FORWARD, BACKWARD AND SYMMETRIC SOLUTIONS OF DISCRETE ARMA - REPRESENTATIONS

by

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1. Introduction

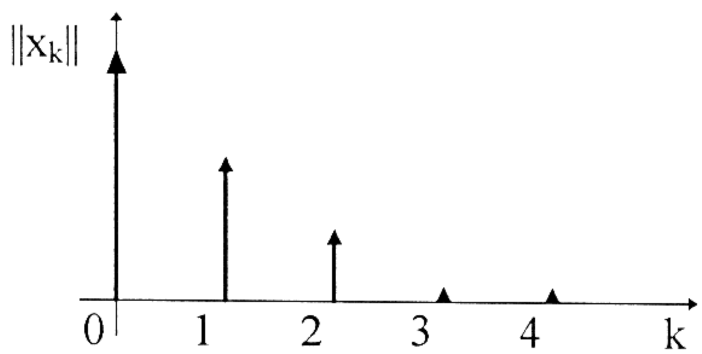
Consider the discrete time equation

$$x_{k+1} = Jx_k + Bu_k$$

where

$$J = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

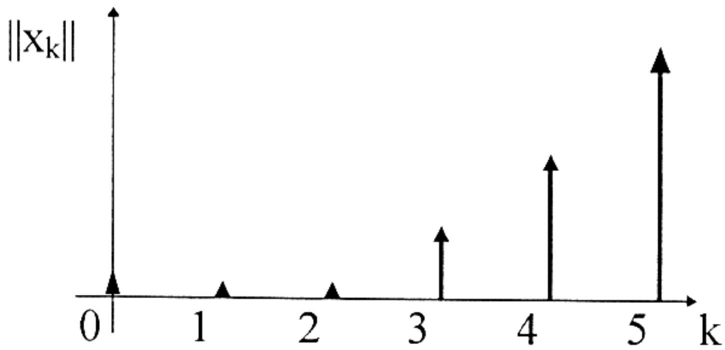
Then for $x_0=[1,1,1]^T$ and $u_k=0$ the response of the system will be



Now consider the 'backward' system

$$Jx_{k+1} = x_k + Bu_k$$

Then for final condition $x_5=[1,1,1]^T$ and $u_k=0$ we have the 'backward response'



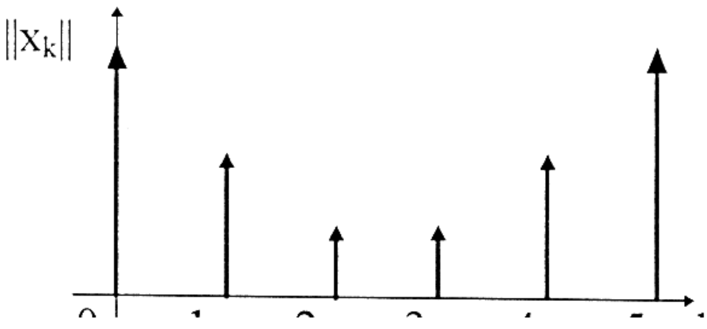
If we compose the above two systems we obtain a singular descriptor system having the form

$$\begin{bmatrix} I & 0 \\ 0 & J \end{bmatrix} x_{k+1} = \begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix} x_k + \begin{bmatrix} B \\ B \end{bmatrix} u_k$$

or in general

$$Ex_{k+1}=Ax_k+Bu_k$$

The response of such a system given $x_0=[1,1,1]^T$, $x_5=[1,1,1]^T$ and $u_k=0$ will be



Such systems are described by

$$(\sigma E - A)x_k = Bu_k$$

where $\sigma x_k = x_{k+1}$ is the forward shift operator.
A more general model examined here has the form

