On the Reduction of a Polynomial Matrix Model of a Linear Multivariable System to Generalised State Space Form

by

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Abstract. The main purpose of this work is to review some old and new reduction algorithms which reduce a general polynomial system matrix to an "equivalent" system matrix in "generalised state space form" in a matrix transformation sense via the concept of *Full System Equivalence*.

Keywords : linear systems, realisations, equivalence, generalised state space systems, full system equivalence.

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1. Introduction.

Consider a linear multivariable system Σ described by a polynomial matrix model [13] :

$$\begin{aligned} A(\rho)\beta(t) &= B(\rho)u(t) \\ y(t) &= C(\rho)\beta(t) + D(\rho)u(t) \end{aligned} \tag{1.1}$$

which equivalently can be written in its *normalized form* [20] :

$$\underbrace{\begin{pmatrix} A(\rho) & B(\rho) & 0 \\ -C(\rho) & D(\rho) & I_p \\ 0 & -I_m & 0 \end{pmatrix}}_{T(\rho)} \underbrace{\begin{pmatrix} \beta(t) \\ -u(t) \\ y(t) \end{pmatrix}}_{\xi(t)} = \underbrace{\begin{pmatrix} 0 \\ 0 \\ I_m \end{pmatrix}}_{U} u(t)$$

$$y(t) = \underbrace{\begin{pmatrix} 0 & 0 & I_p \end{pmatrix}}_{V} \underbrace{\begin{pmatrix} \beta(t) \\ -u(t) \\ y(t) \end{pmatrix}}_{\xi(t)} \tag{1.2}$$

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where $\rho := d/dt$ is the differential operator, $A(\rho) \in \Re[\rho]^{r \times r}, rank_{\Re(\rho)} A(\rho) = r$, $B(\rho) \in \Re[\rho]^{r \times m}$, $C(\rho) \in \Re[\rho]^{p \times r}$, $D(\rho) \in \Re[\rho]^{p \times m}$, $\beta(t):[0-,+\infty) \to \Re^r$ is the *pseudostate* of Σ, $u(t):[0-,+\infty) \to \Re^m$ is the *input vector* of Σ and $y(t):[0-,+\infty) \to \Re^p$ is the *output vector* of Σ. (1.2) has the advantage over (1.1) of permitting consistent definitions of finite and infinite characteristics.

An important problem in Linear System Theory is the following : Given the polynomial matrix description (PMD) $[A(\rho), B(\rho), C(\rho), D(\rho)]$ of Σ determine :

(i) a positive integer $\lambda \in Z^+$,

and

(ii) a quintuple of matrices $E, A \in \Re^{\lambda \times \lambda}$, $B \in \Re^{\lambda \times m}$, $C \in \Re^{p \times \lambda}$, $D \in \Re^{p \times m}$ such that the system Σ_R in generalised state space (gss) form [22], [16] defined by :

$$\Sigma_R : \quad E\dot{x}(t) = Ax(t) + Bu(t) \qquad y(t) = Cx(t) + Du(t) \tag{1.3}$$

is "equivalent" to system Σ or equivalently satisfies the following requirements :

(i) Σ and Σ_R give rise to the same *transfer function matrix* and so have the same finite and infinite *transmission zeros and poles* [14], [15] ,[21],[23] i.e.

$$G(s) := C(s)A(s)^{-1}B(s) + D(s) \equiv C(sE - A)^{-1}B + D =: G_R(s) \tag{1.4}$$

(ii) Σ and Σ_R have the same finite and infinite *system poles and zeros* [14], [15] [21], [23].

(iii) Σ and Σ_R have the same finite and infinite *input (output) decoupling zeros* [13], [21].

(iv) Σ and Σ_R have the same *generalised order* $f := \delta_M(T(s))$ and the same *Rosenbrock degree* d_r [13], [14], [21].

(v) Σ and Σ_R have the same *input (output) dynamical indices* [9].

This problem has been considered by Wolovich [24] in the state space case and Verghese [21], Bosgra & Van Der Weiden [2], Anderson, Coppel & Cullen [1] and Vardulakis [19] in the generalised state space case. The approach given by most of the above authors is separated in two steps (a) determine the model (1.2) and (b) prove that the properties (i)-(v) are satisfied. A more direct approach using system equivalence transformation which preserve the properties (i)-(v) is given in this work. The advantage of this approach is that instead to prove the invariance of the properties (i)-(v), you have to relate the systems (1.1) and (1.3) with a system equivalence transformation which preserves these properties. The Furhmann system equivalence transformation (or extended strict system equivalence transformation) [4] is one of the most known system equivalence transformations which preserves all the *finite* frequency characteristics of equivalent systems, while the Strong System Equivalence [1] and Full System Equivalence [7] transformations are system transformations which preserve both the *finite* and *infinite* frequency characteristics of equivalent systems. However between these last two systems transformations which preserve both finite and infinite frequency characteristics the transformation of Full System Equivalence has an advantage over the transformation of Strong System Equivalence. This advantage consists on the fact that while Full System Equivalence can be performed on a system model by one single step of transformation, Strong System Equivalence needs two steps of transformations. Another advantage of the Full System Equivalence transformation is that the transforming matrices involved in the Full System Equivalence transformation play an essential role in the relation between the solution-input and initial condition-output spaces [9], [12], of the associated models. In this paper we present five reduction algorithms of a PMD to a "full system equivalent" PMD in generalised state space form. In the sequel we sort the proposed algorithms according to the dimension $\lambda \in Z^{+}$ of the pseudostate "x(t)" of the equivalent generalised state space systems. All the proposed algorithms are illustrated via examples.

2. Preliminary Results.

Consider the set $P(p,m)$ of $(r+p)\times(r+m)$ polynomial matrices where the integer $r \geq \max\{-p,-m\}$. Two matrix transformations important is systems theory are

Definition 1. [4]-[6] $T_1(s), T_2(s) \in P(p,m)$ are said to be *Furhmann equivalent* in case there exist polynomial matrices $M(s), N(s)$ such that :

$$\begin{pmatrix} M(s) & T_2(s) \end{pmatrix} \begin{pmatrix} T_1(s) \\ -N(s) \end{pmatrix} = 0 \tag{2.1}$$

where the compound matrices in (2.1) are such that :

(i) they have full normal rank, (2.2a)

(ii) they have no finite zeros. (2.2b)

$T_1(s), T_2(s) \in P(p,m)$ are said to be *fully equivalent* in case when the compound matrices in (2.1) are such that :

(i) they have full normal rank, (2.3a)

(ii) they have no finite nor infinite zeros, (2.3b)

(iii) the following McMillan degree conditions hold

$$\delta_M\begin{pmatrix} M(s) & T_2(s) \end{pmatrix} = \delta_M(T_2(s)) \quad ; \quad \delta_M\begin{pmatrix} T_1(s) \\ -N(s) \end{pmatrix} = \delta_M(T_1(s)) \tag{2.3c}$$ •

Let $\boldsymbol{P}(p,m)$ be the set of of $(r+p)\times(r+m)$ Rosenbrock system matrices. Then

Definition 2. [4], [7] $P_1(s), P_2(s) \in \boldsymbol{P}(p,m)$ are said to be *Furhmann system equivalent* (*full system equivalent (f.s.e.)*) in case there exist polynomial matrices $M(s), N(s), X(s), Y(s)$ such that :

$$\begin{pmatrix} M(s) & 0 \\ X(s) & I \end{pmatrix} \begin{pmatrix} A_1(s) & B_1(s) \\ -C_1(s) & D_1(s) \end{pmatrix} = \begin{pmatrix} A_2(s) & B_2(s) \\ -C_2(s) & D_2(s) \end{pmatrix} \begin{pmatrix} N(s) & Y(s) \\ 0 & I \end{pmatrix} \tag{2.4}$$

where (2.4) is a Furhmann (full) equivalence transformation. •

Some interesting results concerning full system equivalence transformation are

Theorem 3. [8], [9], [23]

(i) (f.s.e.) is an equivalence relation on $\boldsymbol{P}(m,m)$.

(ii) Under (f.s.e) the following are invariant :

(a) the generalised order $f := \delta_M(T(s))$, the order $n = \deg(T(s))$ and the Rosenbrock degree d_r

(b) the transfer function matrix and thus the finite and infinite transmission zeros and poles ,

(c) the finite and infinite system poles and zeros,

(d) the finite and infinite invariant zeros,

(d) the finite and infinite input (output) decoupling zeros,

(e) the input (output) dynamical indices.

(iii) (f.s.e.) defines the same equivalence class with complete system equivalence [11] in the special case of generalised state space systems. •

3. Generalised State Space Realisations of Rosenbrock System Matrices.

Although our main scope is to propose generalised state space reduction algorithms of general PMDs, a brief description of at least one state space realization algorithm is needed so that to clearly see the already known results concerning only the finite frequency characteristics of systems in conjugtion to our main results which handle both the finite and infinite frequency characteristics in the same manner. For this reason we present the known state space reduction algorithm presented by Wolovich [24] and an illustrative example where it is easily seen that the infinite frequency characteristics of a system do not remain invariant under the Furhmann System Equivalence transformation. In the sequel using the same example and different generalised state space reduction algorithms we can easily see the differences.

3.1 Wolovich equivalent model (1973).

An algorithm which gives rise to a Furhmann System Equivalent state space model of a PMD of the form (1.1), was proposed by Wolovich (1973) and is given by the following algorithm :

Algorithm 4.

Step 1. Find $U_L(\rho)$ such that $U_L(\rho)A(\rho)=\tilde{A}(\rho)$ is row proper with row degrees

$$\rho_i, \ i = 1,2,\ldots,r \quad \sum_{i=1}^{r} \rho_i = n \tag{3.1.1}$$

and let $\tilde{B}(\rho) = U_L(\rho)B(\rho)$.

Step 2. If $\tilde{A}(\rho)^{-1}\tilde{B}(\rho)$ is not stricly proper then find $Y(\rho) \in \Re[\rho]^{r\times m}$ and $\hat{B}(\rho) \in \Re[\rho]^{r\times m}$ such that

$$\tilde{B}(\rho) = \tilde{A}(\rho)Y(\rho) + \hat{B}(\rho) \tag{3.1.2a}$$

where

$$\tilde{A}(\rho)^{-1}\hat{B}(\rho) \text{ is stricly proper} \tag{3.1.2b}$$

Step 3. Let

$$\begin{aligned} \Psi_L(s) &= \text{block diag}\{[s^{\rho_i - 1} \cdots s \ 1]^T, \ \mathrm{i} = 1,2,..,\mathrm{r}\} \\ \tilde{A}(s) &= S(s)\tilde{A}_{hr} + \Psi_L(s)\tilde{A}_{lr}, \quad S(s) = \text{diag}\{s^{\rho_i}\} \end{aligned} \tag{3.1.3}$$

Then define

$$A_c^o = \text{block diag}\left\{ \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \in \Re^{\rho_i \times \rho_i}, \ i = 1,2,\ldots,r \right\}$$

$$B_c^0 = I_n \ ; \ C_c^0 = \text{block diag}\left\{[1 \ 0 \ \cdots \ 0] \in \Re^{1\times\rho_i}, \ i = 1,2,\ldots,r\right\} \tag{3.1.4}$$

$$A_0 = A_c^0 - \tilde{A}_{lr}\tilde{A}_{hr}^{-1}C_c^0 \ ; \ C_0 = \tilde{A}_{hr}^{-1}C_c^0$$

and B_0 such that $\Psi_L(s)B_0 = \hat{B}(s)$.

Step 4. Find $\Lambda(s) \in \Re[s]^{p\times n}$ and $C \in \Re^{p\times n}$ such that

$$C(s)C_0 = \Lambda(s)(sI_n - A_0) + C \tag{3.1.5}$$

Step 5. The state space system

$$\begin{aligned} \dot{x}(t) &= A_0 x(t) + B_0 u(t) \\ y(t) &= Cx(t) + J(\rho)u(t) \end{aligned} \tag{3.1.6}$$

where

$$J(\rho) = D(\rho) + C(\rho)Y(\rho) + \Lambda(\rho)B_0 \tag{3.1.7}$$

is Furhmann system equivalent to the state space system (1.1) under the following Furhmann system equivalent transformation :

$$\left(\begin{array}{c|c} U_L^{-1}(s)\Psi_L(s) & 0 \\ \hline -\Lambda(s) & I_p \end{array}\right)\left(\begin{array}{c|c} sI_n - A_0 & B_0 \\ \hline -C & J(s) \end{array}\right) = \left(\begin{array}{c|c} A(s) & B(s) \\ \hline -C(s) & D(s) \end{array}\right)\left(\begin{array}{c|c} C_0 & -Y(s) \\ \hline 0 & I_m \end{array}\right) \quad (3.1.8) \bullet$$

Example 5. [2] Consider the following general dynamical system Σ :

$$\Sigma : \underbrace{\begin{pmatrix} \rho+1 & \rho^3+2\rho^2 \\ \rho^2+3\rho+2 & \rho^4+4\rho^3+4\rho^2+\rho+2 \end{pmatrix}}_{A(\rho)}\underbrace{\begin{pmatrix} \beta_1(t) \\ \beta_2(t) \end{pmatrix}}_{\beta(t)} = \underbrace{\begin{pmatrix} \rho^2+1 \\ \rho^3+2\rho^2+\rho+3 \end{pmatrix}}_{B(\rho)} u(t) \quad \text{(E.1)}$$

$$y(t) = \underbrace{\begin{pmatrix} -\rho^2-3\rho-1 & -\rho^4-4\rho^3-4\rho^2+1 \end{pmatrix}}_{C(\rho)}\begin{pmatrix} \beta_1(t) \\ \beta_2(t) \end{pmatrix} + \underbrace{\begin{pmatrix} \rho^3+2\rho^2+\rho+2 \end{pmatrix}}_{D(\rho)} u(t)$$

where $\rho := d/dt$.

