# The Output Zeroing Problem for General Polynomial Descriptions

by

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## Abstract.

This paper gives a physical interpretation of invariant zeros and indices in terms of the general zero-output behaviour of a linear dynamical system.

## 1. Introduction.

Consider a linear, time invariant, system described by

$$\Sigma : \ A(\rho)\beta(t) = B(\rho)u(t) \qquad y(t) = C(\rho)\beta(t) + D(\rho)u(t) \tag{1.1}$$

where $\rho := d/dt$ is the differential operator, $A(\rho) \in \Re[\rho]^{n\times n}$ with $\det[A(\rho)] \neq 0$, $B(\rho) \in \Re[\rho]^{n\times m}$, $C(\rho) \in \Re[\rho]^{p\times n}$, $D(\rho) \in \Re[\rho]^{p\times m}$, $\beta(t):(0-,+\infty) \to \Re^n$ is the *pseudostate* of the system, $u(t):(0-,+\infty) \to \Re^m$ is the *input* of the system, and $y(t):(0-,+\infty) \to \Re^p$ is the *output* of the system. $\Sigma$ may be rewritten as :

$$\begin{pmatrix} A(\rho) & B(\rho) \\ -C(\rho) & D(\rho) \end{pmatrix} \begin{pmatrix} \beta(t) \\ -u(t) \end{pmatrix} = \begin{pmatrix} 0 \\ -I \end{pmatrix} y(t) \tag{1.2}$$

The general output-zero problem for $\Sigma$ may be stated [6] as follows : Find the set of initial conditions (or pseudostates) and control inputs such that the output is identically zero. Using the system description (1.2), this problem is reduces to studying the structure and properties of the vector space of solutions of the system

$$\underbrace{\begin{pmatrix} A(\rho) & B(\rho) \\ -C(\rho) & D(\rho) \end{pmatrix}}_{P(\rho)} \underbrace{\begin{pmatrix} \beta(t) \\ -u(t) \end{pmatrix}}_{z(t)} = \mathbf{0}_{(n+p)\times(n+m)} \tag{1.3}$$

where

$$P(\rho) = P_0 + P_1\rho + \cdots + P_{q_1}\rho^{q_1} \in \Re[\rho]^{(n+p)\times(n+m)} \tag{1.4}$$

In the case where $\Sigma$ is in state-space form the zero-output problem has been studied by [4]-[7] and the relevance of the finite and infinite invariant zeros of the system was demonstrated. However questions still remain concerning the solution of the general output-zero problem when $\Sigma$ is in the general form (1.1). This question will be considered in this paper. Specifically in sections 2 and 3 we give a geometric interpretation of the finite and infinite invariant zeros, while in section 4 we introduce the notions of left and right invariant indices and give a geometric interpretation of these indices. These interpretations coincide exactly with those given for state-space systems by [5], [7]. In section 5 we give a general solution to the zero-output problem.

## 2. Geometric Interpretation of the Finite Invariant Zeros.

Consider the system $\Sigma$ in (1.1). Define

$$Z = \left\{ \begin{pmatrix} \beta(t) \\ -u(t) \end{pmatrix} = L^{-1}\begin{pmatrix} \hat{\beta}(s) \\ -\hat{u}(s) \end{pmatrix} ; \begin{pmatrix} \mathrm{A(s)} & \mathrm{B(s)} \\ \mathrm{-C(s)} & \mathrm{D(s)} \end{pmatrix}\begin{pmatrix} \hat{\beta}(s) \\ -\hat{u}(s) \end{pmatrix} = 0 \right\} \tag{2.1a}$$

$$Z_u = -\pi_u Z \quad \text{and} \quad Z_\beta = \pi_\beta Z \tag{2.1b}$$

$$\begin{aligned} [a] &= \{a + z \text{ where } a \text{ is a solution of (1.3) and } z \in Z\} = \{a\} \oplus Z \\ [a]_u &= \{a + z \text{ where } a \text{ is a solution of (1.3) and } z \in Z_u\} = \pi_u\{a\} \oplus Z_u \\ [a]_\beta &= \{a + z \text{ where } a \text{ is a solution of (1.3) and } z \in Z_\beta\} = \pi_\beta\{a\} \oplus Z_\beta \end{aligned} \tag{2.1c}$$

where $L[z(t)]$ denotes the Laplace transform of the vector $z(t)$, $\pi_\beta : (\beta(t)^T, (-u(t))^T)^T \in Z \to \beta(t)$ and $\pi_u : (\beta(t)^T, (-u(t))^T)^T \in Z \to -u(t)$. Note also that Z is the solution space of the system (1.3) under zero, input and pseudostate, initial conditions and that its elements are represented by the equivalence class [0].