$$A(\sigma)y(k) = B(\sigma)u(k)$$

or

$$A_q y(k+q) + \ldots + A_0 y(k) = B_q u(k+q) + \ldots + B_0 u(k)$$

where

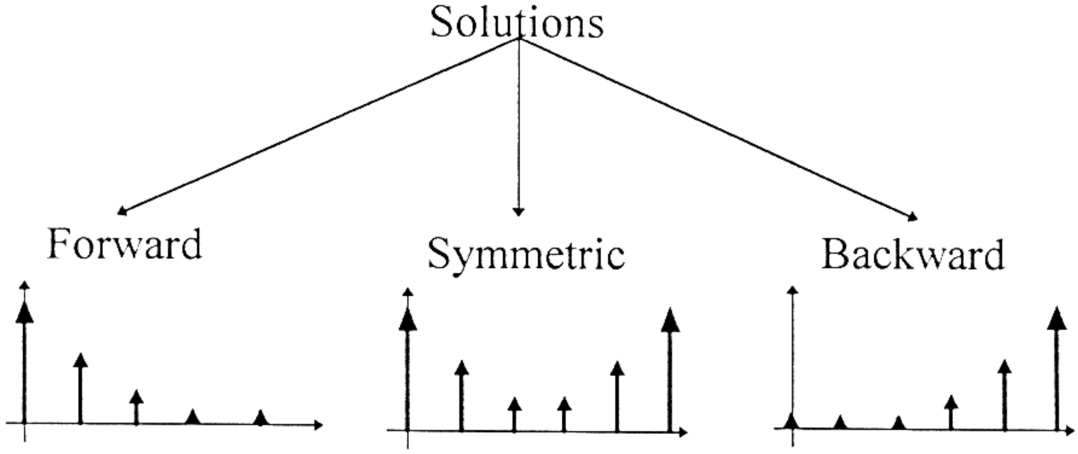
$$A(\sigma) = A_q\sigma^q + A_{q-1}\sigma^{q-1} + \ldots + A_0 \in \Re^{r\times r}[\sigma]$$

$$B(\sigma) = B_q\sigma^q + B_{q-1}\sigma^{q-1} + \ldots + B_0 \in \Re^{r\times m}[\sigma]$$

$y(k) \in \Re^r$ is the output vector

$u(k) \in \Re^m$ is the input vector

There are three kinds of solutions



2. Forward Solution

We solve the equation for k=0,1,2,... given the initial values of $y(k)$

$$y(0), y(1), \ldots, y(q-1)$$

and the input

$$u(k) \text{ for } k=0,1,2,\ldots.$$

Let

$$A^{-1}(\sigma) = H_{\mu}\sigma^{\mu} + H_{\mu-1}\sigma^{\mu-1} + \ldots + H_0 + H_{-1}\frac{1}{\sigma} + H_{-2}\frac{1}{\sigma^2} + \ldots$$

be the Laurent expansion of $A^{-1}(\sigma)$ at $\sigma=\infty$.

The Forward solution is

$$y(k) = \begin{bmatrix} H_{-k-q} & \cdots & H_{-k-1} \end{bmatrix} \begin{bmatrix} A_q & 0 & \cdots & 0 \\ A_{q-1} & A_q & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ A_1 & \cdots & A_{q-1} & A_q \end{bmatrix} \begin{bmatrix} y(0) \\ y(1) \\ \vdots \\ y(q-1) \end{bmatrix} +$$

$$+ \begin{bmatrix} H_{-k} & \cdots & H_{\mu} \end{bmatrix} \begin{bmatrix} B_0 & \cdots & B_q & 0 & \cdots & 0 \\ 0 & B_0 & & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & & B_q & 0 \\ 0 & \cdots & 0 & B_0 & \cdots & B_q \end{bmatrix} \begin{bmatrix} u(0) \\ u(1) \\ \vdots \\ u(k+\mu+q) \end{bmatrix}$$

A recursive Forward solution is given by

$$y(k) = -\begin{bmatrix} H_{-1} & \cdots & H_{-q} \end{bmatrix} \begin{bmatrix} A_0 & 0 & \cdots & 0 \\ A_1 & A_0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ A_{q-1} & \cdots & A_1 & A_0 \end{bmatrix} \begin{bmatrix} y(k-1) \\ y(k-2) \\ \vdots \\ y(k-q) \end{bmatrix} +$$

$$+\begin{bmatrix} H_{-q} & \cdots & H_{\mu} \end{bmatrix} \begin{bmatrix} B_0 & \cdots & B_q & 0 & \cdots & 0 \\ 0 & B_0 & & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & & B_q & 0 \\ 0 & \cdots & 0 & B_0 & \cdots & B_q \end{bmatrix} \begin{bmatrix} u(k-q) \\ u(k-q+1) \\ \vdots \\ u(k+\mu+q) \end{bmatrix}$$