Step 1. There exist unimodular matrix

$$U_L(s) = \begin{pmatrix} s^3+2s^2+1 & -s^2 \\ -s-2 & 1 \end{pmatrix} \quad \text{(E.2)}$$

such that

$$\tilde{A}(s) = U_L(s)A(s) = \begin{pmatrix} s+1 & 0 \\ 0 & s+2 \end{pmatrix} \text{ is row proper } (\rho_1 = 1, \rho_2 = 1)$$

(E.3)

$$\tilde{B}(s) = U_L(s)B(s) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Step 2. $\tilde{A}(s)^{-1}\tilde{B}(s)$ is stricly proper and thus the second step is omitted i.e. $(Y(s) = 0,\ \hat{B}(s) = \tilde{B}(s))$.

Step 3. Let

$$\Psi_L(s) := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad ; \quad S(s) := \begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix}$$

(E.4)

$$\tilde{A}(s) = S(s)\underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{\tilde{A}_{hr}} + \Psi_L(s)\underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}}_{\tilde{A}_{lr}}$$

Then define

$$A_c^0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad ; \quad B_c^0 = C_c^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

(E.5)

$$A_0 = A_c^0 - \tilde{A}_{lr}\tilde{A}_{hr}^{-1}C_c^0 = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix} \quad ; \quad C_0 = \tilde{A}_{hr}^{-1}C_c^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{\Psi_L(s)} B_0 = \underbrace{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}_{\tilde{B}(s)} \Rightarrow B_0 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Step 4. There exist

$$\Lambda(s) = \begin{pmatrix} -s-2 & -s^3-2s^2 \end{pmatrix} \tag{E.6}$$

such that

$$\underbrace{\begin{pmatrix} -s^2-3s-1 & -s^4-4s^3-4s^2+1 \end{pmatrix}}_{C(s)} \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{C_0} =$$

$$= \underbrace{\begin{pmatrix} -s-2 & -s^3-2s^2 \end{pmatrix}}_{\Lambda(s)} \underbrace{\begin{pmatrix} s+1 & 0 \\ 0 & s+2 \end{pmatrix}}_{sI_2 - A_0} + \underbrace{\begin{pmatrix} 1 & 1 \end{pmatrix}}_{C} \tag{E.7}$$

Step 5. The state space system

$$\begin{aligned} \dot{x}(t) &= \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix} x(t) + \begin{pmatrix} 1 \\ 1 \end{pmatrix} u(t) \\ y(t) &= \begin{pmatrix} 1 & 1 \end{pmatrix} x(t) \end{aligned} \tag{E.8}$$

($J(s) = D(s) + C(s) \times 0 + \Lambda(s) B_0 = 0$) is Furhmann system equivalent to the state space system (1.1) under the following Furhmann system equivalent transformation :

$$\left(\begin{array}{cc|c} 1 & s^2 & 0 \\ s+2 & s^3+2s^2+1 & 0 \\ \hline s+2 & s^3+2s^2 & 1 \end{array}\right) \left(\begin{array}{cc|c} s+1 & 0 & 1 \\ 0 & s+2 & 1 \\ \hline -1 & -1 & 0 \end{array}\right) = \left(\begin{array}{cc|c} s+1 & s^3+2s^2 & s^2+1 \\ s^2+3s+2 & s^4+4s^3+4s^2+s+2 & s^3+2s^2+s+3 \\ \hline s^2+3s+1 & s^4+4s^3+4s^2-1 & s^3+2s^2+s+2 \end{array}\right) \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right) \tag{E.9}$$

The following bijective map between the **smooth** solutions of both systems exists [9]

$$\begin{pmatrix} \beta(t) \\ u(t) \end{pmatrix} = \begin{pmatrix} I_2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x(t) \\ u(t) \end{pmatrix} \tag{E.10}$$

Consider now the normalized system matrix of Σ

$$\mathcal{P}(s) = \begin{pmatrix} T(s) & U \\ -V & 0 \end{pmatrix} = \left(\begin{array}{ccc|c} A(s) & B(s) & 0 & 0 \\ -C(s) & D(s) & I & 0 \\ 0 & -I & 0 & I \\ \hline 0 & 0 & I & 0 \end{array}\right) = \left(\begin{array}{cccc|c} s+1 & s^3+2s^2 & s^2+1 & 0 & 0 \\ s^2+3s+2 & s^4+4s^3+4s^2+s+2 & s^3+2s^2+s+3 & 0 & 0 \\ s^2+3s+1 & s^4+4s^3+4s^2-1 & s^3+2s^2+s+2 & 1 & 0 \\ 0 & 0 & -1 & 0 & 1 \\ \hline 0 & 0 & 0 & -1 & 0 \end{array}\right) \tag{E.11}$$

It is easily seen that

$$S^{\infty}_{T(s)}(s)=\begin{pmatrix} s^4 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{s^2} \end{pmatrix} \; ; \; S^{\infty}_{(T(s)\;U)}(s)=\begin{pmatrix} s^4 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{s^2} & 0 \end{pmatrix} \qquad \text{(E.12)}$$

Smith form of T(s) and (T(s) U) (at s=∞)

Thus we can see that the system Σ has one infinite system pole of order 2 and one infinite input decoupling zero of order 2. However the state space system proposed in step 5 doesn't possess infinite frequency characteristics. •

Therefore we conclude from the above example that the infinite frequency characteristics do not remain invariant under the Furhmann's system equivalence transformation and the proposed state space realisation is not desirable if both the finite and infinite frequency characteristics are to be treated in the same manner. For these reasons we shall examine in the sequel generalised state space reduction methods which preserve both the finite and infinite frequency characteristics of the equivalent systems.

3.2 Verghese's equivalent model (1978).

A generalised state space reduction algorithm proposed by Verghese (1978), is described and connected with the transformation of full system equivalence in the following

Algorithm 6. [21], [8]

Step 1. Find $C_\infty \in \Re^{\tilde{r}\times\mu}, J_\infty \in \Re^{\mu\times\mu}, B_\infty \in \Re^{\mu\times\tilde{r}}, D_\infty \in \Re^{\tilde{r}\times\tilde{r}}, \tilde{r}=r+p+m$ a strongly irreducible realization with nondynamic variables of $T(s)\in\Re[s]^{\tilde{r}\times\tilde{r}}$ i.e.

$$T(s)=C_\infty\left(I_\mu - sJ_\infty\right)^{-1}B_\infty + D_\infty$$

where the compound matrices

$$\begin{pmatrix} I_\mu - sJ_\infty & B_\infty \end{pmatrix} \; ; \; \begin{pmatrix} I_\mu - sJ_\infty \\ -C_\infty \end{pmatrix}$$

have no finite nor infinite zeros and $I_\mu - sJ_\infty$ doesn't include unit blocks.

Step 2. Define the matrices

$$E=\begin{pmatrix}-J_\infty & 0\\ 0 & 0\end{pmatrix}\in\Re^{(\tilde r+\mu)\times(\tilde r+\mu)} \quad ; \quad A=\begin{pmatrix}-I_\mu & -B_\infty\\ C_\infty & -D_\infty\end{pmatrix}\in\Re^{(\tilde r+\mu)\times(\tilde r+\mu)}$$

$$B=\begin{pmatrix}0\\ U\end{pmatrix}\in\Re^{(\tilde r+\mu)\times m} \quad ; \quad C=\begin{pmatrix}0 & V\end{pmatrix}\in\Re^{p\times(\tilde r+\mu)}$$

Step 3. The generalised state space system

$$\begin{aligned}E\dot x(t)&=Ax(t)+Bu(t)\\ y(t)&=Cx(t)+Du(t)\end{aligned} \tag{3.2.1}$$

is full system equivalent to the system Σ in (1.1) under the following full system equivalence transformations :

$$\left(\begin{array}{cc|c}C_\infty\left(I_\mu-sJ_\infty\right)^{-1} & I_{\tilde r} & 0\\ \hline 0 & 0 & I_p\end{array}\right)\left(\begin{array}{c|c}sE-A & B\\ \hline -C & D\end{array}\right)=\left(\begin{array}{c|c}T(s) & U\\ \hline -V & 0\end{array}\right)\left(\begin{array}{cc|c}0 & I_{\tilde r} & 0\\ \hline 0 & 0 & I_m\end{array}\right)$$

$$\left(\begin{array}{c|c}0 & 0\\ I_{\tilde r} & 0\\ \hline 0 & I_p\end{array}\right)\left(\begin{array}{c|c}T(s) & U\\ \hline -V & 0\end{array}\right)=\left(\begin{array}{c|c}sE-A & B\\ \hline -C & D\end{array}\right)\left(\begin{array}{c|c}-\left(I_\mu-sJ_\infty\right)^{-1}B_\infty & 0\\ I_{\tilde r} & 0\\ \hline 0 & I_m\end{array}\right) \tag{3.2.2}$$

•

Remark 7. According to our assumption, that the realisation $C_\infty\in\Re^{\tilde r\times\mu}, J_\infty\in\Re^{\mu\times\mu}, B_\infty\in\Re^{\mu\times\tilde r}, D_\infty\in\Re^{\tilde r\times\tilde r}, \tilde r=r+p+m$ is strongly irreducible with nondynamic variables, we obtain [8] that $\mu=\delta_M(T)+v$ and thus the dimension of the pseudostate x(t) the generalised state space system (3.2.1) is :

$$\lambda_v=r+p+m+\delta_M(T)+v$$

where $\delta_M(T)$ denotes the McMillan degree of T(s) and v is the total number of infinite poles of T(s) (of strictly positive orders) i.e.

$$S_{T(s)}^{\infty}(s)=\text{block diag}\left(s^{q_1},\ s^{q_2},\ \cdots,\ s^{q_v},\ I_{k-v},\ \frac{1}{s^{\hat q_{k+1}}},\ \frac{1}{s^{\hat q_{k+2}}},\ \cdots,\ \frac{1}{s^{\hat q_{\tilde r}}}\right)$$

where $q_1\ge q_2\ge\cdots\ge q_v>0$ and $\hat q_{\tilde r}\ge\hat q_{\tilde r-1}\ge\cdots\ge\hat q_{k+1}>0$ are respectively the orders of the infinite poles and zeros of $T(s)$. •

Remark 8. [8] The results of algorithm 6 still holds in case where the realisation $C_\infty\in\Re^{\tilde r\times\mu}, J_\infty\in\Re^{\mu\times\mu}, B_\infty\in\Re^{\mu\times\tilde r}, D_\infty\in\Re^{\tilde r\times\tilde r}$ is only strongly observable i.e. $\left(\left(I_\mu-sJ_\infty^T\right)^{-1} \quad -C_\infty^T\right)^T$ contains no finite nor infinite zeros. However then the dimension of the pseudostate x(t) of the generalised state space system (3.2.1) is greater than λ_v. •

Remark 9. [8] Any other strongly irreducible realisation with nondynamic variables $\tilde{C}_\infty \in \Re^{\tilde{r}\times\mu}, \tilde{J}_\infty \in \Re^{\mu\times\mu}, \tilde{B}_\infty \in \Re^{\mu\times\tilde{r}},\ \tilde{D}_\infty \in \Re^{\tilde{r}\times\tilde{r}}$ of T(s), gives rise to a generalised state space system which is similar to the generalised state space system (3.2.1). •

Example 10. Consider the PMD of the Example 5. Then :