Assume that $P(s) \in \Re[s]^{(n+p)\times(n+m)}$ of (1.3) has

$$U_L(s)P(s)U_R(s) = \text{blockdiag}\left(1,..,1, f_\mu(s),\ldots, f_r(s), \mathbf{0}_{n+p-r,n+m-r}\right) \tag{2.2}$$

$1 \le \mu \le r$ as its Smith form (in **C**) where $f_i(s) \in \Re[s]$ are the nonunit invariant polynomials of P(s) and $f_i(s)/f_{i+1}(s)$., $i = \mu, \mu+1,\ldots,r-1$. Assume that the nonunit invariant polynomials $f_i(s) \in \Re[s]$ have $\ell$ distinct zeros $\lambda_1, \lambda_2, \ldots, \lambda_\ell$ (where for simplicity of notation we assume that $\lambda_i \in \Re$) with partial multiplicities $\sigma_{1,i}, \sigma_{2,i}, \ldots, \sigma_{\ell,i}$ $i = \mu, \mu+1,\ldots,r$ where $0 \le \sigma_{i,\mu} \le \sigma_{i,\mu+1} \le \cdots \le \sigma_{i,r}$.

[8] originally defined the notion of the finite invariant zeros for general systems of the form (1.1). Then [7] utilised this definition for the solution of the zeroing output problem in state space systems. An extension of the definition of generalized invariant zero-direction vectors for general systems of the form (1.1) is

**Definition 1.** Let P(s) be the Rosenbrock system matrix of $\Sigma$ defined in (1.3)-(1.4). Then the *finite invariant zeros* of $\Sigma$ are the finite zeros of P(s). if associated with each invariant zero $\lambda_i$, $i \in \ell$ there is a composite vector $z_0^i = \left(\beta_0^{i\,T}, -u_0^{i\,T}\right)^T$ which lies in the kernel or null space of $P(\lambda_i)$ i.e. $P(\lambda_i)z_0^i = 0$, and such that $P(s)z_0^i \neq 0$, and a sequence of vectors $z_q^i = \left(\beta_q^{i\,T}, -u_q^{i\,T}\right)^T$, $i \in \ell$, $q = 1,2,\ldots,\nu$ such that

$$\begin{aligned} &P(\lambda_i)z_0^i = 0 \\ &P^{(1)}(\lambda_i)z_0^i + P(\lambda_i)z_1^i = 0 \qquad (2.3 \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &\frac{1}{\nu!}P^{(\nu)}(\lambda_i)z_0^i + \frac{1}{(\nu-1)!}P^{(\nu-1)}(\lambda_i)z_1^i + \cdots + \\ &+P^{(1)}(\lambda_i)z_{\nu-1}^i + P(\lambda_i)z_\nu^i = 0 \end{aligned}$$

and

$$P(s)\left(z_0^i + (s-\lambda_i)z_1^i + \cdots + (s-\lambda_i)^\nu z_\nu^i\right) \neq 0 \qquad (2.4)$$

where $P^{(j)}(s)$ denotes the jth derivative of $P(s)$, then $z_0^i$ is called an *invariant zero direction vector for* $\lambda_i$ and $z_q^i, i \in \ell, q = 0,1,..,\nu$ *a sequence of generalised invariant zero direction vectors for* $\lambda_i$. ¤

The existence of sequences of generalized invariant zero-direction vectors is given by the following

**Theorem 1.** Consider the Rosenbrock system matrix (1.3) of $\Sigma$ with Smith form (2.2). Then for any invariant zero $\lambda_i$, $i \in \ell$ of partial multiplicity $\sigma_{i,j}$, $j = \mu, \mu+1, \ldots, r$ $\exists$ an invariant zero-direction vector $z_{j,0}^i = \left(\beta_{j,0}^{i\;T}, -u_{j,0}^{i\;T}\right)^T$ and a sequence of generalized invariant zero-direction vectors $z_{j,q}^i = \left(\beta_{j,q}^{i\;T}, -u_{j,q}^{i\;T}\right)^T$, $i \in \ell$, $j = \mu, \mu+1, \ldots, r$, $q = 1, 2, \ldots, \sigma_{i,j} - 1$

**Proof.** Let $u_j(s)$, with $j = \mu, \mu+1, \ldots, r$ be the columns of $U_R(s)$, defined in (2.2) and $u_j^{(q)}(s) := (d^q/ds^q)u_j(s)$, $q \in (\sigma_{i,j}-1)$.. Then it is easily seen that the vectors

$$z_{j,q}^i := \begin{pmatrix} \beta_{j,q}^i \\ -u_{j,q}^i \end{pmatrix} = \frac{1}{q!}u_j^{(q)}(\lambda_i) \qquad (2.5)$$

with $i \in \ell$ and $j = \mu, \mu+1, \ldots, r$ form a sequence of generalized invariant zero-direction vectors.¤

