Abstract

Admissible initial conditions (AICs) for regular polynomial matrix descriptions (PMDs) are defined. The AICs of equivalent PMDs are shown to be isomorphic.

1. Introduction

This paper considers the question of characterizing those initial conditions which can be viewed as "admissible" from the point of view that they uniquely characterize a solution, sometimes purely in a distributional sense, of the system equations of a regular PMD. It is shown that there is a particularly neat characterization of the set of AICs in terms of a matrix invariant from the normalized form of the PMD. The "admissibility" of this characterization is confirmed by subsequent considerations which establish that equivalent PMDs (appropriately defined) possess AICs which are isomorphic. An explicit construction is also given of this induced isomorphism of AICs.

2. Preliminary Results

Consider the PMD of the system $S$:

\[ T(p) \nu(t) = U(p) u(t) \tag{1.a} \]

where $p = zL(t)$, $T(p) \in \mathbb{R}[p]^{I \times M}$ with $T(p) \neq 0$, $U(p) \in \mathbb{R}[p]^{I \times I}$, $V(p) \in \mathbb{R}[p]^{E \times I}$, $W(p) \in \mathbb{R}[p]^{E \times E}$, $\nu(t) : [0, \infty) \to \mathbb{R}^I$ is the input, and $\nu(t)$ is the output of $S$. An alternative representation of $S$ is:

\[
\begin{bmatrix}
T(p) & U(p) \\
-V(p) & W(p)
\end{bmatrix}
\begin{bmatrix}
\nu(t) \\
\nu(t)
\end{bmatrix}
= \begin{bmatrix}
I & 0 \\
0 & I
\end{bmatrix}
\begin{bmatrix}
\eta(t) \\
\eta(t)
\end{bmatrix}
\tag{1.b}
\]

where $\nu(t)$ is the output, $\eta(t)$ is the output, and $\xi(t)$ is the output.

3. Admissible Initial Conditions

The question of what constitutes the initial condition set for (1) has been covered by [2], [4], [5] for the systems. In the general case of (1) the situation is less well-known although [2], [3] provide indications. Consider [5] the Laplace transform of (2):

\[ T(s) \mathcal{L}(\nu(t)) = \mathcal{L}(u(t) + \alpha(t)) \tag{2.a} \]

where $\alpha(t)$ is a polynomial vector defined as:

\[ \alpha(t) = [I \cdot 1, \ldots, I \cdot 1] \begin{bmatrix}
\xi(0) \\
\xi(0) \\
\xi(0)
\end{bmatrix} \tag{2.b} \]

and $\xi(t)$ is the state variable of $\xi$ in the output polynomial $T(s)$. $\alpha(t)$ contains all the information on the initial conditions for (2.a), but what actually constitutes the AICs necessary to determine a solution of (2.a) is the subject of the following results. We use the term "admissible" to describe the initial conditions which give rise to a unique distributional solution of (2.a) and so (1.a).

This is different from [5] where "admissible" describes the initial conditions which give rise to a solution in the strictly regular sense.

Theorem 1. The set of AICs of $\xi(t)$ (and hence $S$) is:

\[ \eta(t) = \mathcal{L}(\xi(t)) \tag{3} \]

where $\mathcal{L}$ and $\xi(t)$ are defined in (5).

Proof. Suppose $\xi(t) = \mathcal{L}(\xi(t))$, $\xi(t)$ is a two AICs which give rise to the same solution $\xi(t)$ for a given input $u(t)$. From (4.a):

\[ T(s) \mathcal{L}(\nu(t)) = \mathcal{L}(u(t) + \alpha(t)) \tag{4.a} \]

Thus:

\[ \eta(t) = [I \cdot 1, \ldots, I \cdot 1] \begin{bmatrix}
\xi(0) \\
\xi(0) \\
\xi(0)
\end{bmatrix} \tag{4.b} \]