Admissible Initial Condition Space

$$H_{iu} := \{ y(i),\ i = 0,1,\ldots,q-1 :$$

$$\widetilde{A}_1 \begin{bmatrix} H_0 & H_1 & \cdots & H_{q-1} \\ H_{-1} & H_0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & H_1 \\ H_{-q+1} & \cdots & H_{-1} & H_0 \end{bmatrix} \widetilde{A}_1 \begin{bmatrix} y(0) \\ y(1) \\ \vdots \\ y(q-1) \end{bmatrix} = \widetilde{A}_1 \begin{bmatrix} H_0 & \cdots & H_{\mu} & 0 \\ H_{-1} & & & \vdots \\ \vdots & & & 0 \\ H_{-q+1} & & H_{\mu-1} & H_{\mu} \end{bmatrix} \times$$

$$\times \begin{bmatrix} B_0 & \cdots & B_q & 0 & \cdots & 0 \\ 0 & B_0 & & B_q & \ddots & \vdots \\ \vdots & \ddots & \ddots & & \ddots & 0 \\ 0 & \cdots & 0 & B_0 & \cdots & B_q \end{bmatrix} \begin{bmatrix} u(0) \\ u(1) \\ \vdots \\ u(2q+\mu-1) \end{bmatrix} \}$$

where $$\widetilde{A}_1 = \begin{bmatrix} A_0 & 0 & \cdots & 0 \\ A_1 & A_0 & \ddots & \vdots \\ \vdots & & \ddots & 0 \\ A_{q-1} & \cdots & & A_0 \end{bmatrix}$$

3. Backward Solution

We solve the equation for $k=N,N-1,N-2,.$ given the final values of $y(k)$

$$y(N), y(N-1), \ldots, y(N-q+1)$$

and the input

$$u(k) \text{ for } k=N,N-1,N-2,\ldots.$$

Let

$$A^{-1}(\sigma) = V_{-l}\sigma^{-l} + V_{-l+1}\sigma^{-l+1} + \ldots + V_0 + V_1\sigma + V_2\sigma^2 + \ldots$$

be the Laurent expansion of $A^{-1}(\sigma)$ at $\sigma=0$.

The Backward solution is

$$y(k) = \begin{bmatrix} V_{N-k} & \cdots & V_{N-k-q+1} \end{bmatrix} \begin{bmatrix} A_0 & 0 & \cdots & 0 \\ A_1 & A_0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ A_{q-1} & \cdots & A_1 & A_0 \end{bmatrix} \begin{bmatrix} y(N) \\ y(N-1) \\ \vdots \\ y(N-q+1) \end{bmatrix} +$$

$$+ \begin{bmatrix} V_{N-k-q} & \cdots & V_{-l} \end{bmatrix} \begin{bmatrix} B_q & \cdots & B_0 & 0 & \cdots & 0 \\ 0 & B_q & & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & & B_0 & 0 \\ 0 & \cdots & 0 & B_q & \cdots & B_0 \end{bmatrix} \begin{bmatrix} u(N) \\ u(N-1) \\ \vdots \\ u(k-l) \end{bmatrix}$$

A recursive Backward solution is given by

$$y(k)=\begin{bmatrix} V_q & \cdots & V_1 \end{bmatrix}\begin{bmatrix} A_0 & 0 & \cdots & 0 \\ A_1 & A_0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ A_{q-1} & \cdots & A_1 & A_0 \end{bmatrix}\begin{bmatrix} y(k+q) \\ y(k+q-1) \\ \vdots \\ y(k+1) \end{bmatrix}+$$

$$+\begin{bmatrix} V_0 & \cdots & V_{-l} \end{bmatrix}\begin{bmatrix} B_q & \cdots & B_0 & 0 & \cdots & 0 \\ 0 & B_q & & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & & B_0 & 0 \\ 0 & \cdots & 0 & B_q & \cdots & B_0 \end{bmatrix}\begin{bmatrix} u(k+q) \\ u(k+q-1) \\ \vdots \\ u(k-l) \end{bmatrix}$$