**Theorem 2.** Let P(s) be the Rosenbrock system matrix of $\Sigma$ defined in (1.4) and let $\lambda_i$, $i \in \ell$ be an invariant zero of $\Sigma$. Then to an input of the form

$$[u_q^i(t)]_u := \left[\sum_{k=0}^{\nu-q} u_k^i \frac{t^{\nu-q-k}}{(\nu-q-k)!} e^{\lambda_i t}\right]_u \quad i \in \ell \qquad (2.6a)$$

$$\text{and } q = 0,1,\ldots,\nu$$

and with initial conditions of the form :

$$u_q^{i(j)}(0-) = \lambda_i^j u_q^i + j\lambda_i^{j-1}u_{q-1}^i + \cdots + \\ + \cdots + \frac{j(j-1)\cdots(j-q+1)}{q!}\lambda_i^{j-q}u_0^i \qquad (2.6b)$$

with $j = 0,1,\ldots,q_1 - 1$, there corresponds a pseudostate of $\Sigma$ of the form

$$[\beta_q^i(t)]_\beta := \left[\sum_{k=0}^{\nu-q} \beta_k^i \frac{t^{\nu-q-k}}{(\nu-q-k)!} e^{\lambda_i t}\right]_\beta \quad i \in \ell \qquad (2.7a)$$

$$\text{and } q = 0,1,\ldots,\nu$$

with initial conditions of the form

$$\beta_q^{i(j)}(0-) = \lambda_i^j \beta_q^i + \cdots + \frac{j(j-1)\cdots(j-q+1)}{q!}\lambda_i^{j-q}\beta_0^i \qquad (2.7b)$$

for $j = 0,1,\ldots,q_1 - 1$ which produces zero output of $\Sigma$

$$y(t) = 0 \qquad (2.8)$$

where the vectors $\left(\beta_k^{i\,T}, -u_k^{i\,T}\right)^T$ are generalized invariant zero-direction vectors for $\lambda_i$.

**Proof.** See [10]. ¤

**Corollary 1.** From Theorems 1 and 2 we conclude that there exist an input vector space

$$U_0^C = \{[u_{j,q}^i(t)]_u := \left[\sum_{k=0}^{\sigma_{i,j}-1-q} u_{j,k}^i \frac{t^{\sigma_{i,j}-q-k-1}}{(\sigma_{i,j}-q-k-1)!} e^{\lambda_i t}\right]_u ; \qquad (2.9)$$

$$i \in \ell,\ j = \mu, \mu+1, \ldots, r \text{ and } q = 0,1,\ldots,\sigma_{i,j} - 1\}$$

which gives rise through the relation (1.1) to the pseudostate vector space

$$B_0^C = \{[\beta_{j,q}^i(t)]_\beta := \left[\sum_{k=0}^{\sigma_{i,j}-1-q} \beta_{j,k}^i \frac{t^{\sigma_{i,j}-q-k-1}}{(\sigma_{i,j}-q-k-1)!} e^{\lambda_i t}\right]_\beta ; \qquad (2.10)$$

$$i \in \ell,\ j = \mu, \mu+1, \ldots, r \text{ and } q = 0,1,\ldots,\sigma_{i,j} - 1\}$$

and consequently to the output vector space

$$Y_0 = \{0\} \qquad (2.11)$$

**Proof.** See [10]. ¤

## 3. Geometric Interpretation of the Infinite Invariant Zeros.

Consider the Rosenbrock matrix P(s) in (1.4) and define its "dual" polynomial matrix $\tilde{P}(w)$ [1] as

$$\tilde{P}(w):=w^{q_1}P\left(\frac{1}{w}\right)=P_{q_1}+P_{q_1-1}w+\cdots+P_0w^{q_1} \quad (3.1)$$

Then $\exists$ unimodular matrices $\tilde{U}_L(w)$, $\tilde{U}_R(w)$, [1] s.t.

$$\tilde{U}_L(w)\tilde{P}(w)\tilde{U}_R(w)= \quad (3.2)$$
$$=\text{blockdiag}\{\tilde{f}_1(w), w^{q_1-q_2}\tilde{f}_2(w),..,w^{q_1-q_k}\tilde{f}_k(w), w^{q_1+\hat{q}_{k+1}}\tilde{f}_{k+1}(w),\dots,w^{q_1+\hat{q}_r}\tilde{f}_r(w), 0_{n+p-r,n+m-r}\}$$

where $\tilde{f}_i(0)\neq 0$, $q_1\geq\cdot\cdot\geq q_k\geq 0$ and $0<\hat{q}_{k+1}\leq\cdot\cdot\leq\hat{q}_r$ are respectively the order of the poles and zeros at $s=\infty$ of P(s).

The notion of the infinite invariant zeros was presented by [6] (state space systems), [9] (PMDs of the form (1.1)), while the notion of generalized infinite zero-direction vectors for state space systems was utilised by [6] for the solution of the output zeroing problem in the state space case. In what follows we present an extension of the generalized infinite zero-direction vectors to the case of general systems of the form (1.1).

