An Algorithm for the Computation of the Generalized Inverse of a Rational Matrix

J.Jones[†], N.Karampetakis[‡] and A.C.Pugh[†]

[†]*Department of Mathematical Sciences*
Loughborough University Of Technology
Loughborough, LE11 3TU
U.K.

[‡]*Department of Mathematics*
Aristotle University of Thessaloniki
54006 GREECE

Abstract

Decell(1965) has given an algorithm for computing the generalized inverse of a non-regular constant matrix. This has been extended by Karampetakis(1995) for the polynomial case. Here we show that the same algorithm holds for a non-regular rational matrix defined over the field of rational functions.

1 Introduction

The inverse of a constant matrix $A \in \mathbb{R}^{n\times m}$ exists iff it is non-singular, or in other words, iff it is square ($n = m$) and it possesses a non-zero determinant. Equally, in the case of a polynomial matrix, its inverse exists iff it is square and its determinant is not the zero polynomial.

Consider now a constant non-regular matrix $A \in \mathbb{R}^{n\times m}$, i.e. A is either non-square and/ or has zero determinant. For such a case Penrose(1955) has defined the *generalized inverse* of A, denoted by $A^{\dagger} \in \mathbb{R}^{m\times n}$. This is a unique matrix which satisfies certain governing equations. In the special case where A is non-singular the generalized inverse of A is simply its inverse. i.e. $A^{\dagger} = A^{-1}$. These governing conditions similarly hold when we define the generalized inverse of a matrix over the ring of polynomials and over the ring of rational functions respectively.

For the constant non-regular matrix $A \in \mathbb{R}^{n\times m}$ Decell(1965) has given a constructive proof on how to form such a generalized inverse. He later implemented this to form a numerical algorithm for its construction. This algorithm has been recently extended by Karampetakis(1995) for the more general non-regular polynomial matrix case

$$A(s) \stackrel{\text{def}}{=} A_q s^q + A_{q-1} s^{q-1} + \cdots + A_1 s + A_0 \in \mathbb{R}[s]^{n\times m} \qquad (1.1)$$

Here we show that this algorithm indeed holds for the rational matrix case where $A(s) \in \mathbb{R}(s)^{n\times m}$ is defined over the field of rational functions. We subsequently implement this algorithm in some numerical examples via the symbolic computational language MAPLE[1].

2 Preliminaries

Here we give the definition of the generalized inverse for the rational matrix $A(s)$. This definition was originally given by Penrose[5] for the constant matrix case.

Definition 2.1 *For every rational matrix $A(s) \in \mathbb{R}(s)^{n\times m}$ $\exists$ a unique matrix $A(s)^{\dagger} \in \mathbb{R}(s)^{m\times n}$, termed the generalized inverse, which satisfies the following conditions*

i) $A(s)A(s)^{\dagger}A(s) = A(s)$

ii) $A(s)^{\dagger}A(s)A(s)^{\dagger} = A(s)^{\dagger}$

iii) $\left(A(s)A(s)^{\dagger}\right)^T = A(s)A(s)^{\dagger}$

iv) $\left(A(s)^{\dagger}A(s)\right)^T = A(s)^{\dagger}A(s)$

where $A(s)^T$ denotes the transpose of $A(s)$. In the special case where the matrix $A(s)$ is square and non-singular the generalized inverse of $A(s)$ is simply its inverse. i.e. $A(s)^{\dagger} = A(s)^{-1}$. □

3 Generalized Inverse of a Rational Matrix.

In this section we show that the algorithm presented by Decell[2] for the determination of the generalized inverse of a constant matrix case holds for the rational matrix case. To show this an intermediate result for the solution of the equation $P(s)X(s)Q(s) = C(s)$ is needed. This result was originally given by Penrose[5] for when P, X, Q and C are constant matrices.