Admissible Final Condition Space

$$\overline{H}_{iu}:=\{y(i),\ i=N,N-1,\dots,N-q+1:$$

$$\widetilde{A}_2\begin{bmatrix} V_{-q} & V_{-q-1} & \cdots & V_{-2q+1} \\ V_{-q+1} & V_{-q} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \\ V_{-1} & \cdots & & V_{-q} \end{bmatrix}\widetilde{A}_2\begin{bmatrix} y(N) \\ y(N-1) \\ \vdots \\ y(N-q+1) \end{bmatrix}=\widetilde{A}_2\begin{bmatrix} V_{-q} & \cdots & V_{-l} & 0 \\ V_{-q+1} & & & \vdots \\ \vdots & & & 0 \\ V_{-1} & \cdots & V_{-l+1} & V_{-l} \end{bmatrix}\times$$

$$\times\begin{bmatrix} B_q & \cdots & B_0 & 0 & \cdots & 0 \\ 0 & B_q & & B_0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & & \ddots & 0 \\ 0 & \cdots & 0 & B_q & \cdots & B_0 \end{bmatrix}\begin{bmatrix} u(N) \\ u(N-1) \\ \vdots \\ u(N-q-l+1) \end{bmatrix}\}$$

where $$\widetilde{A}_2=\begin{bmatrix} A_q & 0 & \cdots & 0 \\ A_{q-1} & A_q & \ddots & \vdots \\ \vdots & & \ddots & 0 \\ A_1 & \cdots & & A_q \end{bmatrix}$$

4. Symmetric Solution

We give the solution for a finite time interval

$$k=0,1,2,...,N\text{-}2,N\text{-}1,N$$

given boundary conditions

$$y(0), y(1),..., y(q\text{-}1) \quad \text{and} \quad y(N), y(N\text{-}1),..., y(N\text{-}q+1)$$

and the input vector

$$u(0), u(1),..., u(N\text{-}1), u(N)$$

The symmetric solution is given by

$$y(k)=\begin{bmatrix} H_{-k-1} & \cdots & H_{-k-q} \end{bmatrix} \begin{bmatrix} A_q & A_{q-1} & \cdots & A_1 \\ 0 & A_q & \ddots & \vdots \\ \vdots & \ddots & \ddots & A_{q-1} \\ 0 & \cdots & 0 & A_q \end{bmatrix} \begin{bmatrix} y(q-1) \\ y(q-2) \\ \vdots \\ y(0) \end{bmatrix} +$$

$$+\begin{bmatrix} H_{N-k} & \cdots & H_{N-k-q+1} \end{bmatrix} \begin{bmatrix} A_0 & 0 & \cdots & 0 \\ A_1 & A_0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ A_{q-1} & \cdots & A_1 & A_0 \end{bmatrix} \begin{bmatrix} y(N) \\ y(N-1) \\ \vdots \\ y(N-q+1) \end{bmatrix} +$$

$$+\begin{bmatrix} H_{N-k-q} & \cdots & H_k \end{bmatrix} \begin{bmatrix} B_q & \cdots & B_0 & 0 & \cdots & 0 \\ 0 & B_q & & B_0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & & \ddots & 0 \\ 0 & \cdots & 0 & B_q & \cdots & B_0 \end{bmatrix} \begin{bmatrix} u(N) \\ u(N-1) \\ \vdots \\ u(0) \end{bmatrix}$$

Forward-Symmetric solution

$$y(k) = -\begin{bmatrix} H_{-1} & \cdots & H_{-q} \end{bmatrix} \begin{bmatrix} A_q & A_{q-1} & \cdots & A_1 \\ 0 & A_q & \ddots & \vdots \\ \vdots & \ddots & \ddots & A_{q-1} \\ 0 & \cdots & 0 & A_q \end{bmatrix} \begin{bmatrix} y(k-1) \\ y(k-2) \\ \vdots \\ y(k-q) \end{bmatrix} +$$

$$+\begin{bmatrix} H_{N-k} & \cdots & H_{N-k-q+1} \end{bmatrix} \begin{bmatrix} A_0 & 0 & \cdots & 0 \\ A_1 & A_0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ A_{q-1} & \cdots & A_1 & A_0 \end{bmatrix} \begin{bmatrix} y(N) \\ y(N-1) \\ \vdots \\ y(N-q+1) \end{bmatrix} +$$

$$+\begin{bmatrix} H_{N-k-q} & \cdots & H_{-q} \end{bmatrix} \begin{bmatrix} B_q & \cdots & B_0 & 0 & \cdots & 0 \\ 0 & B_q & & B_0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & & \ddots & 0 \\ 0 & \cdots & 0 & B_q & \cdots & B_0 \end{bmatrix} \begin{bmatrix} u(N) \\ u(N-1) \\ \vdots \\ u(k-q) \end{bmatrix}$$