**Definition 2.** Let P(s) be the Rosenbrock system matrix of $\Sigma$ defined in (1.3)-(1.4), then the *infinite invariant zeros* of $\Sigma$ are the infinite zeros of P(s). Associated with each invariant infinite zero of $\Sigma$ there is a composite vector $z_0^\infty=\left(\beta_0^{\infty T},-u_0^{\infty T}\right)^T$ which lies in the kernel or null space of $\tilde{P}(0)\equiv P_{q_1}$ i.e. $\tilde{P}(0)z_0^\infty=0\Leftrightarrow P_{q_1}z_0^\infty=0$, and such that $P(s)z_0^\infty\neq 0$, and a sequence of vectors $z_j^\infty=\left(\beta_j^{\infty T},-u_j^{\infty T}\right)^T$, $j=1,\dots,q_1+q$ such that

$$P_{q_1}z_0^\infty=0$$
$$P_{q_1-1}z_0^\infty+P_{q_1}z_1^\infty=0 \quad (3.3)$$
$$\dots\dots\dots\dots$$
$$P_0z_0^\infty+P_1z_1^\infty+\cdots+P_{q_1-1}z_{q_1-1}^\infty+P_{q_1}z_{q_1}^\infty=0$$
$$P_0z_1^\infty+P_1z_2^\infty+\cdots+P_{q_1-1}z_{q_1}^\infty+P_{q_1}z_{q_1+1}^\infty=0$$
$$\dots\dots\dots\dots$$
$$P_0z_q^\infty+P_1z_{q+1}^\infty+\cdots+P_{q_1-1}z_{q_1+q-1}^\infty+P_{q_1}z_{q_1+q}^\infty=0$$

and

$$P(s)\left(z_0^\infty s^{q_1+q}+z_1^\infty s^{q_1+q-1}+\cdots+z_{q_1+q}^\infty\right)\neq 0 \quad (3.4)$$

Then $z_0^\infty$ is called an *invariant infinite zero-direction vector for the infinite zero of order* q and $z_j^\infty$, $j=1,..,q_1+q$ *a sequence of generalised invariant infinite zero-direction vectors for the infinite zero of order* q. ¤

The existence of such a chain of generalized infinite zero-direction vectors is given by the following

**Theorem 3.** Consider the Rosenbrock system matrix (1.3) of $\sum$ and the local Smith form of its dual polynomial matrix at w=0 in (3.2). Then for every infinite invariant zero of order $\hat{q}_j$ with $j=k+1,k+2,\dots,r$ $\exists$ an invariant infinite zero-direction vector $z_{i,0}^\infty=\left(\beta_{i,0}^{\infty T},-u_{i,0}^{\infty T}\right)^T$, and a sequence of generalized invariant infinite zero-direction vectors $z_{i,j}^\infty=\left(\beta_{i,j}^{\infty T},-u_{i,j}^{\infty T}\right)^T$, $j=1,\dots,q_1+\hat{q}_i-1$.

**Proof.** Let $\tilde{u}_i(w)$ be the ith column of $\tilde{U}_R(w)$ and $\tilde{u}_i^{(q)}(w)$, $\tilde{P}^{(q)}(w)$ be the qth derivatives of $\tilde{u}_i(w)$, $\tilde{P}(w)$ with respect to w. Then its is easily seen that the vectors

$$z_{i,q}:=\begin{pmatrix}\beta_{i,q}^\infty\\ -u_{i,q}^\infty\end{pmatrix}=\frac{1}{q!}\tilde{u}_i^{(q)}(0)=\tilde{u}_{i,q} \quad (3.5)$$

with $q=0,1,\dots,q_1+\hat{q}_i$, i=k+1,k+2,..,r form a sequence of generalized invariant infinite zero-direction vectors. ¤

**Theorem 4.** Let P(s) be the Rosenbrock system matrix P(s) of $\Sigma$ defined in (1.4). Then to an input of the form

$$[u_q^\infty(t)]_u:=\sum_{j=0}^{q}\left[u_j^\infty\delta^{(q-j)}(t)\right]_u \quad (3.6)$$

with $u_q^{\infty(i)}(0-)=-u_{j+i+1}^\infty$ for $i=0,1,\dots,q_1-1$

there corresponds a pseudostate of $\Sigma$ of the form

$$[\beta_q^\infty(t)]_\beta:=\sum_{j=0}^{q}\left[\beta_j^\infty\delta^{(q-j)}(t)\right]_\beta \quad (3.7)$$

with $\beta_q^{\infty(i)}(0-)=-\beta_{j+i+1}^\infty$ for $i=0,1,\dots,q_1-1$

and a zero output of $\Sigma$ i.e.

$$y(t)=0 \quad (3.8)$$

where the vectors $\left(\beta_j^{\infty T},-u_j^{\infty T}\right)^T$ are generalized invariant infinite zero-direction vectors for the infinite zero of order q

**Proof.** See [10]. ¤

**Corollary 2.** From Theorems 3 and 4 we conclude that there exist an input vector space

$$U_0^\infty=\{[u_{i,q}^\infty(t)]_u:=\sum_{j=0}^{q}\left[u_{i,j}^\infty\delta^{(q-j)}(t)\right]_u; \quad (3.9)$$
$$i=k+1,k+2,\dots,r \text{ and } q=0,1,\dots,\hat{q}_i-1\}$$

which gives rise through the relation (1.1) to the pseudostate vector space

$$B_0^\infty=\{[\beta_{i,q}^\infty(t)]_\beta:=\sum_{j=0}^{q}\left[\beta_{i,j}^\infty\delta^{(q-j)}(t)\right]_\beta; \quad (3.10)$$
$$i=k+1,k+2,\dots,r \text{ and } q=0,1,\dots,\hat{q}_i-1\}$$