Theorem 3.1 [3] *(Solution of the Equation $P(s)X(s)Q(s) = C(s)$)*

The equation $P(s)X(s)Q(s) = C(s)$ where $P(s) \in \mathbb{R}(s)^{n\times m}$, $Q(s) \in \mathbb{R}(s)^{k\times l}$ and $C(s) \in \mathbb{R}(s)^{n\times l}$ has a solution $X(s) \in \mathbb{R}(s)^{m\times k}$ iff

$$P(s)P(s)^{\dagger}C(s)Q(s)^{\dagger}Q(s) = C(s)$$

in which case all the solutions are given by

$$X(s) = P(s)^{\dagger}C(s)Q(s)^{\dagger} + Y(s) - P(s)^{\dagger}P(s)Y(s)Q(s)Q(s)^{\dagger} \tag{3.1}$$

where $P(s)^{\dagger}, Q(s)^{\dagger}$ are the generalized inverses of $P(s)$ and $Q(s)$ respectively and $Y(s)$ is arbitrary to within having the dimension of $X(s)$. □

Theorem 3.2 *(Generalized Inverse of a Rational Matrix)*

Let $A(s) \in \mathbb{R}(s)^{n\times m}$ and let $f(\lambda, s)$ denote the characteristic polynomial of $A(s)A(s)^T$. i.e.

$$\begin{aligned} f(\lambda, s) &\stackrel{\text{def}}{=} \det\left[\lambda I_n - A(s)A(s)^T\right] \\ &= a_0(s)\lambda^n + a_1(s)\lambda^{n-1} + \dots + a_{n-1}(s)\lambda + a_n(s), \quad a_0(s) = 1 \end{aligned} \tag{3.2}$$

Let $a_n(s) = \ldots = a_{k+1}(s) = 0$ while $a_k(s) \neq 0$. Then the generalized inverse $A(s)^{\dagger}$ of $A(s) \in \mathbb{R}(s)^{m\times n}$ is given by

$$A(s)^{\dagger} = -a_k(s)^{-1}A(s)^T\left[\left(A(s)A(s)^T\right)^{k-1} + a_1(s)\left(A(s)A(s)^T\right)^{k-2} + \dots + a_{k-1}(s)I\right] \tag{3.3}$$

If $k = 0$, i.e. $a_n(s) = \ldots = a_1(s) = 0$, then $A(s)^{\dagger} = 0_{m,n}$, the $(m \times n)$ zero matrix. □

Proof According to the Cayley-Hamilton Theorem, $A(s)A(s)^T$ satisfies its own characteristic polynomial, that is

$$\left(A(s)A(s)^T\right)^n + a_1(s)\left(A(s)A(s)^T\right)^{n-1} + \dots + a_{n-1}(s)\left(A(s)A(s)^T\right) + a_{k-1}(s)I_n = 0_{n,n} \tag{3.4}$$

Let $k \neq 0$ be the largest integer such that $a_k(s) \neq 0$ and define $B(s) = A(s)A(s)^T$. Hence (3.4) can be rewritten in the form

$$B^n(s) + a_1(s)B^{n-1}(s) + \cdots + a_k(s)B^{n-k}(s) = O_{n,n}$$
$$\Rightarrow \quad B^{n-k}(s)\left(B^k(s) + a_1(s)B^{k-1}(s) + \cdots + a_{k-1}(s)B(s) + a_k(s)I_n\right) = 0_{n,n}$$

or equivalently

$$B^{n-k}(s)X(s) = Z \tag{3.5}$$

where $X(s) = B^k(s) + a_1(s)B^{k-1}(s) + \cdots + a_{k-1}(s)B(s) + a_k(s)I_n$ and $B^0(s) = I_n$. From Theorem 3.1, (3.5) clearly satisfies the solvablity condition given and hence all solutions are given by

$$\begin{aligned} X(s) &= B^{n-k}(s)O_{n,n} + Y(s) - \left(B^{n-k}(s)\right)^{\dagger} B^{n-k}(s)Y(s) \\ &= Y(s) - \left(B^{n-k}(s)\right)^{\dagger} B^{n-k}(s)Y(s) \end{aligned} \tag{3.6}$$

