Proceedings of the 34th
Conference on Decision & Control
New Orleans, LA - December 1995

ON THE ADMISSIBLE INITIAL CONDITIONS OF A REGULAR PMD

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Abstract

Admissible initial conditions (AICs) for regular polynomial matrix descriptions (PMDs) are defined. The AICs of equivalent PMDs are shown to be isomorphic.

1. Introduction

This paper considers the question of characterising those initial conditions which can be viewed as "admissible" from the point of view that they uniquely characterise a solution, sometimes merely in a distributional sense, of the system equations of a regular PMD. It is shown that there is a particularly neat characterisation of the set of AICs in terms of information derivable from the normalised form of the PMD. The "admissibility" of this characterisation is confirmed by subsequent considerations which establish that equivalent PMDs (appropriately defined) possess AICs sets which are isomorphic. An explicit construction is also given of this induced isomorphism of AICs.

2. Preliminary Results

Consider the PMD of the system S

$$T(\rho)\beta(t) = U(\rho)u(t) \qquad (1.a)$$

$$y(t) = V(\rho)\beta(t) + W(\rho)u(t) \qquad (1.b)$$

where $(\rho = d/dt)$, $T(\rho) \in \mathbb{R}[\rho]^{r\times r}$ with $|T(\rho)| \neq 0$, $U(\rho) \in \mathbb{R}[\rho]^{r\times m}$, $V(\rho) \in \mathbb{R}[\rho]^{p\times r}$, $W(\rho) \in \mathbb{R}[\rho]^{p\times m}$, $\beta(t) : [0-,\infty) \to \mathbb{R}^r$ is the *pseudo state* of Σ, $u(t) : [0-,\infty) \to \mathbb{R}^m$ is the *control input* and $y(t)$ the *output* of S. An alternative representation of S is

$$\underbrace{\begin{pmatrix} T(\rho) & U(\rho) & 0 \\ -V(\rho) & W(\rho) & I \\ 0 & -I & 0 \end{pmatrix}}_{\mathcal{T}(\rho)} \underbrace{\begin{pmatrix} \beta(t) \\ -u(t) \\ y(t) \end{pmatrix}}_{\xi(t)} = \underbrace{\begin{pmatrix} 0 \\ 0 \\ I \end{pmatrix}}_{\mathcal{U}} u(t) \qquad (2.a)$$

$$y(t) = \underbrace{\begin{pmatrix} 0 & 0 & I \end{pmatrix}}_{\mathcal{V}} \xi(t) \qquad (2.b)$$

denoted Σ, and termed [2] the *normalized form* of S.

Definition 1 [1]. Let Σ_1, Σ_2 be the normalized form of two systems S_1, S_2. S_1, S_2 are said to be **fundamentally equivalent** (f.e.) iff the two following conditions hold:

(i) $\exists$ a *bijection* between the $\xi_i(t)$, $i = 1, 2$ of Σ_1, Σ_2:

$$\xi_2(t) = N(\rho)\xi_1(t) \qquad (3)$$

(ii) for the given $u(t)$, two pseudostates related as in (3) produce the same output $y(t)$. $\square$

We consider the action of (3) in the context of the map it induces on the AIC sets of equivalent PMDs.

3. Admissible Initial Conditions

The question of what constitutes the initial condition set for (1) has been covered by [2], [4], [5] for gss systems. In the general case of (1) the situation is less well-known although [2], [3] provide indications. Consider [3] the Laplace transform of (2)

$$\mathcal{T}(s)\xi(s) = \mathcal{U}u(s) + \alpha_{\mathcal{T}}(s) \qquad (4.a)$$

$$y(s) = \mathcal{V}\xi(s) \qquad (4.b)$$

where $\alpha_{\mathcal{T}}(s)$ is a polynomial vector defined as

$$\alpha_{\mathcal{T}}(s) = [s^{k-1}I, \ldots, sI, I] \begin{bmatrix} \mathcal{T}_k & 0 & \ldots & 0 \\ \vdots & \vdots & \ldots & \vdots \\ \mathcal{T}_2 & \mathcal{T}_3 & \ddots & 0 \\ \mathcal{T}_1 & \mathcal{T}_2 & \ldots & \mathcal{T}_k \end{bmatrix} \begin{bmatrix} \xi(0-) \\ \vdots \\ \xi^{(k-2)}(0-) \\ \xi^{(k-1)}(0-) \end{bmatrix}$$

$$:= S_{k-1}\mathcal{X}_{\mathcal{T}}\hat{\xi}(0-). \qquad (5)$$

and $\mathcal{T}_i$ is the matrix coefficient of s^i in the matrix polynomial $\mathcal{T}(s)$. $\alpha_{\mathcal{T}}(s)$ contains all the information on the initial conditions for (2.a), but what actually constitutes the AICs necessary to determine a solution of (2.a) is the subject of the following result. We use the term "admissible" to describe the initial conditions which give rise to a unique distributional solution of (2.a) and so (1.a). This is different to [5] where "admissible" describes that the initial conditions which give rise to a solution in the strictly regular sense.