and consequently to the output vector space

$$Y_0=\{0\} \quad (3.11)$$

**Proof.** See [10]. ¤

## 4. Geometric Interpretation of the Invariant Indices.

The Rosenbrock system matrix $P(s) \in \Re[s]^{(n+p)\times(n+m)}$ of the system (1.1) is assumed to have rank $r \le \min(n+p, n+m)$ and therefore the dimension of the right null space of P(s) is equal to n+m-r. Consider a minimal polynomial basis of the right (left) null space of P(s), denoted

$$\left[\bar{u}_{r+1}(s),\ \bar{u}_{r+2}(s),\ \ldots,\ \bar{u}_{n+m}(s)\right] \qquad (4.1)$$

$$\left(\left[v_{r+1}(s),\ v_{r+2}(s),\ \cdots,\ v_{n+p}(s)\right]\right)$$

The greatest degrees of the columns $\bar{u}_i(s),\ i = r+1,\ldots,n+m$, denoted $\{\varepsilon_{r+1}, \varepsilon_{r+2},\ldots,\varepsilon_{n+m}\}$ are called the *right minimal indices* of P(s), while the greatest degrees of the rows $v_i(s),\ i = r+1,\ldots,n+p$, denoted $\{\eta_{r+1}, \eta_{r+2},\ldots,\eta_{n+p}\}$ are called the *left minimal indices* of P(s).

**Definition 3.** Let P(s) be the Rosenbrock system matrix of $\Sigma$ defined in (1.3)-(1.4). Then the *invariant right (left) indices* of $\Sigma$, are the right (left) minimal indices of P(s) if associated with each invariant right (left) index there is a sequence of vectors $z_j^\varepsilon = \left(\beta_j^{\varepsilon T}, -u_j^{\varepsilon T}\right)^T$, $j = 0,1,\ldots,q$, $\left(z_j^\eta,\ j = 0,1,\ldots,q\right)$ such that

$$\begin{aligned}
&P_{q_1} z_0^\varepsilon = 0\\
&P_{q_1-1} z_0^\varepsilon + P_{q_1} z_1^\varepsilon = 0\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
&(\text{if } q < q_1)\\
&P_0 z_0^\varepsilon + P_1 z_1^\varepsilon + \cdots + P_{q-1} z_{q-1}^\varepsilon + P_q z_q^\varepsilon = 0\\
&(\text{if } q \ge q_1)\\
&P_0 z_0^\varepsilon + P_1 z_1^\varepsilon + \cdots + P_{q_1-1} z_{q_1-1}^\varepsilon + P_{q_1} z_{q_1}^\varepsilon = 0\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
&P_0 z_q^\varepsilon = 0
\end{aligned} \qquad (4.2a)$$

$$\left(\begin{aligned}
&z_0^\eta P_{q_1} = 0\\
&z_0^\eta P_{q_1-1} + z_1^\eta P_{q_1} = 0\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
&(\text{if } q < q_1)\\
&z_0^\eta P_0 + z_1^\eta P_1 + \cdots + z_{q-1}^\eta P_{q-1} + z_q^\eta P_q = 0\\
&(\text{if } q \ge q_1)\\
&z_0^\eta P_0 + z_1^\eta P_1 + \cdots + z_{q_1-1}^\eta P_{q_1-1} + z_{q_1}^\eta P_{q_1} = 0\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
&z_q^\eta P_0 = 0
\end{aligned}\right) \qquad (4.2b)$$

Then $z_j^\varepsilon$, $j = 0,1,\ldots,q$, $\left(z_j^\eta,\ j = 0,1,\ldots,q\right)$ are called *a sequence of invariant right (left) index direction vectors for the right (left) index* q. ¤

**Theorem 5.** Consider the Rosenbrock system matrix (1.3) of $\sum$ defined in (1.3) and the minimal basis (4.1) for its right (left) null space. Then for every right (left) index of order $\varepsilon_i,\ i = r+1,\ldots,n+m$, $(\eta_i,\ i = r+1, \ldots, p+m)$ $\exists$ a sequence of invariant right (left) index-direction vectors $z_{i,j}^\varepsilon = \left(\beta_{i,j}^{\varepsilon\,T}, -u_{i,j}^{\varepsilon\,T}\right)^T$, $j = 0,1,\ldots,\varepsilon_i$, $\left(z_{i,j}^\eta,\ j = 0,1,\ldots,\eta_i\right)$.

**Proof.** It is easily seen that the vectors

$$z_{i,q}^\varepsilon := \begin{pmatrix} \beta_{i,q}^\varepsilon \\ -u_{i,q}^\varepsilon \end{pmatrix} = \bar{u}_{i,\varepsilon_i - q} \quad \left( z_{i,q}^\eta := \begin{pmatrix} \beta_{i,q}^\eta \\ -u_{i,q}^\eta \end{pmatrix} = \bar{v}_{i,\eta_i - q} \right) \qquad (4.3)$$

where

$$\bar{u}_i(s) = \bar{u}_{i,0} + \bar{u}_{i,1} s + \cdots + \bar{u}_{i,\varepsilon_i} s^{\varepsilon_i}$$

$$\left(\bar{v}_i(s) = \bar{v}_{i,0} + \bar{v}_{i,1} s + \cdots + \bar{v}_{i,\eta_i} s^{\eta_i}\right)$$