Step 1. There exist a strongly irreducible realization $C_\infty \in \Re^{4\times 5}, J_\infty \in \Re^{5\times 5}$, $B_\infty \in \Re^{5\times 4}, D_\infty \in \Re^{4\times 4}, \tilde{r} = r + p + m = 4$ of T(s) such that :

$$T(s) = \begin{pmatrix} s+1 & s^3+2s^2 & s^2+1 & 0 \\ s^2+3s+2 & s^4+4s^3+4s^2+s+2 & s^3+2s^2+s+3 & 0 \\ s^2+3s+1 & s^4+4s^3+4s^2-1 & s^3+2s^2+s+2 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} =$$

$$= \underbrace{\begin{pmatrix} 0 & -1 & -2 & -2 & -2 \\ -1 & -4 & -6 & -7 & -6 \\ -1 & -4 & -6 & -6 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}}_{C_\infty} \underbrace{\begin{pmatrix} 1 & -s & 0 & 0 & 0 \\ 0 & 1 & -s & 0 & 0 \\ 0 & 0 & 1 & -s & 0 \\ 0 & 0 & 0 & 1 & -s \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}^{-1}}_{(I_5 - sJ_\infty)^{-1}} \underbrace{\begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & -2 & -3 & 0 \\ -1 & 2 & 2 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}}_{B_\infty} + \underbrace{\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -3 & 0 \\ -1 & 2 & -3 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}}_{D_\infty} \tag{E.1}$$

Step 2. Define the matrices

$$E := \begin{pmatrix} -J_\infty & 0 \\ 0 & 0 \end{pmatrix} = \left(\begin{array}{ccccc|cccc} 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \in \Re^{9\times 9} \quad ; \quad B := \begin{pmatrix} 0 \\ U \end{pmatrix} = \left(\begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \hline 0 \\ 0 \\ 0 \\ 1 \end{array}\right) \in \Re^{9\times 1} \tag{E.2}$$

$$A := \begin{pmatrix} -I_\mu & -B_\infty \\ C_\infty & -D_\infty \end{pmatrix} = \left(\begin{array}{ccccc|cccc} -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 & 0 & -1 & 2 & 3 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 & -2 & -2 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 \\ \hline 0 & -1 & -2 & -2 & -2 & 0 & 0 & 0 & 0 \\ -1 & -4 & -6 & -7 & -6 & 0 & 0 & 3 & 0 \\ -1 & -4 & -6 & -6 & -1 & 1 & -2 & 3 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right) \in \Re^{9\times 9}$$

$$C := \begin{pmatrix} 0 & V \end{pmatrix} = \left(\begin{array}{ccccc|cccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array} \right) \in \Re^{1\times 9}$$

Step 3. The generalised state space system

$$\begin{aligned} E\dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t) \end{aligned} \tag{E.3}$$

is full system equivalent to the PMD Σ in Example 5 under the following full system equivalent transformations :

$$\left(\begin{array}{ccccccccc|c} 0 & -1 & -2-s & -2-2s-s^2 & -2-2s-2s^2-s^3 & 1 & 0 & 0 & 0 & 0 \\ -1 & -4-s & -6-4s-s^2 & -7-6s-4s^2-s^3 & -6-7s-6s^2-4s^3-s^4 & 0 & 1 & 0 & 0 & 0 \\ -1 & -4-s & -6-4s-s^2 & -6-6s-4s^2-s^3 & -1-6s-6s^2-4s^3-s^4 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array} \right) \times \left(\begin{array}{c|c} sE-A & B \\ \hline -C & D \end{array} \right) =$$

$$= \left(\begin{array}{c|c} T(s) & U \\ \hline -V & 0 \end{array} \right) \left(\begin{array}{cc|c} 0_{4,5} & I_4 & 0_{4,1} \\ \hline 0_{1,5} & 0_{1,4} & I_1 \end{array} \right)$$

$$\left(\begin{array}{c|c} 0_{5,4} & 0_{5,1} \\ I_4 & 0_{4,1} \\ \hline 0_{1,4} & I_1 \end{array} \right) \left(\begin{array}{c|c} T(s) & U \\ \hline -V & 0 \end{array} \right) = \left(\begin{array}{c|c} sE-A & B \\ \hline -C & D \end{array} \right) \left(\begin{array}{cccc|c} -s+s^2 & 2s-2s^2+s^4 & -1+3s-2s^2+s^3 & 0 & 0 \\ -1+s & 2-2s+s^3 & 3-2s+s^2 & 0 & 0 \\ 1 & -2+s^2 & -2+s & 0 & 0 \\ 0 & s & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \end{array} \right) \tag{E.4}$$

•

3.3. Bosgra & Van Der Weiden equivalent model (1981).

A generalised state space reduction algorithm proposed by Bosgra & Van Der Weiden [2], is shown [8] to be a full system equivalent reduction method in the following :

Algorithm 11. [2], [8], [9]

Step 1. Let P(s) be the Rosenbrock system matrix of the system Σ in (1.1)

$$P(s) := \begin{pmatrix} A(s) & B(s) \\ -C(s) & D(s) \end{pmatrix} := P_0 + P_1 s + \cdots + P_q s^q \tag{3.3.1}$$

where q is the greatest degree of all the polynomial elements included in the matrix P(s).

Step 2. Define the Hankel matrices

$$\Pi_E := \begin{pmatrix} P_2 & P_3 & \cdots & P_q \\ P_3 & P_4 & \cdots & 0 \\ \vdots & \vdots & \iddots & \vdots \\ P_{q-1} & P_q & \cdots & 0 \\ P_q & 0 & \cdots & 0 \end{pmatrix} \;;\; \Pi_A := \begin{pmatrix} P_3 & P_4 & \cdots & P_q & 0 \\ P_4 & P_5 & \cdots & 0 & 0 \\ \vdots & \vdots & \iddots & \vdots & \vdots \\ P_q & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix} \;;\; \Pi_B := \begin{pmatrix} P_2 \\ P_3 \\ \vdots \\ P_{q-1} \\ P_q \end{pmatrix}$$

$$\Pi_C := \begin{pmatrix} P_2 & P_3 & \cdots & P_{q-1} & P_q \end{pmatrix} \tag{3.3.2}$$

where $\operatorname{rank}_{\mathfrak{R}} \Pi_E = r_E$.

Step 3. Let the sets of positive integers $I = \{i_1, \; i_2, \; \cdots, \; i_{r_E}\}$ and $J = \{j_1, \; j_2, \; \cdots, \; j_{r_E}\}$ define a *row and column selection* of Π_E such that the rows $i_1, i_2, \ldots, i_{r_E}$ and columns $j_1, j_2, \ldots, j_{r_E}$ of Π_E are linearly independent. Let P_E, P_A, P_B, P_C be submatrices of $\Pi_E, \Pi_A, \Pi_B, \Pi_C$ respectively such that P_E, P_A, P_B formed by the rows I and P_E, P_A, P_C are formed by the columns J.

Step 4. Let $\lambda_{lnv} := r + r_E + p + m$. Define the matrices

$$E := \begin{pmatrix} P_1 & P_C & 0 \\ P_B & P_A & 0 \\ 0 & 0 & 0 \end{pmatrix} \in \mathfrak{R}^{\lambda_{lnv} \times \lambda_{lnv}} \;;\; A := \begin{pmatrix} -P_0 & 0 & \begin{pmatrix} 0 \\ -I_p \end{pmatrix} \\ 0 & P_E & 0 \\ \begin{pmatrix} 0 & I_m \end{pmatrix} & 0 & 0 \end{pmatrix} \in \mathfrak{R}^{\lambda_{lnv} \times \lambda_{lnv}}$$

$$B := \begin{pmatrix} 0_{\bar{r}+p+r_E, m} \\ I_m \end{pmatrix} \in \mathfrak{R}^{\lambda_{lnv} \times m} \;;\; C := \begin{pmatrix} 0_{p, \bar{r}+m+r_E} & I_p \end{pmatrix} \in \mathfrak{R}^{p \times \lambda_{lnv}} \tag{3.3.3}$$

Then the generalised state space system

$$\begin{aligned} E\dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t) \end{aligned} \tag{3.3.4}$$

is full system equivalent to the system Σ in (1.1) under the following full system equivalent transformations :

$$\left(I_{r+p} \quad P_C s(P_E - sP_A)^{-1} \quad \begin{pmatrix} B(s) \\ D(s) \end{pmatrix} \middle| \begin{pmatrix} 0 \\ I_p \end{pmatrix} \right) \left(\begin{array}{c|c} sE - A & B \\ \hline -C & D \end{array} \right) = \left(\begin{array}{c|c} A(s) & B(s) \\ -C(s) & D(s) \end{array} \right) \left(\begin{array}{cccc|c} I_r & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & I_m \end{array} \right) \tag{3.3.5}$$

$$\begin{pmatrix} I_r & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & I_p \end{pmatrix} \left(\begin{array}{c|c} A(s) & B(s) \\ -C(s) & D(s) \end{array} \right) = \left(\begin{array}{c|c} sE-A & B \\ \hline -C & D \end{array} \right) \begin{pmatrix} I_{r+m} \\ (P_E - sP_A)^{-1} P_B s \\ \hline (C(s) \quad -D(s)) \\ \hline (0 \;|\; I_m) \end{pmatrix} \qquad \bullet$$

Remark 12. [8], [9] The dimension of the pseudostate x(t) of the generalised state space system (3.2.1) is :

$$\lambda_{bvv} = r + p + m + \delta_M(T) - v = \deg[A(s)] + \sum_{i=v+1}^{r+p+m} (\hat{q}_i(T)+1)$$

where $\delta_M(T)$ denotes the McMillan degree of the polynomial matrix T(s) and $\hat{q}_{\bar{r}}(T) \geq \hat{q}_{\bar{r}-1}(T) \geq \cdots \geq \hat{q}_v(T) > 0$ are the orders of the infinite zeros of $T(s)$ at $s=\infty$ (including the zero orders). •

Remark 13. [8], [9] Any other selection of the index sets I and J gives rise to system similar generalised state space systems of the generalised state space system (3.3.4). •

Example 14. Consider the PMD of the Example 5. Then :

Step 1. Let P(s) be the Rosenbrock system matrix of the system Σ in Example 5

$$P(s) = \left(\begin{array}{cc|c} s+1 & s^3+2s^2 & s^2+1 \\ s^2+3s+2 & s^4+4s^3+4s^2+s+2 & s^3+2s^2+s+3 \\ \hline s^2+3s+1 & s^4+4s^3+4s^2-1 & s^3+2s^2+s+2 \end{array} \right) =$$

(E.1)

$$= \underbrace{\begin{pmatrix} 1 & 0 & 1 \\ 2 & 2 & 3 \\ 1 & -1 & 2 \end{pmatrix}}_{P_0} + \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 1 \\ 3 & 0 & 1 \end{pmatrix}}_{P_1} s + \underbrace{\begin{pmatrix} 0 & 2 & 1 \\ 1 & 4 & 2 \\ 1 & 4 & 2 \end{pmatrix}}_{P_2} s^2 + \underbrace{\begin{pmatrix} 0 & 1 & 0 \\ 0 & 4 & 1 \\ 0 & 4 & 1 \end{pmatrix}}_{P_3} s^3 + \underbrace{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}}_{P_4} s^4$$

Step 2. Define the Hankel matrices

$$\Pi_E := \begin{pmatrix} P_2 & P_3 & P_4 \\ P_3 & P_4 & 0 \\ P_4 & 0 & 0 \end{pmatrix} = \left(\begin{array}{ccc|ccc|ccc} 0 & 2 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 4 & 2 & 0 & 4 & 1 & 0 & 1 & 0 \\ 1 & 4 & 2 & 0 & 4 & 1 & 0 & 1 & 0 \\ \hline 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 4 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right) \; ; \; \Pi_B := \begin{pmatrix} P_2 \\ P_3 \\ P_4 \end{pmatrix} = \left(\begin{array}{ccc} 0 & 2 & 1 \\ 1 & 4 & 2 \\ 1 & 4 & 2 \\ \hline 0 & 1 & 0 \\ 0 & 4 & 1 \\ 0 & 4 & 1 \\ \hline 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{array} \right)$$

$$\Pi_A := \begin{pmatrix} P_3 & P_4 & 0 \\ P_4 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \left(\begin{array}{ccc|ccc|ccc} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 4 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\Pi_C := \begin{pmatrix} P_2 & P_3 & P_4 \end{pmatrix} = \left(\begin{array}{ccc|ccc|ccc} 0 & 2 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 4 & 2 & 0 & 4 & 1 & 0 & 1 & 0 \\ 1 & 4 & 2 & 0 & 4 & 1 & 0 & 1 & 0 \end{array}\right) \tag{E.2}$$

where $\operatorname{rank}_{\Re}\Pi_E = 3$.

