\begin{bmatrix} A_q & 0 & \cdots & 0 \\ A_{q-1} & A_q & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_1 & A_2 & \cdots & A_q \end{bmatrix}\begin{bmatrix} y(0-) \\ y^{(1)}(0-) \\ \vdots \\ y^{(q-1)}(0-) \end{bmatrix} \tag{5.3}$$

and $u_s(0-)$, Ω defined in (4.7b) and (4.5) respectively. We define now the following admissible initial condition space H_{iu}.

Definition 2. Consider the following space of initial conditions :

$$H_{iu}:=\left\{\begin{bmatrix} y(0-) \\ y^{(1)}(0-) \\ \vdots \\ y^{(q-1)}(0-) \end{bmatrix} ; \begin{bmatrix} u(0-) \\ u^{(1)}(0-) \\ \vdots \\ u^{(s-1)}(0-) \\ \vdots \\ u^{(\hat{q}_r+s-1)}(0-) \end{bmatrix} :\right.$$

<u>If $q<\hat{q}_r$</u>

$$\begin{bmatrix} H_{\hat{q}_r} & 0 & \cdots & 0 \\ H_{\hat{q}_r-1} & H_{\hat{q}_r} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ H_1 & H_2 & \cdots & H_{\hat{q}_r} \end{bmatrix}\times$$

$$\begin{bmatrix} B_0 & B_1 & \cdot & B_l & & \\ 0 & B_0 & & & \ddots & \\ \vdots & \vdots & \ddots & & & \\ 0 & 0 & \cdots & B_0 & B_1 \cdots B_{l-1} & B_l \end{bmatrix}\begin{bmatrix} u(0-) \\ u^{(1)}(0-) \\ \vdots \\ u^{(l-1)}(0-) \\ \vdots \\ u^{(\hat{q}_r+l-1)}(0-) \end{bmatrix}$$

$$-\begin{bmatrix} H_{\hat{q}_r} & 0 & & \\ H_{\hat{q}_r-1} & H_{\hat{q}_r} & & 0 \\ \vdots & \vdots & \ddots & \\ H_{\hat{q}_r-q+1} & H_{\hat{q}_r-q+2} & \cdots & H_{\hat{q}_r} \\ \vdots & \vdots & & \vdots \\ H_1 & H_2 & & H_q \end{bmatrix}\times$$

$$\times\begin{bmatrix} A_0 & A_1 & \cdots & A_{q-1} \\ 0 & A_0 & \cdots & A_{q-2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_0 \end{bmatrix}\begin{bmatrix} y(0-) \\ y^{(1)}(0-) \\ \vdots \\ y^{(q-1)}(0-) \end{bmatrix}=0 \tag{5.4a}$$

<u>If $q>\hat{q}_r$</u>

$$\begin{bmatrix} H_{\hat{q}_r} & 0 & \cdots & 0 \\ H_{\hat{q}_r-1} & H_{\hat{q}_r} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ H_1 & H_2 & \cdots & H_{\hat{q}_r} \end{bmatrix}\times$$

$$\begin{bmatrix} B_0 & B_1 & \cdots & B_s & & \\ 0 & B_0 & & & \ddots & \\ \vdots & \vdots & \ddots & & & \\ 0 & 0 & \cdots & B_0 & B_1 \cdots B_{s-1} & B_s \end{bmatrix}\times\begin{bmatrix} u(0-) \\ u^{(1)}(0-) \\ \vdots \\ u^{(s-1)}(0-) \\ \vdots \\ u^{(\hat{q}_r+s-1)}(0-) \end{bmatrix}$$

$$-\begin{bmatrix} H_{\hat{q}_r} & 0 & \cdots & 0 \\ H_{\hat{q}_r-1} & H_{\hat{q}_r} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ H_1 & H_2 & \cdots & H_{\hat{q}_r} \end{bmatrix} \tag{5.4b}$$

$$\left.\times\begin{bmatrix} A_0 & A_1 \cdots A_{\hat{q}_r-1} & \cdots & A_{q-1} \\ 0 & A_0 \cdots A_{\hat{q}_r-2} & \cdots & A_{q-2} \\ \vdots & \vdots \quad \ddots \quad \vdots & & \vdots \\ 0 & 0 \cdots A_0 & \cdots & A_{q-\hat{q}_r} \end{bmatrix}\begin{bmatrix} y(0-) \\ y^{(1)}(0-) \\ \vdots \\ y^{(q-1)}(0-) \end{bmatrix}=0\right\}$$

or equivalently

$$H_{iu}:=\left\{\begin{matrix} y^{(i)}(0-) \; i=0,1,\ldots,q-1 \\ u^{(i)}(0-) \; i=0,1,..,\hat{q}_r+s-1 \end{matrix} :\right.$$

$$\sum_{i=k}^{\hat{q}_r} H_i\left[\sum_{j=0}^{s} B_j u^{(i+j-k)}(0-) - \sum_{j=0}^{q_1-i+k-1} A_j y^{(i+j-k)}(0-)\right]=0,$$

$$\left. k=1,2,..,\hat{q}_r \right\} \tag{5.5}$$

H_{iu} will be called the *admissible initial condition space* of the ARMA—representation (2.1) for every $u(t)$. □

It is easily seen from (5.3) that under the above set of initial conditions, the solution (5.3) of the ARMA—representation (2.1) will be free of impulses. This is the reason we call these conditions admissible. Under the above initial conditions the solution of the ARMA—representation (2.1) will be the following

$$y(t)= C e^{Jt} [y_s(0-) - u_s(0-)] + \\ + \int_0^t Ce^{J(t-\tau)}\Omega\, u(\tau)d\tau + \\ + [H_{-s}\ H_{-s+1}\ldots H_0 \ldots H_{\hat{q}_r}] \times$$

$$\begin{bmatrix} B_s & 0 & & \\ B_{s-1} & B_s & & 0 \\ \vdots & \vdots & \ddots & \\ B_1 & B_2 \cdots & & \ddots \\ B_0 & B_1 & & \\ 0 & B_0 & & \\ \vdots & \vdots & \ddots & \\ 0 & 0 & \cdots B_0\ B_1 \cdots B_{s-1} & B_s \end{bmatrix} \begin{bmatrix} u(t) \\ u^{(1)}(t) \\ \vdots \\ u^{(s-1)}(t) \\ \vdots \\ \vdots \\ u^{(\hat{q}_r+s)}(t) \end{bmatrix} \tag{5.6a}$$

or equivalently

$$y(t) = C e^{Jt} [y_s(0-) - u_s(0-)] + \\ + \int_0^t Ce^{J(t-\tau)}\Omega\, u(\tau)d\tau + \\ + \sum_{i=0}^{s} \sum_{j=i}^{s} H_{-i} B_j u^{(k+j-1)}(t) + \\ + \sum_{i=1}^{\hat{q}_r} \sum_{j=0}^{s} H_i B_j u^{(j+i)}(t) \tag{5.6b}$$

6. Conclusions

A closed formula is given for the solution of an ARMA representation, of the form $A(\rho)y(t)=B(\rho)u(t)$, in terms of the fundamental matrix H_k of $A(s)^{-1}$ which are easily implemented (Fragulis et al 1990) and the finite and infinite Jordan pairs of A(s). The above closed formula of the solution is very important for various analysis, synthesis and design problems because it is easily implemented.

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