$q = 0,1,\ldots,\varepsilon_i\,(\eta_i)$ and $i = r+1, r+2,\ldots,n+m$ $(i = r+1, r+2,\ldots,p+m)$ satisfies relation (4.2) and thus form a sequence of invariant right (left) index-direction vectors). •

**Theorem 6.** Consider the system $\sum$ defined in (1.1) with Rosenbrock system matrix $P(s) \in \Re[s]^{(n+p)\times(n+m)}$ where $\operatorname{rank}_{\Re(s)} P(s) = r \le \min(n+p, n+m)$. Then to an input of the form

$$[u_k^\varepsilon(t)]_u := \sum_{j=0}^{k} \left[u_j^\varepsilon \delta^{(k-j)}(t)\right]_u \ \text{with}\ u_q^{\varepsilon(i)}(0-) = -u_{j+i+1}^\varepsilon$$

$$i = 0,1,\ldots,q_1 - 1 \text{ and } k = 0,1,\ldots,q-1 \qquad (4.4)$$

there corresponds a pseudostate of $\Sigma$ of the form

$$[\beta_k^\varepsilon(t)]_\beta := \sum_{j=0}^{k} \left[\beta_j^\varepsilon \delta^{(k-j)}(t)\right]_\beta \quad \text{with}\ \beta_q^{\varepsilon(i)}(0-) = -\beta_{j+i+1}^\varepsilon$$

$$\text{for } i = 0,1,\ldots,q_1 - 1 \text{ and } k = 0,1,\ldots,q-1 \qquad (4.5)$$

and a zero output i.e.

$$y(t) = 0 \qquad (4.6)$$

where the vectors $\left(\beta_j^{\varepsilon T}, -u_j^{\varepsilon T}\right)^T$ $j = 0,1,\ldots,q-1$ are genralised invariant right index-direction vectors for a right (left) index of order q..

**Proof.** See [10]. ¤

**Corollary 3.** From the above theorem we conclude that there exist an input vector space

$$U_0^\varepsilon = \{[u_{i,q}^\varepsilon(t)]_u := \sum_{j=0}^{q} \left[u_{i,j}^\varepsilon \delta^{(q-j)}(t)\right]_u; \qquad (4.7)$$

$$i = r+1, r+2,\ldots,n+m \text{ and } q = 0,1,\ldots,\varepsilon_i - 1\}$$

which gives rise through the relation (1.1) to the pseudostate vector space

$$B_0^\varepsilon = \{[\beta_{i,q}^\varepsilon(t)]_\beta := \sum_{j=0}^{q} \left[\beta_{i,j}^\varepsilon \delta^{(q-j)}(t)\right]_\beta; \qquad (4.8)$$

$$i = r+1, r+2,\ldots,n+m \text{ and } q = 0,1,\ldots,\varepsilon_i - 1\}$$

and consequently to the output vector space

$$Y_0 = \{0\} \qquad (4.9)$$

**Proof.** See [10]. ¤

Let now

$$v_i(s) = v_{i,0} + v_{i,1}s + \cdots + v_{i,\eta_i} s^{\eta_i}$$
$$\left( \overset{(4.3)}{=} z^{\eta}_{i,\eta_i} + z^{\eta}_{i,\eta_i-1} s + \cdots + z^{\eta}_{i,0} s^{\eta_i} \right) \tag{4.10}$$

with $i = r+1,\ldots,n+p$, be the vectors of a left minimal polynomial basis of the Rosenbrock system matrix P(s). It has been suggested [6] that the left minimal basis of the Rosenbrock system matrix P(s) in state space systems plays the same role as the right minimal basis for the "dual" system of (1.1). However this is not exactly the case and the next Theorem reveals the precise connection between the invariant left indices and solution to the zero-output problem.

**Theorem 7.** The zero-output problem has a solution iff the following $\eta := \eta_{r+1} + \cdots + \eta_{n+p}$ constraints between the initial conditions $\left(\beta^{(q)}(0-)^T, -u^{(q)}(0-)^T\right)^T$, $q = 0,1,\ldots,q_1-1$ are satisfied

$$\begin{pmatrix} v_{i,\eta_i} & 0 & \cdots & 0 \\ v_{i,\eta_i-1} & v_{i,\eta_i} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ v_{i,\eta_i-q_1+1} & v_{i,\eta_i-q_1+2} & \cdots & v_{i,\eta_i} \\ \vdots & \vdots & \cdots & \vdots \\ v_{i,1} & v_{i,2} & \cdots & v_{i,q_1} \end{pmatrix} \times$$
$$\times \begin{pmatrix} P_0 & P_1 & \cdots & P_{q_1-1} \\ 0 & P_0 & \cdots & P_{q_1-2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & P_0 \end{pmatrix} \begin{pmatrix} \begin{pmatrix} \beta(0-) \\ -u(0-) \end{pmatrix} \\ \begin{pmatrix} \beta^{(1)}(0-) \\ -u^{(1)}(0-) \end{pmatrix} \\ \vdots \\ \begin{pmatrix} \beta^{(q_1-1)}(0-) \\ -u^{(q_1-1)}(0-) \end{pmatrix} \end{pmatrix} = 0_{\eta_i,1} \tag{4.11}$$