Step 3. Define I={1,2,4}, J={1,2,3} the sets of 3 indices of rows and columns such that the I rows and J columns of the matrix Π_E are linearly independent respectively. Let P_E, P_A, P_B, P_C be submatrices of $\Pi_E, \Pi_A, \Pi_B, \Pi_C$ respectively such that the matrices P_E, P_A, P_B are constructed by the I rows and the matrices P_E, P_A, P_C by the columns J.

$$P_E := \begin{pmatrix} 0 & 2 & 1 \\ 1 & 4 & 2 \\ 0 & 1 & 0 \end{pmatrix} \; ; \; P_A := \begin{pmatrix} 0 & 1 & 0 \\ 0 & 4 & 1 \\ 0 & 0 & 0 \end{pmatrix} \; ; \; P_B := \begin{pmatrix} 0 & 2 & 1 \\ 1 & 4 & 2 \\ 0 & 1 & 0 \end{pmatrix} \; ; \; P_C := \begin{pmatrix} 0 & 2 & 1 \\ 1 & 4 & 2 \\ 1 & 4 & 2 \end{pmatrix} \tag{E.3}$$

Step 4. Let $\lambda_{bvv} := r + r_E + p + m = 2 + 3 + 1 + 1 = 7$. Define the matrices

$$E := \begin{pmatrix} P_1 & P_C & 0 \\ P_B & P_A & 0 \\ 0 & 0 & 0 \end{pmatrix} = \left(\begin{array}{ccc|ccc|c} 1 & 0 & 0 & 0 & 2 & 1 & 0 \\ 3 & 1 & 1 & 1 & 4 & 2 & 0 \\ 3 & 0 & 1 & 1 & 4 & 2 & 0 \\ \hline 0 & 2 & 1 & 0 & 1 & 0 & 0 \\ 1 & 4 & 2 & 0 & 4 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \in \Re^{7\times 7}$$

$$A := \begin{pmatrix} -P_0 & 0 & \begin{pmatrix} 0 \\ -I_p \end{pmatrix} \\ 0 & P_E & 0 \\ \begin{pmatrix} 0 & I_m \end{pmatrix} & 0 & 0 \end{pmatrix} = \left(\begin{array}{ccc|ccc|c} -1 & 0 & -1 & 0 & 0 & 0 & 0 \\ -2 & -2 & -3 & 0 & 0 & 0 & 0 \\ -1 & 1 & -2 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 1 & 4 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{array}\right) \in \Re^{7\times 7} \tag{E.4}$$

$$B := \begin{pmatrix} 0_{6,1} \\ 1 \end{pmatrix} \in \Re^{7\times 1} \quad ; \quad C := \begin{pmatrix} 0_{1,6} & 1 \end{pmatrix} \in \Re^{1\times 7}$$

Then the generalised state space system

$$
\begin{aligned}
E\dot{x}(t) &= Ax(t) + Bu(t) \\
y(t) &= Cx(t) + Du(t)
\end{aligned}
\tag{E.5}
$$

is full system equivalent to the system Σ in Example 5 under the following full system equivalent transformations :

$$
\left(\begin{array}{ccccccc|c}
1 & 0 & 0 & s & 0 & s^2 & s^2+1 & 0 \\
0 & 1 & 0 & s^2 & s & s^3+2s^2 & s^3+2s^2+s+3 & 0 \\
\hline
0 & 0 & 1 & s^2 & s & s^3+2s^2 & s^3+2s^2+s+2 & 1
\end{array}\right)
\left(\begin{array}{c|c}
sE-A & B \\
\hline
-C & D
\end{array}\right)
=
\left(\begin{array}{c|c}
A(s) & B(s) \\
\hline
-C(s) & D(s)
\end{array}\right)
\left(\begin{array}{ccccccc|c}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}\right)
$$

(E.6)

$$
\left(\begin{array}{cc|c}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
\hline
0 & 0 & 1
\end{array}\right)
\left(\begin{array}{c|c}
A(s) & B(s) \\
\hline
-C(s) & D(s)
\end{array}\right)
=
\left(\begin{array}{c|c}
sE-A & B \\
\hline
-C & D
\end{array}\right)
\left(\begin{array}{cc|c}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
s & s^3+2s^2 & s^2 \\
0 & s & 0 \\
0 & s^2 & s \\
-s^2-3s-1 & -s^4-4s^3-4s^2+1 & -s^3-2s^2-s-2 \\
\hline
0 & 0 & 1
\end{array}\right)
$$

•

3.4. Zhang's equivalent model (1989).

Zhang [25] has proposed an algorithm which reduce a matrix fraction description to a generalised state space description. However the reduced generalised state space model doesn't possess necessarily the same infinite frequency characteristics with the original matrix fraction description. In the algorithm which follows we add some new steps in Zhang's algorithm with the consequence the generalised state space model which we propose to be full system equivalent to the polynomial matrix description (1.1).

Algorithm 15. [8] Consider the PMD of the example 5. Then

Step 1. Find a unimodular matrix $U(s) \in \Re[s]^{\bar{r}\times\bar{r}}$, $\bar{r} = r + p + m$ which transforms the compound matrix $\begin{pmatrix} T(s)^T & I_{\bar{r}} \end{pmatrix}^T$ to a column reduced form i.e. such that the matrix

$$
\begin{pmatrix} Q(s) \\ R(s) \end{pmatrix} := \begin{pmatrix} T(s)U(s) \\ U(s) \end{pmatrix} \in \Re[s]^{2\bar{r}\times\bar{r}}
\tag{3.4.1}
$$

is column reduced.

Step 2. Let

$$S^{\infty}_{\binom{Q}{R}}(s)=\begin{pmatrix}\text{diag}\begin{pmatrix}s^{q_1} & s^{q_2} & \cdots & s^{q_{\bar{r}}}\end{pmatrix}\\ 0_{\bar{r}\times\bar{r}}\end{pmatrix} \tag{3.4.2}$$

be the Smith form at s=∞ of the compound matrix (3.4.1), where $q_1 \geq q_2 \geq \cdots \geq q_{\bar{r}} \geq 0$ are the orders of the infinite poles of this compound matrix which coincide with the column degrees of the same compound matrix because of column reduceness [18]. We can now write the compound matrix (3.4.1) as :

$$\begin{pmatrix}Q(s)\\R(s)\end{pmatrix}:=\begin{pmatrix}Q_0\\R_0\end{pmatrix}\text{diag}\{s^{q_i}\}+\begin{pmatrix}Q_1\\R_1\end{pmatrix}\text{diag}\{s^{q_i-1}\}+\cdots+\begin{pmatrix}Q_{q_1}\\R_{q_1}\end{pmatrix}\text{diag}\{s^{q_i-q_1}\} \tag{3.4.3}$$

where q_1 is the greatest order of the infinite poles of the compound matrix (3.4.1) or equivalently [18] the greatest degree of all the polynomials of the compound matrix (3.4.1).

Step 3. Define the matrices

$$E:=\begin{pmatrix}I_{\bar{r}} & 0 & \cdots & 0 & 0\\ 0 & I_{\bar{r}} & \cdots & 0 & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & \cdots & I_{\bar{r}} & 0\\ 0 & 0 & \cdots & 0 & 0\end{pmatrix}\in\Re[s]^{(q+1)\bar{r}\times(q+1)\bar{r}}$$

$$A:=\begin{pmatrix}0 & I_{\bar{r}} & 0 & \cdots & 0\\ 0 & 0 & I_{\bar{r}} & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \cdots & I_{\bar{r}}\\ -Q_q & -Q_{q-1} & -Q_{q-2} & \cdots & -Q_0\end{pmatrix}\in\Re[s]^{(q+1)\bar{r}\times(q+1)\bar{r}} \tag{3.4.4}$$

$$B:=\begin{pmatrix}0\\-I_{\bar{r}}\end{pmatrix}\in\Re[s]^{(q+1)\bar{r}\times\bar{r}}\quad ; \quad C:=\begin{pmatrix}-R_q & -R_{q-1} & \cdots & -R_0\end{pmatrix}$$

Cross out all the columns of E, A, C and the respective rows of E, A, B which correspond to the negative powers of s and thus construct the matrices $E^{'}, A^{'}, B^{'}$ and $C^{'}$. Then the generalised state space system

$$\begin{aligned}E^{'}\dot{x}(t)&=A^{'}x(t)+(B^{'}U)u(t)\\ y(t)&=(VC^{'})x(t)\end{aligned} \tag{3.4.5}$$

is full system equivalent to the system Σ in (1.1) under the following full system equivalent transformation :

$$\left(\begin{array}{c|c} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ I_{\bar{r}} & 0 \\ \hline 0 & I_p \end{array}\right)\left(\begin{array}{c|c} T(s) & U \\ \hline -V & 0 \end{array}\right)=\left(\begin{array}{c|c} sE' - A' & B'U \\ \hline -VC' & 0 \end{array}\right)\left(\begin{array}{c|c} \mathrm{diag}\{s^{q_i-q_1}\}R(s)^{-1} & 0 \\ \mathrm{diag}\{s^{q_i-q_1+1}\}R(s)^{-1} & 0 \\ \vdots & \vdots \\ \mathrm{diag}\{s^{q_i}\}R(s)^{-1} & 0 \\ \hline 0 & I_m \end{array}\right) \tag{3.4.6}$$

where in the compound matrix of the right term in (3.4.6) are included only the positive powers of s in the terms $\left\{\mathrm{diag}\{s^{q_i-q_1}\},\mathrm{diag}\{s^{q_i-q_1+1}\},\ldots,\mathrm{diag}\{s^{q_i}\}\right\}$. •

Remark 16. [8] The dimension of the pseudostate x(t) the generalised state space system (3.4.5) is :

$$\lambda_z = \bar{r} + \delta_M(T) \tag{3.4.7}$$

where $\delta_M(T)$ denotes the McMillan degree of the polynomial matrix T(s). •

Remark 17. [8] Any other selection of the unimodular matrix U(s) which makes the compound matrix $\left(I_{\bar{r}} \quad T(s)^T\right)^T$ column reduced, gives rise to complete system equivalent generalised state space systems of the generalised state space system (3.4.5). •

An alternative algorithm of the proposed we give in the sequel.

Algorithm 18. [8]

Step 1. Find a unimodular matrix $U(s)\in\Re[s]^{\bar{r}\times\bar{r}}$, $\bar{r}=r+p+m$ which transforms the compound matrix $\left(I_{\bar{r}} \quad T(s)\right)$ to a row reduced form i.e. such that the matrix

$$\left(Q(s) \quad R(s)\right):=\left(U(s) \quad T(s)U(s)\right)\in\Re[s]^{\bar{r}\times 2\bar{r}} \tag{3.4.8}$$

is row reduced.

Step 2. Let

$$S^{\infty}_{(Q\ \ R)}(s)=\left(\mathrm{diag}\left(s^{q_1} \quad s^{q_2} \quad \cdots \quad s^{q_{\bar{r}}}\right) \quad 0_{\bar{r}\times\bar{r}}\right) \tag{3.4.9}$$

be the Smith form at s=∞ of the compound matrix (3.4.8), where $q_1 \ge q_2 \ge \cdots \ge q_{\bar{r}} \ge 0$ are the orders of the infinite poles of this compound matrix which coincide with the row degrees of the same compound matrix because of row reduceness [18]. We can now write the compound matrix (3.4.8) as :

$$(Q(s) \quad R(s)) := \mathrm{diag}\{s^{q_i}\}(Q_0 \quad R_0) + \mathrm{diag}\{s^{q_i-1}\}(Q_1 \quad R_1) + \cdots + \mathrm{diag}\{s^{q_i-q_1}\}(Q_{q_1} \quad R_{q_1}) \tag{3.4.10}$$

where q_1 is the greatest order of the infinite poles of the compound matrix (3.4.8) or equivalently [18] the greatest degree of all the polynomials of the compound matrix (3.4.8).