Let $Y_1(s)$ determine a particular solution. From (3.5) and (3.6) we can write

$$B^k(s) + a_1(s)B^{k-1}(s) + \cdots + a_{k-1}(s)B(s) + a_k(s)I_n = Y_1(s) - \left(B^{n-k}(s)\right)^{\dagger} B^{n-k}(s)Y_1(s) \tag{3.7}$$

We note that $B(s) = A(s)A(s)^T$ is normal, i.e. $B(s)^T B(s) = B(s)B(s)^T$, so that $B(s)^{\dagger}B(s) = B(s)B(s)^{\dagger}$ and also for each integer p, $(B(s)^p)^{\dagger} = \left(B(s)^{\dagger}\right)^p$. By also noting that $B(s)^{\dagger}B(s)$ is idempotent, i.e. $B(s)^{\dagger}B(s) = \left(B(s)^{\dagger}B(s)\right)^2$ we have

$$\left(B^{n-k}(s)\right)^{\dagger} B^{n-k}(s) = \left(B(s)^{\dagger}\right)^{n-k} B^{n-k}(s) = \left(B(s)^{\dagger}B(s)\right)^{n-k} = B(s)^{\dagger}B(s) \tag{3.8}$$

From [6] $B(s)^{\dagger}B(s) = \left(A(s)A(s)^T\right)^{\dagger}\left(A(s)A(s)^T\right) = A(s)A(s)^{\dagger}$ and hence substituting for (3.8) in (3.7) we obtain

$$B^k(s) + a_1(s)B^{k-1}(s) + \cdots + a_{k-1}(s)B(s) + a_k(s)I_n = Y_1(s) - A(s)A(s)^{\dagger}Y_1(s) \tag{3.9}$$

Pre-multiplying (3.9) by $A(s)^{\dagger}$ and noting that $A(s)^{\dagger}A(s)A(s)^{\dagger} = A(s)^{\dagger}$, (see Definition 2.1), we obtain

$$A(s)^{\dagger}B^k(s) + a_1(s)A(s)^{\dagger}B^{k-1}(s) + \cdots + a_{k-1}(s)A(s)^{\dagger}B(s) + a_k(s)I_n = 0_{n,n}$$

From our definition $a_k(s) \neq 0$ and from $A(s)^{\dagger}A(s)A(s)^T = A(s)^{\dagger}B(s) = A(s)^T$[6]

$$A(s)^{\dagger} = -a_k(s)^{-1}A(s)^T\left[\left(A(s)A(s)^T\right)^{k-1} + a_1(s)\left(A(s)A(s)^T\right)^{k-2} + \cdots + a_{k-1}(s)I\right]$$

as required.

If $k = 0$ then from (3.4) we have that $\left(A(s)A(s)^T\right)^n = 0_{n,n} \Rightarrow A(s) = 0_{n,m} \Rightarrow A(s)^{\dagger} = 0_{m,n}$ as required. □

Under the hypotheses of Theorem 3.2 a recursive algorithm to form the generalized inverse of a rational matrix can be produced. This is shown below and is an extension of that given by Decell[2].

Algorithm 3.3(Computation of the Generalized Inverse of a Rational Matrix)

<u>Step 1:</u>

Let $A(s) \in \mathbb{R}(s)^{n \times m}$. Consider the following sequences $\{a_0(s), a_1(s), \ldots, a_n(s)\}$ and

$\{B_0(s), B_1(s), \ldots, B_n(s)\}$ which are constructed in the following recursive way.

$$\begin{array}{lll} A_0(s) = 0 & a_0(s) = 1 & B_0(s) = I_n \\ A_1(s) = \left(A(s)A(s)^T\right)B_0(s) & a_1(s) = -\dfrac{trace\left(A_1(s)\right)}{1} & B_1(s) = A_1(s) + a_1(s)I_n \\ \vdots & \vdots & \vdots \\ A_n(s) = \left(A(s)A(s)^T\right)B_{n-1}(s) & a_n(s) = -\dfrac{trace\left(A_n(s)\right)}{n} & B_n(s) = A_n(s) + a_n(s)I_n \end{array} \tag{3.10}$$