Backward-Symmetric solution

$$y(k) = \begin{bmatrix} H_{-k-1} & \cdots & H_{-k-q} \end{bmatrix} \begin{bmatrix} A_q & A_{q-1} & \cdots & A_1 \\ 0 & A_q & \ddots & \vdots \\ \vdots & \ddots & \ddots & A_{q-1} \\ 0 & \cdots & 0 & A_q \end{bmatrix} \begin{bmatrix} y(q-1) \\ y(q-2) \\ \vdots \\ y(0) \end{bmatrix} -$$

$$-\begin{bmatrix} H_0 & \cdots & H_{-q+1} \end{bmatrix} \begin{bmatrix} A_0 & 0 & \cdots & 0 \\ A_1 & A_0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ A_{q-1} & \cdots & A_1 & A_0 \end{bmatrix} \begin{bmatrix} y(k+q) \\ y(k+q-1) \\ \vdots \\ y(k+1) \end{bmatrix} +$$

$$+\begin{bmatrix} H_0 & \cdots & H_{-k} \end{bmatrix} \begin{bmatrix} B_q & \cdots & B_0 & 0 & \cdots & 0 \\ 0 & B_q & & B_0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & & \ddots & 0 \\ 0 & \cdots & 0 & B_q & \cdots & B_0 \end{bmatrix} \begin{bmatrix} u(k+q) \\ u(k+q-1) \\ \vdots \\ u(0) \end{bmatrix}$$

Admissible Boundary condition space

$$\widetilde{H}_{iu} = \{ y_{0,q-1}, y_{N-q+1,N} : \begin{bmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{bmatrix} \begin{bmatrix} X_A y_{N-q+1,N} \\ X_{\widetilde{A}} y_{0,q-1} \end{bmatrix} = \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix} B_N u_{0,N} \}$$

where

$$W_{11} = \begin{bmatrix} H_{-q} & \cdots & H_{-2q+1} \\ \vdots & \ddots & \vdots \\ H_{-1} & \cdots & H_{-q} \end{bmatrix}, W_{12} = \begin{bmatrix} H_{-N+q-1} & \cdots & H_{-N} \\ \vdots & \ddots & \vdots \\ H_{-N+2q-2} & \cdots & H_{-N+q-1} \end{bmatrix}$$

$$W_{21} = \begin{bmatrix} H_{N-2q+1} & \cdots & H_{N-3q+2} \\ \vdots & \ddots & \vdots \\ H_{N-q} & \cdots & H_{N-2q+1} \end{bmatrix}, W_{22} = \begin{bmatrix} H_0 & \cdots & H_{-q+1} \\ \vdots & \ddots & \vdots \\ H_{q-1} & \cdots & H_0 \end{bmatrix}$$

$$X_A = \begin{bmatrix} A_q & \cdots & A_1 \\ & \ddots & \vdots \\ 0 & & A_q \end{bmatrix}, X_{\widetilde{A}} = \begin{bmatrix} A_0 & & 0 \\ \vdots & \ddots & \\ A_{q-1} & \cdots & A_0 \end{bmatrix}$$

$$Z_1 = \begin{bmatrix} H_{-q} & \cdots & H_{-N} \\ \vdots & \ddots & \vdots \\ H_{-1} & \cdots & H_{-N+q-1} \end{bmatrix}, Z_2 = \begin{bmatrix} H_{N-2q+1} & \cdots & H_{-q+1} \\ \vdots & \ddots & \vdots \\ H_{N-q} & \cdots & H_0 \end{bmatrix}$$

$$B_N = \begin{bmatrix} B_q & \cdots & B_0 & & 0 \\ & \ddots & & \ddots & \\ 0 & & B_q & \cdots & B_0 \end{bmatrix}$$

$$Y_{N-q+1,N} = \begin{bmatrix} y(N) \\ \vdots \\ y(N-q+1) \end{bmatrix}, y_{0,q-1} = \begin{bmatrix} y(q-1) \\ \vdots \\ y(0) \end{bmatrix}, u_{0,N} = \begin{bmatrix} u(N) \\ \vdots \\ u(0) \end{bmatrix}$$