Step 3. Define the matrices

$$E := \begin{pmatrix} I_{\bar{r}} & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & I_{\bar{r}} & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & I_{\bar{r}} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & I_{\bar{r}} & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \end{pmatrix} \in \Re[s]^{(q+2)\bar{r}\times(q+2)\bar{r}}$$

$$A := \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 & -Q_q & -R_q \\ I_{\bar{r}} & 0 & \cdots & 0 & 0 & -Q_{q-1} & -R_{q-1} \\ 0 & I_{\bar{r}} & \cdots & 0 & 0 & -Q_{q-2} & -R_{q-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & I_{\bar{r}} & 0 & -Q_1 & -R_1 \\ 0 & 0 & \cdots & 0 & I_{\bar{r}} & -Q_0 & -R_0 \\ 0 & 0 & \cdots & 0 & 0 & I_{\bar{r}} & 0 \end{pmatrix} \in \Re[s]^{(q+2)\bar{r}\times(q+2)\bar{r}}$$

$$B := \begin{pmatrix} 0 \\ U \end{pmatrix} \in \Re[s]^{(q+2)\bar{r}\times\bar{r}} \quad ; \quad C := \begin{pmatrix} 0 & V \end{pmatrix} \in \Re[s]^{\bar{r}\times(q+2)\bar{r}} \tag{3.4.11}$$

Cross out all the columns of E, A, C and the respective rows of E, A, B which corresponds to the negative powers of s and thus construct the matrices E', A', B' and C'. Then the generalised state space system

$$\begin{aligned} E'\dot{x}(t) &= A'x(t) + B'u(t) \\ y(t) &= C'x(t) \end{aligned} \tag{3.4.12}$$

is full system equivalent to the system Σ in (1.1) under the following full system equivalent transformation :

$$\left(\begin{array}{ccccc|c} Q(s)^{-1}\{\operatorname{diag}\{s^{q_i-q_1}\}, & \operatorname{diag}\{s^{q_i-q_1+1}\}, & \cdots, & \operatorname{diag}\{s^{q_i}\}\} & I_{\bar{r}} & 0 \\ \hline & 0 & & & 0 & I_p \end{array}\right)\left(\begin{array}{c|c} sE' - A' & B' \\ \hline -C' & 0 \end{array}\right) =$$

$$= \left(\begin{array}{c|c} T(s) & U \\ \hline -V & 0 \end{array}\right)\left(\begin{array}{cccc|c} 0 & 0 & \cdots & I_{\bar{r}} & 0 \\ \hline 0 & 0 & \cdots & 0 & I_m \end{array}\right) \tag{3.4.13}$$

where in the compound matrix of the left term in (3.4.13) are included only the positive powers of s in the terms $\left\{\operatorname{diag}\{s^{q_i-q_1}\}, \operatorname{diag}\{s^{q_i-q_1+1}\}, \ldots, \operatorname{diag}\{s^{q_i}\}\right\}$. •

Remark 19. [8] The dimension of the pseudostate x(t) the generalised state space system (3.4.12) is :

$$\lambda_z' = 2\bar{r} + \delta_M(T) \tag{3.4.14}$$

where $\delta_M(T)$ denotes the McMillan degree of the polynomial matrix T(s). •

Note that any other selection of the unimodular matrix U(s) which makes the compound matrix $(I_{\bar{r}} \quad T(s))$ row reduced, gives rise to complete system equivalent generalised state space systems of the generalised state space system (3.4.12).

Example 20. Consider the PMD of the Example 5. Then :

Step 1. There exists a unimodular matrix

$$U(s) = \begin{pmatrix} 1-s & s & -s & 0 \\ 0 & 1 & 0 & 0 \\ 1 & -s-3 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in \Re[s]^{4\times 4} \tag{E.1}$$

such that the compound matrix $\left([T(s)U(s)]^T \quad U(s)^T\right)^T$ is column reduced.

Step 2.

$$\begin{pmatrix} T(s)U(s) \\ U(s) \end{pmatrix} = \left(\begin{array}{cccc} 2 & -3 & -s+1 & 0 \\ 2s+5 & -3s-7 & -s^2-s+3 & 0 \\ 3s+3 & -4s-7 & -s^2+2 & 1 \\ -1 & s+3 & -1 & 0 \\ \hline 1-s & s & -s & 0 \\ 0 & 1 & 0 & 0 \\ 1 & -s-3 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right) = \begin{pmatrix} Q(s) \\ R(s) \end{pmatrix} = \tag{E.2}$$

$$
= \underbrace{\left(\begin{array}{cccc} 0 & 0 & 0 & 0 \\ 2 & -3 & -1 & 0 \\ 3 & -4 & -1 & 1 \\ 0 & 1 & 0 & 0 \\ \hline -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right)}_{\begin{pmatrix} Q_0 \\ R_0 \end{pmatrix}} \begin{pmatrix} s & 0 & 0 & 0 \\ 0 & s & 0 & 0 \\ 0 & 0 & s^2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + \underbrace{\left(\begin{array}{cccc} 2 & -3 & -1 & 0 \\ 5 & -7 & -1 & 0 \\ 3 & -7 & 0 & 0 \\ -1 & 3 & 0 & 0 \\ \hline 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & -3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)}_{\begin{pmatrix} Q_1 \\ R_1 \end{pmatrix}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & s & 0 \\ 0 & 0 & 0 & s^{-1} \end{pmatrix} + \underbrace{\left(\begin{array}{cccc} 0 & 0 & 1 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & -1 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)}_{\begin{pmatrix} Q_2 \\ R_3 \end{pmatrix}} \begin{pmatrix} s^{-1} & 0 & 0 & 0 \\ 0 & s^{-1} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & s^{-2} \end{pmatrix}
$$

Step 3. Define the matrices

$$
E := \begin{pmatrix} I_8 & 0_{8,4} \\ 0_{4,8} & 0_{4,4} \end{pmatrix} \in \Re^{12\times12} \quad ; \quad A := \begin{pmatrix} 0 & I_4 & 0 \\ 0 & 0 & I_4 \\ -Q_2 & -Q_1 & -Q_0 \end{pmatrix} = \left(\begin{array}{cccc|cccc|cccc} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ \hline 0 & 0 & -1 & 0 & -2 & 3 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -3 & 0 & -5 & 7 & 1 & 0 & -2 & 3 & 1 & 0 \\ 0 & 0 & -2 & 0 & -3 & 7 & 0 & 0 & -3 & 4 & 1 & -1 \\ 0 & 0 & 1 & 0 & 1 & -3 & 0 & 0 & 0 & -1 & 0 & 0 \end{array}\right) \in \Re^{12\times12}
$$

(E.3)

$$
B := \begin{pmatrix} 0_{8,4} \\ -I_4 \end{pmatrix} \quad ; \quad C := \begin{pmatrix} -R_2 & -R_1 & -R_0 \end{pmatrix} = \left(\begin{array}{cccc|cccc|cccc} 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & -1 & 3 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{array}\right)
$$

Cross out the columns {1,2,4,8} of E, A, C and the respective rows of E, A, B which corresponds to the negative powers of s and thus construct the matrices $E^{'}, A^{'}, B^{'}$ and $C^{'}$.

$$
E^{'} = \begin{pmatrix} I_4 & 0_{4,4} \\ 0_{4,4} & 0_{4,4} \end{pmatrix} \in \Re^{8\times8} \quad ; \quad A^{'} = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ -1 & -2 & 3 & 1 & 0 & 0 & 0 & 0 \\ -3 & -5 & 7 & 1 & -2 & 3 & 1 & 0 \\ -2 & -3 & 7 & 0 & -3 & 4 & 1 & -1 \\ 1 & 1 & -3 & 0 & 0 & -1 & 0 & 0 \end{pmatrix} \in \Re^{8\times8}
$$

(E.4)

$$
B^{'} = \begin{pmatrix} 0_{4,4} \\ -I_4 \end{pmatrix} \in \Re^{8\times4} \quad ; \quad C^{'} = \begin{pmatrix} 0 & -1 & 0 & 1 & 1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & 3 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix} \in \Re^{4\times8}
$$

Then the generalised state space system

$$E^{'}\dot{x}(t) = A^{'}x(t) + (B^{'}U)u(t)$$
$$y(t) = (VC^{'})x(t) \tag{E.5}$$

is full system equivalent to the system Σ in Example 5 under the following full system equivalent transformation :

$$\left(\begin{array}{c|c} 0_{4,4} & 0_{4,1} \\ I_4 & 0_{4,1} \\ \hline 0_{4,1} & 1 \end{array}\right)\left(\begin{array}{c|c} T(s) & U \\ \hline -V & 0 \end{array}\right) = \left(\begin{array}{c|c} sE^{'} - A^{'} & B^{'}U \\ \hline -VC^{'} & 0 \end{array}\right)\left(\begin{array}{cccc|c} -1 & 3-s-s^2 & 1-s & 0 & 0 \\ 1 & 2s+s^2 & s & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ -s & 3s-s^2-s^3 & s-s^2 & 0 & 0 \\ s & 2s^2+s^3 & s^2 & 0 & 0 \\ 0 & s & 0 & 0 & 0 \\ -s^2 & 3s^2-s^3-s^4 & s^2-s^3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right) \tag{E.6}$$

Using the algorithm 18 we can easily find that the generalised state space system

$$E\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) \tag{E.7}$$

where

$$E = \begin{pmatrix} I_4 & 0_{4,8} \\ 0_{8,4} & 0_{8,8} \end{pmatrix} \in \Re^{12\times 12} \;;\; A = \begin{pmatrix} 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & 1 & -\frac{1}{2} & 0 & 0 & -1 & -1 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 2 & -1 & 0 & 0 & 0 & -2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & -1 & 0 & 1 & 1 & 0 & -1 \\ 0 & 1 & 0 & 0 & \frac{1}{2} & 1 & -1 & 0 & 1 & \frac{1}{2} & 0 & -1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix} \in \Re^{12\times 12} \;;\; B = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \in \Re^{12\times 1}$$

$$C = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \in \Re^{1\times 12} \tag{E.8}$$

is full system equivalent to the system Σ in Example 5 under the following full system equivalent transformation :

$$\left(\begin{array}{ccccccccccc|c} 1 & s & -\frac{s}{2}+s^2 & -s^2 & s^2 & -\frac{s^2}{2}+s^3 & -s^3 & 0 & 1 & 0 & 0 & 0 & 0 \\ 2+s & 2s+s^2 & 1-s+3\frac{s^2}{2}+s^3 & -2s^2-s^3 & 2s^2+s^3 & s-s^2+3\frac{s^3}{2}+s^4 & -2s^3-s^4 & 0 & 0 & 1 & 0 & 0 & 0 \\ 2+s & 2s+s^2 & -s+3\frac{s^2}{2}+s^3 & 1-2s^2-s^3 & 2s^2+s^3 & -s^2+3\frac{s^3}{2}+s^4 & s-2s^3-s^4 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right) \times \tag{E.9}$$

$$\times\begin{pmatrix} sE-A & B \\ -C & 0 \end{pmatrix} = \begin{pmatrix} T(s) & U \\ -V & 0 \end{pmatrix}\begin{pmatrix} 0_{8,4} & 0_{8,1} \\ I_4 & 0_{4,1} \\ 0_{1,4} & 1 \end{pmatrix}$$

3.5. Tan & Vandewall's equivalent model (1989).

Tan & Vandewalle [17] have proposed a generalised state space realization algorithm for matrix fraction descriptions. However the reduced generalised state space models doesn't possess necessary the same properties with the original matrix fraction description. In the algorithm which follows we add some new steps in Tan & Vandewalle's algorithm and we propose a (f.s.e.) generalised state space reduction algorithm for PMDs of the form (1.1).

Algorithm 21. [8], [10]

Step 1. Find a unimodular matrix $U(s) \in \Re[s]^{\bar{r}\times\bar{r}}$, $\bar{r} = r+p+m$ which transforms the compound matrix $\begin{pmatrix} T(s)^T & I_{\bar{r}} \end{pmatrix}^T$ to a column reduced form i.e. such that the matrix

$$\begin{pmatrix} Q(s) \\ R(s) \end{pmatrix} := \begin{pmatrix} T(s)U(s) \\ U(s) \end{pmatrix} \in \Re[s]^{2\bar{r}\times\bar{r}} \tag{3.5.1}$$

is column reduced.