<u>Step 2:</u>

Let $a_n(s) = \ldots = a_{k+1}(s) = 0$ while $a_k(s) \neq 0$. Then the generalized inverse $A(s)^{\dagger}$ of $A(s)$ is given by

$$A(s)^{\dagger} = -a_k(s)^{-1}A(s)^T B_{k-1}(s) \tag{3.11}$$

If $k = 0$ i.e. $a_n(s) = \ldots = a_1(s) = 0$ then $A(s)^{\dagger} \stackrel{def}{=} 0_{m,n}$, the $(m \times n)$ zero matrix. □

4 Examples

In [4], Algorithm 3.3 has been implemented via the symbolic computational language MAPLE. (see [1]). In this section we will implement this procedure to form the generalized inverses of two rational matrices, $A(s) \in \mathbb{R}(s)^{2\times 2}$ and $B(s) \in \mathbb{R}(s)^{2\times 3}$, which are both in the single indeterminate s. The contents of such a MAPLE worksheet for each case is shown below. The machine used is a SUN SPARC station10 with a 75MHz SuperSPARC II processor. The last line of the output indicates the CPU time used in the computation. This is divided into three parts

i) `bytes used` - *(integer)* Number of bytes of memory that have been requested up to that point in the execution of the session

ii) `alloc` - *(integer)* Number of bytes of memory *actually allocated* for data space during the session

iii) `time` - *(floating point number)* Total CPU time in seconds for the session

4.1 Implementation via MAPLE

i) Read in the linear algebra package contained within MAPLE via `>with(linalg):`

ii) Read in the generalized inverse procedure via `>read 'ginv.m':`

iii) Implement the procedure via `>GINVERSE(G);` where `G` represents the rational matrix whose generalized inverse is to be found.

Example 1

Consider the matrix $A(s)$

$$A(s) = \begin{pmatrix} \frac{1}{s+1} & 2 \\ 3s & \frac{3s}{(s+1)^2} \end{pmatrix} \qquad (4.1)$$

It can be seen that $A(s)$ is a non-singular matrix and therefore its generalized inverse $A(s)^{\dagger}$ is defined to be identical to its normal inverse $A(s)^{-1}$. We show this using MAPLE via the implementation of the two procedures `inverse` and `GINVERSE`. These are used to compute the inverse (if one exists) and the generalized inverse of a given matrix respectively.

```
%maple
    |\^/|     Maple V Release 3 (Loughborough University)
._|\|   |/|_. Copyright (c) 1981-1994 by Waterloo Maple Software and the
 \  MAPLE  /  University of Waterloo. All rights reserved.
 <____ ____>
      |       Type ? for help.
> with(linalg):
Warning: new definition for   norm
Warning: new definition for   trace
> read 'ginv.m':
> A:=matrix(2,2,[1/(s+1),2,3+s,3*s/(s+1)^2]);

                [   1                   ]
                [ -----        2        ]
                [ s + 1                 ]
          A :=  [                       ]
                [  3 s     3 ---------  ]
                [                    2  ]
                [           (s + 1)     ]

> inverse(A);

                [    s + 1           (s + 1)^3 ]
                [  - -----      2/3  --------- ]
                [     %1               s %1    ]
                [                              ]
                [ (s + 1)^3          (s + 1)^2 ]
                [ ---------   - 1/3  --------- ]
                [    %1                s %1    ]

%1 := 1 + 2 s^3 + 6 s^2 + 6 s
```

```
> GINVERSE(A);
```

$$\begin{bmatrix} -\frac{s+1}{\%1} & 2/3\,\frac{(s+1)^3}{s\,\%1} \\ \frac{(s+1)^3}{\%1} & -1/3\,\frac{(s+1)^2}{s\,\%1} \end{bmatrix}$$

$$\%1 := 1 + 2s^3 + 6s^2 + 6s$$

```
>quit;
bytes used=746820, alloc=589716, time=0.87
```