Theorem 1. The set of AICs of Σ (and hence S) is

$$\{\eta : \eta = \mathcal{X}_{\mathcal{T}}\hat{\xi}(0-),\ \hat{\xi}(0-) \in \mathbb{R}^{k(r+m+p)}\} \qquad (6)$$

where $\mathcal{X}_{\mathcal{T}}$ and $\hat{\xi}(0-)$ are defined in (5).

Proof. Suppose $\mathcal{X}_{\mathcal{T}}\hat{\xi}_1(0-) \neq \mathcal{X}_{\mathcal{T}}\hat{\xi}_2(0-)$ are two AICs which give rise to the same solution $\xi(s)$ for a given input $u(t)$. From (4.a)

$$\mathcal{T}(s)\xi(s) = [s^{k-1}I, \ldots\ldots, sI, I]\mathcal{X}_{\mathcal{T}}\hat{\xi}_i(0-) + \mathcal{U}u(s),\ i = 1, 2.$$

Thus

$$0 = [s^{k-1}I, \ldots\ldots, sI, I]\left[\mathcal{X}_{\mathcal{T}}\hat{\xi}_1(0-) - \mathcal{X}_{\mathcal{T}}\hat{\xi}_2(0-)\right] \qquad (7)$$

Now $\ker[s^{k-1}I, \ldots\ldots, sI, I] = 0$, and so

$$\mathcal{X}_{\mathcal{T}}\hat{\xi}_1(0-) = \mathcal{X}_{\mathcal{T}}\hat{\xi}_2(0-) \qquad (8)$$

which is a contradiction. Thus for a given $u(t)$, (6) is in one-to-one correspondence with the solution set of (2.a). This confirms (6) as the set of AICs for Σ, and hence S. $\square$

Note the AICs for the general PMD reduce directly to the form established in [2],[5] when the system is in gss form. Another interesting result arising from (8) is

Corollary 1. The initial conditions $\hat{\xi}(0-)$ are the AICs of (2) iff $\mathcal{X}_{\mathcal{T}}$ (or equivalently $\mathcal{T}_k$) has full column rank. $\square$

Corollary 1 essentially requires that $\mathcal{T}(s)$ be column reduced and thus has no infinite zeros.

The AICs represent a section of the solution space taken at time $t = 0-$, and thus represent a set of points from which solutions emanate. It can therefore be expected that the bijection between the solution spaces of two (f.e.) PMDs would induce a bijection between the corresponding sets of AICs. Further since the AIC sets are merely sets of "points" it is expected that this bijection be a constant map.

Theorem 2. If S_1, S_2 are (f.e.) then their sets of AICs are isomorphic. Further this isomorphism is of the form

$$\mathcal{X}_{T_2}\hat{\xi}_2(0-) = M\mathcal{X}_{T_1}\hat{\xi}_1(0-) \tag{9}$$

where M is a constant matrix.

Proof. Let Σ_1, Σ_2 corresponding to S_1, S_2 be

$$\begin{aligned} T_i(\rho)\xi_i(t) &= \mathcal{U}_i u(t) \\ y(t) &= \mathcal{V}_i\xi_i(t) \end{aligned} \quad i = 1, 2 \tag{10}$$

Let the bijection between the pseudostates be

$$\xi_2(t) = N(\rho)\xi_1(t). \tag{11}$$

Without loss of generality it may be assumed that

$$\begin{aligned} T_i(\rho) &:= T_{i,k}\rho^k + \ldots + T_{i,1}\rho + T_{i,0} \\ N(\rho) &:= N_k\rho^k + \ldots + N_1\rho + N_0 \end{aligned} \tag{12}$$

where at least one of $T_{i,k}, N_k$ is nonzero. Taking Laplace transformations in (11) we obtain as in (5),

$$\xi_2(s) = N(s)\xi_1(s) - S_{q-1}\mathcal{X}_N\hat{\xi}_1(0-). \tag{13}$$

Relation (11) is a map and so according to [1] we obtain

$$\operatorname{Rank}_{\mathbb{R}}\left[\mathcal{X}_{T_1}^T\ \mathcal{X}_N^T\right]^T = \operatorname{Rank}_{\mathbb{R}}[\mathcal{X}_{T_1}] \tag{14}$$

and thus $\mathcal{X}_N = Q\mathcal{X}_{T_1}$. Then (13) may be rewritten as

$\xi_2(s) = N(s)\xi_1(s) - S_{k-1}Q\mathcal{X}_{T_1}\hat{\xi}_1(0-) \Longrightarrow$

$S_{k-1}\mathcal{X}_{T_2}\hat{\xi}_2(0-) + \mathcal{U}_2u(s) = T_2(s)\left(N(s)T_1(s)^{-1}S_{k-1}\right.$

$$\left.\mathcal{X}_{T_1}\hat{\xi}_1(0-) + N(s)T_1(s)^{-1}\mathcal{U}_1u(s) - S_{k-1}Q\mathcal{X}_{T_1}\hat{\xi}_1(0-)\right) \tag{15}$$