Step 2. Let

$$S^{\infty}_{\binom{Q}{R}}(s) = \begin{pmatrix} \operatorname{diag}\begin{pmatrix} s^{q_1} & s^{q_2} & \cdots & s^{q_{\bar{r}}} \end{pmatrix} \\ 0_{\bar{r}\times\bar{r}} \end{pmatrix} \tag{3.5.2}$$

be the Smith form at s=∞ of the compound matrix (3.5.1), where $q_1 \geq q_2 \geq \cdots \geq q_{\bar{r}} \geq 0$ are the orders of the infinite poles of this compound matrix which coincide with the column degrees of the same compound matrix because of column reduceness [18]. Define the matrix

$$S(s) = \text{block diag}\left\{ \begin{pmatrix} s^{q_i} \\ s^{q_i-1} \\ \vdots \\ 1 \end{pmatrix} \quad i = 1,2,..,\bar{r} \right\} \tag{3.5.3}$$

and write the polynomial matrices Q(s) and R(s) as follows

$$Q(s) = Q_c S(s) \quad ; \quad R(s) = R_c S(s) \tag{3.5.4}$$

Step 3. Construct the core realisation

$$E_0 = \text{block diag}\left\{\begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix} \in \Re^{(q_i+1)\times(q_i+1)} \quad i = 1,2,\ldots,\bar{r}\right\}$$

$$A_0 = \text{block diag}\left\{\begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & 0 & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix} \in \Re^{(q_i+1)\times(q_i+1)} \quad i = 1,2,\ldots,\bar{r}\right\} \tag{3.5.5}$$

$$B_0 = \text{block diag}\left\{\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix} \in \Re^{(q_i+1)\times 1} \quad i = 1,2,\ldots,\bar{r}\right\}$$

$$C_0 = I_n \qquad n = \sum_{i=1}^{\bar{r}} [q_i + 1]$$

Step 4. Then the generalised state space system

$$\begin{aligned} E\dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) \end{aligned} \tag{3.5.6}$$

where

$$E = E_0 \quad ; \quad A = A_0 - B_0 Q_c \quad ; \quad B = B_0 U \quad ; \quad C = VR_c \tag{3.5.7}$$

is full system equivalent to the system Σ in (1.1) under the following full system equivalent transformation :

$$\left(\begin{array}{c|c} B_0 & 0 \\ \hline 0 & I_p \end{array}\right)\left(\begin{array}{c|c} T(s) & U \\ \hline -V & 0 \end{array}\right) = \left(\begin{array}{c|c} sE - A & B \\ \hline -C & 0 \end{array}\right)\left(\begin{array}{c|c} S(s)R(s)^{-1} & 0 \\ \hline 0 & I_m \end{array}\right) \tag{3.5.8a}$$

or the symmetric full system equivalence transformation :

$$\left(\begin{array}{c|c} T(s)R_c(sE - A)^{-1} & 0 \\ \hline 0 & I_p \end{array}\right)\left(\begin{array}{c|c} sE - A & B \\ \hline -C & 0 \end{array}\right) = \left(\begin{array}{c|c} T(s) & U \\ \hline -V & 0 \end{array}\right)\left(\begin{array}{c|c} R_c & 0 \\ \hline 0 & I_m \end{array}\right) \tag{3.5.8b}$$ •

Remark 22. [8] The dimension of the pseudostate x(t) in the generalised state space system (3.5.6) is :

$$\lambda_{tv} = \bar{r} + \delta_M(T) \tag{3.5.9}$$

where $\delta_M(T)$ denotes the McMillan degree of the polynomial matrix T(s). •

Remark 23. [8] Any other selection of the unimodular matrix U(s) which makes the compound matrix $\left(I_{\bar{r}} \quad T(s)^T\right)^T$ column reduced, gives rise to complete system equivalent generalised state space systems of the generalised state space system (3.4.6). •

An alternative algorithm of the proposed is the following

Algorithm 24. [8]

Step 1. Find a unimodular matrix $U(s) \in \Re[s]^{\bar{r}\times\bar{r}}$, $\bar{r} = r + p + m$ which transforms the compound matrix $\left(I_{\bar{r}} \quad T(s)^T\right)^T$ to a column reduced form i.e. such that the matrix

$$\begin{pmatrix} Q(s) \\ R(s) \end{pmatrix} := \begin{pmatrix} U(s) \\ T(s)U(s) \end{pmatrix} \in \Re[s]^{2\bar{r}\times\bar{r}} \tag{3.5.10}$$

is column reduced.

Step 2. Let

$$S^{\infty}_{\begin{pmatrix} Q \\ R \end{pmatrix}}(s) = \begin{pmatrix} \text{diag}\left(s^{q_1} \quad s^{q_2} \quad \cdots \quad s^{q_{\bar{r}}}\right) \\ 0_{\bar{r}\times\bar{r}} \end{pmatrix} \tag{3.5.11}$$

be the Smith form at s=∞ of the compound matrix (3.4.1), where $q_1 \geq q_2 \geq \cdots \geq q_{\bar{r}} \geq 0$ are the orders of the infinite poles of this compound matrix which coincides with the column degrees of the same compound matrix because of column reduceness [8]. Define the matrix

$$S(s) = \text{block diag}\left\{ \begin{pmatrix} s^{q_i} \\ s^{q_i - 1} \\ \vdots \\ 1 \end{pmatrix} \quad i = 1,2,..,\bar{r} \right\} \tag{3.5.12}$$

and write the polynomial matrices Q(s) and R(s) as follows

$$Q(s) = Q_c S(s) \quad ; \quad R(s) = R_c S(s) \tag{3.5.13}$$

Step 3. Construct the core realisation

$$E_0 = \text{block diag}\left\{ \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix} \in \Re^{(q_i+1)\times(q_i+1)} \quad i = 1,2,\ldots,\bar{r} \right\}$$

(3.5.14)

$$A_0 = \text{block diag}\left\{\begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & 0 & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix} \in \Re^{(q_i+1)\times(q_i+1)} \quad i = 1,2,\ldots,\bar{r}\right\}$$

$$B_0 = \text{block diag}\left\{\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix} \in \Re^{(q_i+1)\times 1} \quad i = 1,2,\ldots,\bar{r}\right\}$$

$$C_0 = I_n \qquad n = \sum_{i=1}^{\bar{r}} [q_i + 1]$$

Step 4. Then the generalised state space system

$$\begin{aligned} E\dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) \end{aligned} \tag{3.5.15}$$

where

$$E = \begin{pmatrix} E_0 & 0 \\ 0 & 0 \end{pmatrix} \; ; \; A = \begin{pmatrix} A_0 - B_0 Q_c & -B_0 \\ R_c & 0 \end{pmatrix} \; ; \; B = \begin{pmatrix} 0 \\ U \end{pmatrix} \; ; \; C = \begin{pmatrix} 0 & V \end{pmatrix} \tag{3.5.16}$$

is full system equivalent to the system Σ in (1.1) under the following full system equivalent transformation :

$$\left(\begin{array}{c|c} 0 & 0 \\ I_{\bar{r}} & 0 \\ \hline 0 & I_p \end{array}\right)\left(\begin{array}{c|c} T(s) & U \\ \hline -V & 0 \end{array}\right) = \left(\begin{array}{c|c} sE - A & B \\ \hline -C & 0 \end{array}\right)\left(\begin{array}{c|c} -S(s)R(s)^{-1} & 0 \\ I_{\bar{r}} & 0 \\ \hline 0 & I_m \end{array}\right) \tag{3.5.17a}$$

or the symmetric full system equivalence transformation :

$$\left(\begin{array}{cc|c} R_c(sE-A)^{-1} & I_{\bar{r}} & 0 \\ \hline 0 & 0 & I_p \end{array}\right)\left(\begin{array}{c|c} sE - A & B \\ \hline -C & 0 \end{array}\right) = \left(\begin{array}{c|c} T(s) & U \\ \hline -V & 0 \end{array}\right)\left(\begin{array}{cc|c} 0 & I_{\bar{r}} & 0 \\ \hline 0 & 0 & I_m \end{array}\right) \tag{3.5.17b}$$ •

Remark 25. [8] The dimension of the pseudostate x(t) in the generalised state space system (3.5.15) is :

$$\lambda'_{tv} = 2\bar{r} + \delta_M(T) \tag{3.5.18}$$

where $\delta_M(T)$ denotes the McMillan degree of the polynomial matrix T(s). •

Any other selection of the unimodular matrix U(s) which makes the compound matrix $\begin{pmatrix} I_{\bar{r}} & T(s)^T \end{pmatrix}^T$ column reduced, gives rise [8] to complete system equivalent generalised state space systems of the generalised state space system (3.5.15).

Example 26. Consider the PMD of the Example 5. Then :

Step 1. There exist unimodular matrix

$$U(s)=\begin{pmatrix} 1-s & s & -s & 0 \\ 0 & 1 & 0 & 0 \\ 1 & -s-3 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in \Re[s]^{4\times 4} \tag{E.1}$$

such that the compound matrix $\left([T(s)U(s)]^T \quad U(s)^T\right)^T \equiv \left(Q(s)^T \quad R(s)^T\right)^T$ is column reduced.

Step 2. Let

$$S^{\infty}_{\begin{pmatrix} Q(s) \\ R(s) \end{pmatrix}}(s)=\begin{pmatrix} \operatorname{diag}\left(s \quad s \quad s^2 \quad 1\right) \\ 0_{4,4} \end{pmatrix} \tag{E.2}$$

and define the matrix

$$S(s)=\begin{pmatrix} s & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & s & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & s^2 & 0 \\ 0 & 0 & s & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{E.3}$$

Then write the polynomial matrices Q(s) and R(s) as follows

$$Q(s)=\begin{pmatrix} 2 & 3 & -s+1 & 0 \\ 2s+5 & -3s-7 & -s^2-s+3 & 0 \\ 3s+3 & -4s-7 & -s^2+2 & 1 \\ -1 & s+3 & -1 & 0 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 2 & 0 & -3 & 0 & -1 & 1 & 0 \\ 2 & 5 & -3 & -7 & -1 & -1 & 3 & 0 \\ 3 & 3 & -4 & -7 & -1 & 0 & 2 & 1 \\ 0 & -1 & 1 & 3 & 0 & 0 & -1 & 0 \end{pmatrix}}_{Q_C} S(s)$$

(E.4)

$$R(s)=\begin{pmatrix} 1-s & s & -s & 0 \\ 0 & 1 & 0 & 0 \\ 1 & -s-3 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \underbrace{\begin{pmatrix} -1 & 1 & 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & -3 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}}_{R_c} S(s)$$

Step 3. Construct the core realization

$$E_{()}=\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \in \Re^{8\times 8} \quad ; \quad A_{()}=\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \in \Re^{8\times 8} \quad ; \quad B_{()}=\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in \Re^{8\times 4}$$

$$C_0 = I_8 \tag{E.5}$$

Step 4. Then the generalised state space system

$$\begin{aligned} E\dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) \end{aligned} \tag{E.6}$$

where

$$E = E_0 \quad ; \quad A = A_0 - B_0 Q_c = \begin{pmatrix} 0 & -2 & 0 & 3 & 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2 & -5 & 3 & 7 & 1 & 1 & -3 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -3 & -3 & 4 & 7 & 1 & 0 & -2 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & -3 & 0 & 0 & 1 & 0 \end{pmatrix} \in \Re^{8 \times 8} \quad ; \quad B = B_c U = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \in \Re^{8 \times 1}$$

$$C = VR_c = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \in \Re^{1 \times 8} \tag{E.7}$$

is full system equivalent to the system Σ in Example 5 under the following full system equivalence transformation :

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right) \left(\begin{array}{c|c} T(s) & U \\ \hline -V & 0 \end{array}\right) = \left(\begin{array}{c|c} sE - A & B \\ \hline -C & 0 \end{array}\right) \left(\begin{array}{cccc|c} s & 2s^2 + s^3 & s^2 & 0 & 0 \\ 1 & 2s + s^2 & s & 0 & 0 \\ 0 & s & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ -s^2 & 3s^2 - s^3 - s^4 & s^2 - s^3 & 0 & 0 \\ -s & 3s - s^2 - s^3 & s - s^2 & 0 & 0 \\ -1 & 3 - s - s^2 & 1 - s & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right) \tag{E.8}$$

or the symmetric full system equivalence transformation

$$\left(\begin{array}{cccccccc|c} 1 & 1 + s & 0 & -s + s^2 & 0 & 0 & s & 0 & 0 \\ 0 & 4 + 3s + s^2 & 1 & -2 - 2s + s^2 + s^3 & 0 & -1 & (1 + s)^2 & 0 & 0 \\ 0 & 4 + 3s + s^2 & 0 & -3 - 2s + s^2 + s^3 & 1 & -1 & (1 + s)^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right) \left(\begin{array}{c|c} sE - A & B \\ \hline -C & 0 \end{array}\right) =$$

$$= \left(\begin{array}{c|c} T(s) & U \\ \hline -V & 0 \end{array}\right) \left(\begin{array}{cccccccc|c} -1 & 1 & 1 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & -3 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right) \tag{E.9}$$

Using the algorithm 24 we can easily find that the generalised state space system

$$\begin{aligned} E\dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) \end{aligned} \tag{E.10}$$

where

$$E=\begin{pmatrix} E_0 & 0_{8,4} \\ 0_{4,8} & 0_{4,4} \end{pmatrix} \;;\; A=\begin{pmatrix} 1 & -1 & -1 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 3 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 \\ 0 & 2 & 0 & -3 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 2 & 5 & -3 & -7 & -1 & -1 & 3 & 0 & 0 & 0 & 0 & 0 \\ 3 & 3 & -4 & -7 & -1 & 0 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 3 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \in \Re^{12\times 12} \;;\; B=\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \in \Re^{12\times 1}$$

(E.11)

$$C=\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \in \Re^{1\times 12}$$

is full system equivalent to the system Σ in Example 5 under the following full system equivalence transformation :

$$\left(\begin{array}{cc} 0_{8,4} & 0_{8,1} \\ I_4 & 0_{4,1} \\ \hline 0_{1,4} & 1 \end{array}\right) \left(\begin{array}{c|c} T(s) & U \\ \hline -V & 0 \end{array}\right) = \left(\begin{array}{c|c} sE-A & B \\ \hline -C & 0 \end{array}\right) \left(\begin{array}{cccc|c} -s & -2s^2-s^3 & -s^2 & 0 & 0 \\ -1 & -2s-s^2 & -s & 0 & 0 \\ 0 & -s & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ s^2 & -3s^2+s^3+s^4 & -s^2+s^3 & 0 & 0 \\ s & -3s+s^2+s^3 & -s+s^2 & 0 & 0 \\ 1 & -3+s+s^2 & -1+s & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right) \qquad \text{(E.12)}$$

•

3.6. Vardulakis' equivalent model (1989).

Vardulakis [19] proposed an algorithm which reduces a general polynomial matrix description to a generalised state space system with the same finite and infinite properties. As we can show in the following algorithm the proposed generalised state space model is actually a full system equivalent model.