for i=r+1,r+2,...,n+p.
**Proof.** See [10]. ¤

## 5. The Solution Subspace of the Output Zeroing Problem.

Consider the AR-representation (1.3). According to [2], [3] the solution vector space $\hat{B}$ of (1.3) is comprised of equivalence classes and has dimension equal to the number of finite and infinite zeros of P(s) (order accounted for) and the sum of the right minimal indices (order accounted for). More specifically it is known that

$$\hat{B} = \left\langle \left[ \left\{ \sum_{k=0}^{\sigma_{i,j}-1-q} \begin{pmatrix} \beta^i_{j,k} \\ -u^i_{j,k} \end{pmatrix} \frac{t^{\sigma_{i,j}-q-k-1}}{(\sigma_{i,j}-q-k-1)!} \right\} e^{\lambda_i t} \right] \;\middle|\; i \in \ell,\ j = \mu,\mu+1,..,r \text{ and } q = 0,1,..,\sigma_{i,j}-1 \right\rangle \oplus$$
$$\oplus \left\langle \left[ \sum_{j=0}^{q} \begin{pmatrix} \beta^{\infty}_{i,j} \\ -u^{\infty}_{i,j} \end{pmatrix} \delta^{(q-j)}(t) \right] \;\middle|\; i = k+1,k+2,..,r \text{ and } q = 0,1,..,\hat{q}_i - 1 \right\rangle \oplus$$
$$\oplus \left\langle \left[ \sum_{j=0}^{q} \begin{pmatrix} \beta^{\varepsilon}_{i,j} \\ -u^{\varepsilon}_{i,j} \end{pmatrix} \delta^{(q-j)}(t) \right] \;\middle|\; i = r+1,r+2,..,n+m \text{ and } q = 0,1,..,\varepsilon_i - 1 \right\rangle \tag{5.1}$$

where [a] denotes the equivalence class of the solution a of (1.3) as has been defined in (2.1c) and the generalized finite and infinite zero-direction vectors and the generalized right index-direction vectors are the ones presented in Theorems 2, 4 and 6. If we now denote by $\hat{B}_\beta$, $\hat{B}_u$ the projection of the space $\hat{B}$ to the space of $[\beta(t)]_\beta$, $[-u(t)]_u$ respectively i.e.

$$\hat{B}_\beta := \left\{ [\beta(t)]_\beta \;\middle|\; \exists\, u(t) : \left[ \begin{pmatrix} \beta(t) \\ -u(t) \end{pmatrix} \right] \in \hat{B} \right\}$$
$$\hat{B}_u := \left\{ [u(t)]_u \;\middle|\; \exists\, \beta(t) : \left[ \begin{pmatrix} \beta(t) \\ -u(t) \end{pmatrix} \right] \in \hat{B} \right\} \tag{5.2}$$

Then we have the following

**Theorem 8.** Every input vector $[-u(t)]_u \in \hat{B}_u$ gives rise through the relation (1.1) to a pseudostate vector $[\beta(t)]_\beta \in \hat{B}_\beta$ and subsequently to the zero-output vector space. We observe also that

$$\hat{B}_u \equiv U_0^C + U_0^\infty + U_0^\varepsilon \tag{5.3}$$

and

$$\hat{B}_\beta \equiv B_0^C + B_0^\infty + B_0^\varepsilon \tag{5.4}$$

**Proof.** See [10]. ¤

**Definition 4.** The pseudostate vector solutions of the system (1.3) define a subspace of the pseudostate space X of the system (1.1) :

$$B_\beta := \left\{ \beta(t) \;\middle|\; [\beta(t)]_\beta \in \hat{B}_\beta \right\} \tag{5.5}$$

which is called the *general output zeroing subspace*, or the *solution subspace of the output zeroing problem*. ¤

Thus the output-zero problem may be described through the following diagram :

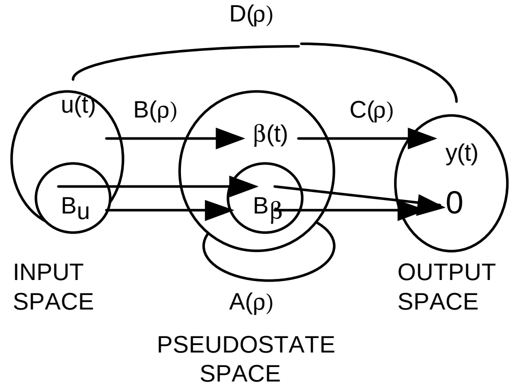

**Diagram 1.** The zero-output problem.

## 6. Illustrative Example.

Consider the following system $\Sigma$ :

$$\begin{pmatrix} 1 & \rho^3 \\ 0 & \rho+1 \end{pmatrix}\begin{pmatrix} \beta_1(t) \\ \beta_2(t) \end{pmatrix}=\begin{pmatrix} \rho+1 \\ 0 \end{pmatrix}u(t) \qquad \text{(E.1a)}$$

$$y(t)=\begin{pmatrix} -\rho & -\rho^4 \end{pmatrix}\begin{pmatrix} \beta_1(t) \\ \beta_2(t) \end{pmatrix}+\left(\rho^2+\rho\right)u(t) \qquad \text{(E.1b)}$$

with Rosenbrock system matrix

$$P(s)=\begin{pmatrix} 1 & s^3 & s+1 \\ 0 & s+1 & 0 \\ s & s^4 & s^2+s \end{pmatrix}$$