Assuming without loss of generality that $u(s) = 0$ and

$$T_2NT_1^{-1} = H_qs^q + H_{q-1}s^{q-1} + \ldots + H_0 + H_{-1}\frac{1}{s} + \ldots \tag{16}$$

is the Laurent expansion of $T_2(s)N(s)T_1(s)^{-1}$, and equating the coefficient matrices of s^i, $i = 0, 1, \ldots, k-1$ in (15) yields

$$\mathcal{X}_{T_2}\hat{\xi}_2(0-) = (\mathcal{H} - \hat{T}_2Q)\mathcal{X}_{T_1}\hat{\xi}_1(0-) \tag{17}$$

where

$$\mathcal{H} = \begin{bmatrix} H_0 & \cdots & H_{k-1} \\ \vdots & \ddots & \vdots \\ H_{-k+1} & \cdots & H_0 \end{bmatrix}; \hat{T}_2 = \begin{bmatrix} T_{02} & \cdots & T_{k-1,2} \\ \vdots & \cdots & \vdots \\ 0 & \cdots & T_{02} \end{bmatrix}$$

The relation (17) is a constant mapping (since $(\mathcal{H} - \hat{T}_2Q)$ is constant, and since it assigns a unique image) between the sets of AICs. Further it will be a bijection since the map (11) from which it is derived is a bijection. □

Example 1. In respect of the homogeneous system

$$S_1 : T_1(\rho)\xi(t) := \begin{pmatrix} \rho+1 & 1 & \rho^2+1 \\ 0 & 1 & \rho^2 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} \xi_1(t) \\ \xi_2(t) \\ \xi_3(t) \end{pmatrix} = 0 \quad (E.1)$$

a basis for the solution space is ($e_{p,n}$ is the pth col. of I_n)

$$X_1 = (e_{1,3}e^{-t},\quad e_{2,3}\delta(t),\quad e_{2,3}\delta^{(1)}(t))$$

This corresponds to the basis of the set of AICs

$$\left\langle \mathcal{X}_{T_1}\hat{\xi}_1(0-)\right\rangle = \left\langle \begin{pmatrix} 0_{1,3} \\ e_{1,3} \end{pmatrix}; \begin{pmatrix} 0_{1,3} \\ e_{1,3} - e_{2,3} \end{pmatrix}; \begin{pmatrix} e_{1,3} + e_{2,3} \\ 0_{1,3} \end{pmatrix}\right\rangle \quad (E.2)$$

Consider the bijection

$$\begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \rho \\ 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} \xi_1(t) \\ \xi_2(t) \\ \xi_3(t) \end{pmatrix} =: N(\rho)\xi(t) \quad (E.3)$$

Note the map (E.3) takes the basis X_1 to a basis X_2

$$X_2 = (e_{1,4}e^{-t}, e_{2,4}\delta(t), e_{3,4}\delta^{(1)}(t) - e_{4,4}\delta(t)) \quad (E.4)$$

of the solution space of S_2 where

$$S_2 : T_2(\rho)x(t) := \begin{pmatrix} \rho+1 & 0 & 0 & 0 \\ 0 & 1 & \rho & 0 \\ 0 & 0 & 1 & \rho \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{pmatrix} = 0 \quad (E.5)$$

In this particular case we have

$$T_2(s)N(s)T_1(s)^{-1} = \begin{pmatrix} 1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} = H_0$$

Further (14) holds and thus $\mathcal{X}_N = Q\mathcal{X}_{T_1}$ with

$$Q = (e_{7,8},\quad 0_{8,5})$$

Then

$$\mathcal{X}_{T_2}\hat{\xi}_2(0-) = \begin{pmatrix} 1 & -1 & -1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}\mathcal{X}_{T_1}\hat{\xi}_1(0-)$$

This is a bijection which may be verified directly, and takes the basis (E.2) for the space of AICs for (E.1), to a basis of the space of AICs for (E.5) of the form

$$\left\langle \mathcal{X}_{T_2}\hat{\xi}_2(0-)\right\rangle = \left\langle \begin{pmatrix} e_{1,4} \\ 0_{1,4} \end{pmatrix}; \begin{pmatrix} e_{2,4} \\ 0_{1,4} \end{pmatrix}; \begin{pmatrix} -e_{4,4} \\ 0_{1,4} \end{pmatrix}\right\rangle \quad (E.6)$$

The elements of (E.6) correspond to the initial condition set for the elements of the X_2 of (E.4). □

4. Conclusions

The question of what constitutes the AIC set, in the sense of characterising those initial conditions which are in one-to-one correspondence with the elements of the distributional solution space of (1), has been addressed in the case of general PMDs. It has been seen in Theorem 1 that this set can be described most neatly in terms of information derived from the normalised system (2). Clearly when two equivalent systems are considered it is expected that equivalence would be induced between their corresponding AIC sets. This occurrence has been confirmed in Theorem 2 where the induced bijective map between the AIC sets is explicitly constructed.

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