Algorithm 27. [19], [8], [10]

Step 1. Form T(s) and compute $n=\deg[T(s)]$, $S^{\infty}_{T(s)}(s)$ and $\mu=\sum_{i=v+1}^{r+p+m}\left[\hat{q}_i(T)+1\right]$, where $\hat{q}_i(T)$, i=v+1,..,r+p+m are the strictly positive orders of the infinite zeros of T(s).

Step 2. Compute

$$T(s)^{-1} = H_{spr}(s) + H_{pol}(s) \tag{3.6.1}$$

and find a minimal state space realisation $\left[\tilde{C}\in\Re^{\bar{r}\times n}, \tilde{J}\in\Re^{n\times n}, \tilde{B}\in\Re^{n\times\bar{r}}\right]$ of $H_{spr}(s)\in\Re_{spr}(s)^{\bar{r}\times\bar{r}}$ and a minimal generalised state space realizations $\left[C_\infty\in\Re^{\bar{r}\times\mu}, J_\infty\in\Re^{\mu\times\mu}, B_\infty\in\Re^{\mu\times\bar{r}}, D_\infty\in\Re^{\bar{r}\times\bar{r}}\right]$ of $H_{pol}(s)\in\Re[s]^{\bar{r}\times\bar{r}}$ i.e.

$$H_{spr}(s) := \tilde{C}(sI_n - \tilde{J})^{-1}\tilde{B}$$

such that $\begin{pmatrix} sI_n - \tilde{J} & \tilde{B}\end{pmatrix}$ and $\begin{pmatrix} sI_n - \tilde{J} \\ -\tilde{C}\end{pmatrix}$ have no decoupling zeros in C

(3.6.2)

$$H_{pol}(s) := C_\infty(I_\mu - sJ_\infty)^{-1}B_\infty + D_\infty$$

$I_\mu - sJ_\infty$ has no nondynamic variables

$\begin{pmatrix} I_\mu - sJ_\infty & B_\infty\end{pmatrix}$ and $\begin{pmatrix} I_\mu - sJ_\infty \\ -C_\infty\end{pmatrix}$ have no decoupling zeros at $s=\{\infty\}$

Step 3. The generalised state space system

$$\begin{aligned} E\dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t) \end{aligned} \tag{3.6.3}$$

where

$$E = \begin{pmatrix} I_n & 0_{n,\mu} \\ 0_{\mu,n} & -J_\infty \end{pmatrix} \in \Re^{(n+\mu)\times(n+\mu)} \quad ; \quad A = \begin{pmatrix} \tilde{J} & 0_{n,\mu} \\ 0_{\mu,n} & -I_\mu \end{pmatrix} \in \Re^{(n+\mu)\times(n+\mu)}$$

(3.6.4)

$$C = \begin{pmatrix} V\tilde{C} & VC_\infty \end{pmatrix} \in \Re^{p\times(n+\mu)} \quad ; \quad B = \begin{pmatrix} \tilde{B}U \\ B_\infty U \end{pmatrix} \in \Re^{(n+\mu)\times m} \quad ; \quad D = VD_\infty U \in \Re^{p\times m}$$

is full system equivalent to the system Σ in (1.1) under the following full system equivalent transformations :

$$\left(\begin{array}{c|c} T(s)\begin{pmatrix}\tilde{C} & C_\infty\end{pmatrix}(sE-A)^{-1} & 0 \\ \hline 0 & I_p \end{array}\right)\left(\begin{array}{c|c} sE-A & B \\ \hline -C & D \end{array}\right) = \left(\begin{array}{c|c} T(s) & U \\ \hline -V & 0 \end{array}\right)\left(\begin{array}{c|c} \begin{pmatrix}\tilde{C} & C_\infty\end{pmatrix} & -D_\infty U \\ \hline 0 & I_m \end{array}\right) \tag{3.6.5a}$$

and

$$\left(\begin{array}{c|c} \begin{pmatrix}\tilde{B} \\ B_\infty\end{pmatrix} & 0 \\ \hline VD_\infty & I_p \end{array}\right)\left(\begin{array}{c|c} T(s) & U \\ \hline -V & 0 \end{array}\right) = \left(\begin{array}{c|c} sE-A & B \\ \hline -C & D \end{array}\right)\left(\begin{array}{c|c} (sE-A)^{-1}\begin{pmatrix}\tilde{B} \\ B_\infty\end{pmatrix}T(s) & 0 \\ \hline 0 & I_m \end{array}\right) \tag{3.6.5b}$$

•

Remark 28. [8], [19] The dimension of the pseudostate x(t) in the generalised state space system (3.6.3) is :

$$\lambda_{aiv} = \bar{r} + \delta_M(T) - v \tag{3.5.18}$$

where $\delta_M(T)$ denotes the McMillan degree of the polynomiaal matrix T(s). and v is the total number of infinite poles of T(s) which have strictly positive order. •

Example 29. Consider the PMD of the Example 5. Then :

Step 1. We have that

$$\det[T(s)] = s^2 + 3s + 2 \Rightarrow \deg[T(s)] = 2 \tag{E.1}$$

and

$$S_{T(s)}^{\infty}(s) = \begin{pmatrix} s^4 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{s^2} \end{pmatrix} \Rightarrow \hat{q}_4 = 2 \Rightarrow \mu = 2 + 1 = 3 \tag{E.2}$$

Step 2. Compute

$$T(s)^{-1} = \begin{pmatrix} s+1 & s^3 + 2s^2 & s^2 + 1 & 0 \\ s^2 + 3s + 2 & s^4 + 4s^3 + 4s^2 + s + 2 & s^3 + 2s^2 + s + 3 & 0 \\ s^2 + 3s + 1 & s^4 + 4s^3 + 4s^2 - 1 & s^3 + 2s^2 + s + 2 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}^{-1} =$$

$$= \underbrace{\begin{pmatrix} \frac{2}{s+1} & -\frac{1}{s+1} & 0 & \frac{1}{s+1} \\ 0 & \frac{1}{s+2} & 0 & \frac{1}{s+2} \\ 0 & 0 & 0 & 0 \\ \frac{2}{s+1} & -\frac{1}{s^2+3s+2} & 0 & \frac{2s+3}{s^2+3s+2} \end{pmatrix}}_{H_{spr}(s)} + \underbrace{\begin{pmatrix} s^2 + s - 1 & -s+1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ s^2 - 4 & -s+1 & 1 & 0 \end{pmatrix}}_{H_{pol}(s)} \tag{E.3}$$

and find a minimal state space realisation $\left[\tilde{C} \in \Re^{4\times 2}, \tilde{J} \in \Re^{2\times 2}, \tilde{B} \in \Re^{2\times 4}\right]$ of $H_{spr}(s) \in \Re_{spr}(s)^{4\times 4}$ and a minimal generalised state space realization $\left[C_\infty \in \Re^{4\times 3}, J_\infty \in \Re^{3\times 3}, B_\infty \in \Re^{3\times 4}, D_\infty \in \Re^{4\times 4}\right]$ of $H_{pol}(s) \in \Re[s]^{4\times 4}$ i.e.

$$H_{spr}(s) := \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 1 & 1 \end{pmatrix}}_{\tilde{C}} \underbrace{\begin{pmatrix} s+1 & 0 \\ 0 & s+2 \end{pmatrix}}_{(sI_2 - \tilde{J})}^{-1} \underbrace{\begin{pmatrix} 2 & -1 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{pmatrix}}_{\tilde{B}}$$

$$H_{pol}(s) = \underbrace{\begin{pmatrix} 1 & 1 & -5 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}}_{C_\infty} \underbrace{\begin{pmatrix} 1 & -s & 0 \\ 0 & 1 & -s \\ 0 & 0 & 1 \end{pmatrix}}_{(I_3 - sJ_\infty)}^{-1} \underbrace{\begin{pmatrix} -4 & -2 & 0 & -2 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}}_{B_\infty} + \underbrace{\begin{pmatrix} 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 8 & 1 & -1 & 2 \end{pmatrix}}_{D_\infty} \tag{E.4}$$

Step 3. The generalised state space system

$$
\begin{aligned}
E\dot{x}(t) &= Ax(t) + Bu(t) \\
y(t) &= Cx(t) + Du(t)
\end{aligned}
\qquad \text{(E.5)}
$$

where

$$
E = \begin{pmatrix} I_2 & 0_{2,3} \\ 0_{3,2} & -J_\infty \end{pmatrix} = \left(\begin{array}{cc|ccc} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \in \Re^{5\times 5} \quad ; \quad A = \begin{pmatrix} \tilde{J} & 0_{2,3} \\ 0_{3,2} & -I_3 \end{pmatrix} = \left(\begin{array}{cc|ccc} -1 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 \\ \hline 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{array}\right) \in \Re^{5\times 5}
$$

$$
C = \begin{pmatrix} V\tilde{C} & VC_\infty \end{pmatrix} = \left(\begin{array}{cc|ccc} 1 & 1 & 1 & 0 & 0 \end{array}\right) \in \Re^{1\times 5} \quad ; \quad B = \begin{pmatrix} \tilde{B}U \\ B_\infty U \end{pmatrix} = \left(\begin{array}{c} 1 \\ 1 \\ \hline -2 \\ 0 \\ 0 \end{array}\right) \in \Re^{5\times 1} \quad ; \quad D = VD_\infty U = 2 \in \Re \qquad \text{(E.6)}
$$

is full system equivalent to the system Σ in Example 5 under the following full system equivalence transformations :

$$
\left(\begin{array}{ccccc|c} 1 & s^2 & s+1 & (s+1)^2 & -4s-5 & 0 \\ s+2 & s^3+2s^2+1 & s^2+3s+2 & s^3+4s^2+5s+2 & -4s^2-14s-12 & 0 \\ s+2 & s^3+2s^2 & s^2+3s+2 & s^3+4s^2+5s+1 & -4s^2-14s-4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right) \left(\begin{array}{c|c} sE-A & B \\ \hline -C & D \end{array}\right) =
$$

$$
= \left(\begin{array}{c|c} T(s) & U \\ \hline -V & 0 \end{array}\right) \left(\begin{array}{ccccc|c} 1 & 0 & 1 & 1 & -5 & -2 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 1 & 1 & 1 & 0 & 0 & -2 \\ \hline 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right) \qquad \text{(E.7)}
$$

and

$$
\left(\begin{array}{cccc|c} 2 & -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ -4 & -2 & 0 & -2 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ \hline 8 & 1 & -1 & 2 & 1 \end{array}\right) \left(\begin{array}{c|c} T(s) & U \\ \hline -V & 0 \end{array}\right) = \left(\begin{array}{c|c} sE-A & B \\ \hline -C & D \end{array}\right) \left(\begin{array}{cccc|c} -s & -s^3-s^2+s-2 & -s^2+s-2 & 0 & 0 \\ s+1 & s^3+2s^2+1 & s^2+1 & 0 & 0 \\ -8s-8 & -8s^3-15s^2-4 & -8s^2+s-8 & 0 & 0 \\ 2s+2 & 2s^3+4s^2+s+2 & 2s^2+3 & 0 & 0 \\ -s-1 & -s^3-2s^2 & -s^2-1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right) \qquad \text{(E.8)} \bullet
$$

4. Comparison and classification of the generalised state space reduction methods.

In sections 3.2-3.6 we presented generalised state space reduction methods for general polynomial matrix descriptions (PMDs). Note that a) what we call here the Zhang's and Tan & Vandewalle's models and b) some alterations which we have made in the other models, cannot be found in the recent bibliografy. The reason we call these models with the name of the above authors is because the construction of these algorithms is based on specific algorithms proposed by these authors. A classification of all these models in two generalised state space reduction algorithms is now proposed by the following two algorithms.