Example 2

Consider the matrix $B(s)$

$$B(s) = \begin{pmatrix} 1 & s+1 & s \\ \frac{1}{s} & \frac{3}{s+1} & \frac{1}{s+2} \end{pmatrix} \qquad (5.2)$$

Obviously $B(s)$ is non-square and therefore its normal inverse is not defined. However its generalized inverse can be found via the implementation of the procedure GINVERSE. This is illustrated below. Subsequently it can be shown that the generalized inverse generated for $B(s)$ satisfies the four properties of Definition 2.1 .

```
%maple
    |\^/|     Maple V Release 3 (Loughborough University)
._|\|   |/|_. Copyright (c) 1981-1994 by Waterloo Maple Software and the
 \  MAPLE  /  University of Waterloo.
 <____ ____>
      |       Type ? for help.
> with(linalg):
Warning: new definition for   norm
Warning: new definition for   trace
> read `ginv.m`:
```



```
> B:=matrix(2,3,[1,s+1,s,1/s,3/(s+1),1/(s+2)]);
```

$$B := \begin{bmatrix} 1 & s+1 & s \\ 1/s & \frac{3}{s+1} & \frac{1}{s+2} \end{bmatrix}$$

```
> inverse(B);
Error, (in inverse) expecting a square matrix
> GINVERSE(B);
```

$$\left[-\frac{s\,(12s^3 + 6s^2 + s + 12 + 4s^4)}{\%1},\ \frac{s\,(s+1)^2\,(s+2)\,(-s + 2 + 2s^2 + 2s^3)}{\%1}\right]$$

$$\left[-\frac{(s+1)\,(s^4 + s^3 - 2s^2 - 4)}{\%1},\ \frac{s\,(s+1)\,(s+2)\,(3s^3 - 2s^2 + 2s^4 + s - 2)}{\%1}\right]$$

$$\left[\frac{s\,(7s^4 + 29s^3 + 30s^2 + s + 2)}{\%1},\ -\frac{s^3\,(s+1)^2\,(s+2)\,(5+2s)}{\%1}\right]$$

$$\%1 := 17s^4 - 2s^3 + 7s^2 - 4s + 4 + 6s^6 + 22s^5$$

```
> quit;
bytes used=772680, alloc=655240, time=0.90
```

5 Conclusions and Further Work

In the above we have presented a natural extension to the work of Decell, and more cently Karampetakis, to give an algorithm for the computation of the generalized verse of a singular, rational matrix. This clearly widens the implications of the neralized inverse in linear systems theory, a selection of which can be found in [3].

For the case when s above is a parameter, say s_i, the matrix $A(s_i) \in \mathbb{C}^{n \times m}$ is a nstant matrix. Clearly, the above algorithm for the computation of the generalized verse can also be used in this case. For some values of s_i, this inverse may not ist and a modification to the algorithm is needed to form corresponding generalized verses for these cases. This is an area of work currently being looked upon and with its subsequent applications.

References

[1] Char, B.W., Geddes, K.G., Gonnet, G.H. and Watt, S.M. (1991). *Maple V nguage Reference Manual*. Springer-Verlag.

[2] Decell, H.P. (1965). *SIAM Review* **7**, 526–528.

[3] Karampetakis, N.P. (1995). Computation of the Generalized Inverse of a Poly-mial Matrix and Applications. To appear in Linear Algebra and Applications.

[4] Jones, J., Karampetakis, N.P., Pugh, A.C. Computation of the Generalized verse of a Polynomial Matrix via MAPLE. Submitted to ISCAS-96, New Orleans, SA.

[5] Penrose, R. (1955). A Generalized Inverse for Matrices. *Proc. Cambridge 'ilos. Soc.* **51**, 406–413.

[6] Price, C. (1964) The matrix pseudoinverse and minimal variance estimates. *'AM Review* **6**, 115–120