Define according to (2.1a)

$$Z=\left\{\begin{pmatrix} \int_{0-}^{t}(-\delta(\tau)-\delta^{(1)}(\tau))z(t-\tau)d\tau \\ 0 \\ \int_{0-}^{t}\delta(\tau)z(t-\tau)d\tau \end{pmatrix}\right\}$$

$$Z_u=-\pi_u Z=-\int_{0-}^{t}\delta(\tau)z(t-\tau)d\tau$$

$$Z_\beta=\pi_\beta Z=\begin{pmatrix} \int_{0-}^{t}(-\delta(\tau)-\delta^{(1)}(\tau))z(t-\tau)d\tau \\ 0 \end{pmatrix}$$

$$[a]=a\oplus Z,\quad [a]_u=a\oplus Z_u,\quad [a]_\beta=a\oplus Z_\beta$$

where z(t) is an arbitrary function.

**a) The finite invariant zeros.**

There exist one finite invariant zero $\lambda_1=-1$ of partial multiplicitiy one i.e. $\sigma_{1,2}=1$ which according to Theorems 1, 2 and Corollary 1 implies that

$$U_0^C:=\langle[0]_u\rangle\overset{(E.1a)}{\rightarrow}B_0^C:=\left\langle\left[\begin{pmatrix}1\\1\end{pmatrix}e^{-t}\right]_\beta\right\rangle\overset{(E.1b)}{\rightarrow}Y_0=\{0\}$$

**b) The infinite elementary divisors.**

P(s) has one invariant infinite zero of order 1 and thus according to Theorems 3,4 and Corollary 2 we have that

$$U_0^\infty:=\langle[-\delta(t)]_u\rangle\overset{(E.1a)}{\rightarrow}B_0^\infty:=\left\langle\left[\begin{pmatrix}-1\\0\end{pmatrix}\delta(t)\right]_\beta\right\rangle\overset{(E.1b)}{\rightarrow}Y_0=\{0\}$$

**c) The right and left invariant indices.**

$\Sigma$ has one right invariant index of order 1 i.e. the degree of $u_3(s)$ and thus from Theorems 5,6 and Corollary 3 we have that

$$U_0^\varepsilon:=\langle[0]_u\rangle\overset{(E.1a)}{\rightarrow}B_0^\varepsilon:=\left\langle\left[\begin{pmatrix}-1\\0\end{pmatrix}\delta(t)\right]_\beta\right\rangle\overset{(E.1b)}{\rightarrow}Y_0=\{0\}$$

$\Sigma$ has one left invariant index of order 1 which gives rise according to Theorem 7 to the following necessary and sufficient condition for the existence of solution to the zero-output problem

$$-\beta_1(0-)-\beta_2^{(3)}(0-)-u(0-)-u^{(1)}(0-)=0$$

**d) The Solution Subspace of the Output Zeroing Problem.**

Denote

$$\hat{B}_u=U_0^C+U_0^\infty+U_0^\varepsilon=\langle[\delta(t)]_u\rangle$$

$$\hat{B}_\beta=B_0^C+B_0^\infty+B_0^\varepsilon=\hat{B}_\beta=\left\langle\left[\begin{pmatrix}1\\1\end{pmatrix}e^{-t}\right]_\beta,\ \left[\begin{pmatrix}-1\\0\end{pmatrix}\delta(t)\right]_\beta\right\rangle$$

Then according to Theorem 8 we have that

$$\hat{B}_u\overset{(E.1a)}{\rightarrow}\hat{B}_\beta\overset{(E.1b)}{\rightarrow}\{0\}$$

The space $B_\beta$ :

$$B_\beta=\left\{\beta(t)\ \middle|\ \begin{matrix}\beta(t)=\lambda_1\begin{pmatrix}1\\1\end{pmatrix}e^{-t}+\lambda_2\begin{pmatrix}-1\\0\end{pmatrix}\delta(t)+ \\ +\begin{pmatrix}\int_{0-}^{t}(-\delta(\tau)-\delta^{(1)}(\tau))z(t-\tau)d\tau \\ 0\end{pmatrix}\quad \lambda_1,\lambda_2\in\Re\end{matrix}\right\}$$

is called the *general output zeroing subspace*, or the *solution subspace of the output zeroing problem*. •

## 7. Conclusions.

A geometric interpretation of the finite and infinite invariant zeros and the right and left minimal indices of a system has been given in terms of the solution of the zero-output problem. More specifically it has been shown that while the first three prementioned characteristics of the system give rise to the solution space of the zero-output problem, the fourth gives rise to conditions for the existence of solution to the above problem.

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