Algorithm 30. [10]

Step 1. Given $\left[A(s)\in\Re[s]^{r\times r}, B(s)\in\Re[s]^{r\times m}, C(s)\in\Re[s]^{p\times r}, D(s)\in\Re[s]^{p\times m}\right]$ be the PMD (1.1) of Σ form $T(s)\in\Re[s]^{\bar{r}\times\bar{r}}$, $\bar{r}=r+p+m$.

Step 2. Compute a strongly irreducible realisation $T(s)=C_0(s)A_0^{-1}(s)B_0(s)+D_0(s)$ of $T(s)\in\Re[s]^{\bar{r}\times\bar{r}}$ where the polynomial matrices $A_0(s), B_0(s), C_0(s), D_0(s)$ are matrix pencils.

Step 3. The generalised state space system matrix :

$$P_T(s)=\left(\begin{array}{cc|c} A_0(s) & B_0(s) & 0 \\ -C_0(s) & D_0(s) & U \\ \hline 0 & -V & 0 \end{array}\right) \tag{4.1}$$

is full system equivalent to the system Σ in (1.1) under the following full system equivalence transformations :

$$\left(\begin{array}{c|c} 0 & 0 \\ I_{\bar{r}} & 0 \\ \hline 0 & I_p \end{array}\right)\left(\begin{array}{c|c} T(s) & U \\ \hline -V & 0 \end{array}\right)=\left(\begin{array}{cc|c} A_0(s) & B_0(s) & 0 \\ -C_0(s) & D_0(s) & U \\ \hline 0 & -V & 0 \end{array}\right)\left(\begin{array}{c|c} -A_0^{-1}(s)B_0(s) & 0 \\ I_{\bar{r}} & 0 \\ \hline 0 & I_m \end{array}\right) \tag{4.2a}$$

and

$$\left(\begin{array}{cc|c} C_0(s)A_0^{-1}(s) & I_{\bar{r}} & 0 \\ \hline 0 & 0 & I_p \end{array}\right)\left(\begin{array}{cc|c} A_0(s) & B_0(s) & 0 \\ -C_0(s) & D_0(s) & U \\ \hline 0 & -V & 0 \end{array}\right)=\left(\begin{array}{c|c} T(s) & U \\ \hline -V & 0 \end{array}\right)\left(\begin{array}{cc|c} 0 & I_{\bar{r}} & 0 \\ \hline 0 & 0 & I_m \end{array}\right) \tag{4.2b}$$ •

Algorithm 31. [10]

Step 1. Given $\left[A(s)\in\Re[s]^{r\times r}, B(s)\in\Re[s]^{r\times m}, C(s)\in\Re[s]^{p\times r}, D(s)\in\Re[s]^{p\times m}\right]$ be the PMD (1.1) of Σ form $T(s)\in\Re[s]^{\bar{r}\times\bar{r}}$, $\bar{r}=r+p+m$.

Step 2. Compute a strongly irreducible realisation $T(s)^{-1}=C(sE-A)^{-1}B+D$ of $T(s)^{-1}$ where $sE-A\in\Re[s]^{\lambda\times\lambda}$.

Step 3. The generalised state space system :

$$\Sigma_{T^{-1}}: \quad E\dot{x}(t) = Ax(t) + (BU)u(t)$$
$$y(t) = (VC)x(t) + (VDU)u(t) \qquad (4.3)$$

is full system equivalent to the system Σ in (1.1) under the following full system equivalence transformations :

$$\left(\begin{array}{c|c} T(s)C(sE-A)^{-1} & 0 \\ \hline 0 & I_p \end{array}\right)\left(\begin{array}{c|c} sE-A & BU \\ \hline -VC & VDU \end{array}\right) = \left(\begin{array}{c|c} T(s) & U \\ \hline -V & 0 \end{array}\right)\left(\begin{array}{c|c} C & -DU \\ \hline 0 & I_m \end{array}\right) \qquad (4.4a)$$

and

$$\left(\begin{array}{c|c} B & 0 \\ \hline VD & I_p \end{array}\right)\left(\begin{array}{c|c} T(s) & U \\ \hline -V & 0 \end{array}\right) = \left(\begin{array}{c|c} sE-A & BU \\ \hline -VC & VDU \end{array}\right)\left(\begin{array}{c|c} (sE-A)^{-1}BT(s) & 0 \\ \hline 0 & I_m \end{array}\right) \qquad (4.4b)$$ •

All the proposed generalised state space reductions are full system equivalent to the PMD in (1.1) and thus from the transitivity property (Theorem 3) of full system equivalence are full system equivalent to each other or equivalently (Theorem 3) completely system equivalent. However an interesting question concerns the dimension of the pseudostate of all those proposed generalised state space systems. We shall answer this question in the sequel.

Consider the Smith form at s=∞ of $T(s) \in \Re[s]^{\bar{r}\times\bar{r}}$:

$$S^{\infty}_{T(s)}(s) = \text{block diag}\left(s^{q_1},\ s^{q_2},\ \cdots,\ s^{q_v},\ I_{k-v},\ \frac{1}{s^{\hat{q}_{k+1}}},\ \frac{1}{s^{\hat{q}_{k+2}}},\ \cdots,\ \frac{1}{s^{\hat{q}_{\bar{r}}}}\right) \qquad (4.5)$$

where $q_1 \geq q_2 \geq \cdots \geq q_v > 0$ and $\hat{q}_{\bar{r}} \geq \hat{q}_{\bar{r}-1} \geq \cdots \geq \hat{q}_{k+1} > 0$ are respectively the orders of the infinite poles and zeros of $T(s)$. Then from Remark 7, 12, 16, 19, 22, 25 and 28 we have that

	Generalised state space model	Dimension of the pseudostate
1.	Verghese's model (3.2.1)	$\lambda_v = \bar{r} + \delta_M(T) + v$
2.	Bosgra & Van Der Weiden model (3.3.4)	$\lambda_{bvv} = \bar{r} + \delta_M(T) - v$
3.1	Zhang's model (3.4.5)	$\lambda_z = \bar{r} + \delta_M(T)$
3.2	Zhang's model (3.4.12)	$\lambda_z = 2\bar{r} + \delta_M(T)$
4.1	Tan & Vandewalle's model (3.5.6)	$\lambda_{tv} = \bar{r} + \delta_M(T)$
4.2	Tan & Vandewalle's model (3.5.15)	$\lambda'_{tv} = 2\bar{r} + \delta_M(T)$
5.	Vardulakis' model (3.6.3)	$\lambda_{niv} = \bar{r} + \delta_M(T) - k$

where $\bar{r} = r + p + m$, v={the number of strictly positive orders of infinite poles of T(s) i.e. $q_i > 0$} and k={the number of infinite poles of T(s), including the zero ones i.e. $q_i \geq 0$}.

It is obvious from the above comparison of the dimensions of the pseudostates of the proposed generalised state space reduction models, that the model proposed by A.I.Vardulakis (1991) after the alterations which we have made i.e. extraction of the nondynamic variables of his proposed generalised state space model, has the least dimension among the other models. This is the least dimension for an "equivalent" generalised state space model of the PMD (1.1) [8]. However an obvious disadvantage of Vardulakis' model is the computational effort needed for its construction.

$$\lambda_{aiv} \underset{\substack{\text{nondynamic}\\ \text{variables}}}{\overset{T(s)\text{ has no}}{\leq}} \lambda_{hvv} \underset{\text{infinite zeros}}{\overset{T(s)\text{ has no}}{\leq}} \lambda_z = \lambda_{tv} \underset{\text{infinite zeros}}{\overset{T(s)\text{ has no}}{\leq}} \lambda_v \underset{\substack{\text{nondynamic}\\ \text{variables}}}{\overset{\substack{T(s)\text{ has no}\\ \text{infinite zeros \&}}}{\leq}} \lambda_z^{'} = \lambda_{tv}^{'}$$

5. Conclusions.

In this paper we have examined the problem of transforming a general polynomial matrix model of a linear multivariable system into a model in generalised state space form via the transformation of full system equivalence. We have proposed 5 different reduction algorithms and in the last section we classified them in two seperate theoretical algorithms. A comparison of all the proposed models shows that Vardulakis (1991) model is the best one as concerns the least dimension of the pseudostate of the reduced generalised state space system. The proposed algorithms have been implemented by our colleague J.Jones (1995) in the symbolic language Maple V.3.0.

REFERENCES

[1] B.D.O. Anderson, W.A. Coppel and D.J. Cullen, Strong system equivalence., *J.Austral.Math.Soc.Ser.*, B.**27**, (1985), 194-222.

[2] O.H. Bosgra and A.J.J. Van Der Weiden, Realisations in generalised state-space form for polynomial system matrices and the definition of poles, zeros and decoupling zeros at infinity., *Int. J.Control*, **33**, (1981), 393-411.

[3] P.M.G. Ferreira, Infinite system zeros., *Int. J. Control*, **32,** (1980), 731-735.

[4] P.A. Fuhrmann, On strict system equivalence and similarity., *Int. J.Control*, **25**, (1977), 5-10.

[5] G.E. Hayton, A.B. Walker and A.C. Pugh, The transformation of full equivalence., *Int. Report (I)*, (1988), Dept. of Elec. Eng., Univ. Hull, U.K..

[6] G.E. Hayton, A.C. Pugh and P. Fretwell, Infinite elementary divisors of a matrix polynomial and implications., *Int. J. Control,* **47**, (1988), 53-64.

[7] G.E. Hayton, A.B. Walker and A.C. Pugh, Infinite frequency structure preserving transformations for general polynomial system matrices., *Int. J. Control*, **52**, (1990), 1-14.

[8] N.P. Karampetakis, Notions of equivalence for linear multivariable systems., Ph.D. Thesis, (1993), Department of Mathematics, Aristotle University of Thessaloniki, Greece.

[9] N.P. Karampetakis and A.I.G. Vardulakis, Generalised state-space system matrix equivalents of a Rosenbrock system matrix., *IMA Journal of Control and its Information*, **10**, (1993), 323-344.

[10] N.P. Karampetakis, A.C. Pugh and A.I.G. Vardulakis, A classification of generalised state space reduction methods for linear multivariable systems., *2nd IEEE Mediterranean Symposium on New Directions in Control and Automation*, June 19-22, 1994, Chania, Crete.

[11] A.C. Pugh, G.E. Hayton and P. Fretwell, On transformations of matrix pencils and implications in linear system theory., *Int. J. Control,* **45**, (1987), 529-548.

[12] A.C. Pugh, N.P. Karampetakis, A.I.G. Vardulakis and G.E. Hayton, A fundamental notion of equivalence for linear multivariable systems., *IEEE Trans. on Autom. Control*, Vol.**39**, No.5, (1994), 1141-1145.

[13] H.H. Rosenbrock, *State-Space and Multivariable Theory*, Nelson, (1970), London.

[14] H.H. Rosenbrock, The zeros of a system., *Int. J. Control*, **18**, (1973), 297-299.

[15] H.H. Rosenbrock, Corrections to "The zeros of a system", *Int. J. Control*, **20**, (1974), 525-527.

[16] H.H. Rosenbrock, Structural properties of linear dynamical systems., *Int. J.Control*, Vol.**20**, No.2, (1974), 191-202.

[17] S. Tan and J. Vandewall, A singular system realization for arbitrary matrix fraction descriptions., *ISCAS'88*, (1988), 615-618,

[18] A.I.G. Vardulakis, *Linear Multivariable Control, Algebraic Analysis and Synthesis Methods*., (1991), Nelson-Wiley, London.

[19] A.I.G. Vardulakis, On the transformation of a polynomial matrix model of a linear multivariable system to generalised state space form, *30th IEEE Conference on Decision and Control*, (1991), Brighton, England.

[20] A.I.G. Vardulakis, Model transformations and equivalence., EURACO NETWORK, Robust and Adaptive Control Tutorial Workshop, 29 Aug.-2 Sep. 1994, University of Dublin, Trinity College, Dublin, Ireland.

[21] G.C. Verghese, *Infinite frequency behaviour in general dynamical systems*, Ph.D. Dissertation, (1979), Stanford University, Stanford, U.S.A.

[22] G.C. Verghese, B.C. Levy and T. Kailath, A generalised state space for singular systems., *IEEE Trans. on Automatic Control*, **AC-26**, No.4, (1981), 811-830.

[23] A.B. Walker, *Equivalence transformations for Linear Systems.*, Ph.D. Thesis, (1988), Hull University, Hull, England.

[24] W.A. Wolovich, Determination of state space representations for linear multivariable systems, *Automatica*, Vol.**9**, (1973), 97-106.

[25] Shou-Yuan Zhang, Polynomial matrix linearization and strongly irreducible realisation for singular systems., *Int. J. Control*, Vol.**49**, No.2, (1989),471-479.