Now let $[I \cdot 1, \ldots, I \cdot 1] = 0$, and so

\[ \xi(t) = \mathcal{L}(\xi(t)) \tag{5} \]

which is a contradiction. Thus for a given $u(t)$, (6) is in one-to-one correspondence with the solution of (2.a). This confirms (5) as the set of AICs of $\xi(t)$, and hence $S$. Note the AICs for the general PMD reduce directly to the form established in [2], [5] when the system is in its normal form. Another interesting result arising from (6) is:

Corollary 1. The initial conditions $\xi(t)$ are the AICs of (2) if $\xi(t)$ is in its normal form.

Corollary 1 essentially requires that $T(s)$ be column reduced and thus has no infinite-rank.

The AICs represent a section of the solution space at $t = 0$, and thus represent a set of points from which...
solutions commute. It can therefore be expected that the bijection between the solution spaces of (9) and PMDS would induce a bijection between the corresponding sets of AICs. Further since the AIC sets are merely sets of "points" it is expected that this bijection be a constant map.

Theorem 2. If $S_i$, $S_j$ are (i.e.) then their sets of AICs are isomorphic. Further that isomorphism is of the form

\[ T_\beta \xi(0-1) = N\xi(0-) \]

where $N$ is a constant matrix.

Proof. Let $T_i$, $T_j$ corresponding to $S_i$, $S_j$ be

\[ \xi(0-) = T_j\xi(0-1) \]

Let the bijection between the pseudostates be

\[ \xi(0-) = N\xi(0-1) \]

Without loss of generality it may be assumed that

\[ T_j = T_{1,j} + \ldots + T_{k,j} = T_{0,j} \quad N(j) = N_{1,j} + \ldots + N_{k,j} = N_j \]

where at least one of $T_{k,j}, N_j$ is nonzero. Taking Laplace transformations in (11) we obtain as in (5),

\[ \xi(s) = N(s)\xi(0-) = S_{-1}-N(s)\xi(0-) \]

Relation (11) is a map and so according to [1]

\[ \text{Rank} \{X_j, X_j^\top\} = \text{Rank} \{X_j\} \text{Rank} \{X_j\} \]

and thus $S_i = QX_j$. Thus (13) may be rewritten as

\[ \xi(s) = N(s)\xi(0-) - S_{-1}N(s)\xi(0-) = S_{-1}\xi(0-) \]

\[ S_{-1}\xi(0-) - N(s)\xi(0-) = T_j\xi(0-) \]

Assuming without loss of generality that $\xi(0-) = 0$ and

\[ T_j = 0 \quad T_j = 0 \]

is the Laurent expansion of $T_j\xi(0-) = 1$, and equating the coefficients of the $i,j,k$ in (10) yields

\[ X_i \xi(0-) = (N - T_j)X_j \xi(0-) \]

where

\[ N = \begin{bmatrix}
  N_{1,j} & \cdots & N_{k,j}
\end{bmatrix}
\]

\[ T_j = \begin{bmatrix}
  T_{1,j} & \cdots & T_{k,j}
\end{bmatrix}
\]

The solution (17) is a constant mapping since $\text{det}(T_j) = 0$. This is a bijection, and since it assigns a unique image to each element of $M_i$, it is expected that equivalence would be induced between their corresponding AIC sets. This conclusion has been confirmed in Theorem 2 where the induced bijection map between the AIC sets is explicitly constructed.

4. Conclusions

The question of what constitutes the AIC set, in the sense of characterizing those initial conditions which are in one-to-one correspondence with the elements of the distributional solution space of (1), has been addressed in the case of general PMDS. It has been shown in Theorem 1 that this set can be described exactly in terms of information derived from the normalized system (2). Clearly when two equivalent systems are considered it is expected that equivalence would be induced between their corresponding AIC sets. This conclusion has been confirmed in Theorem 2 where the induced bijection map between the AIC sets is explicitly constructed.

References


