FORWARD, BACKWARD AND
SYMMETRIC SOLUTIONS OF DISCRETE
ARMA - REPRESENTATIONS

by

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1. Introduction

Consider the discrete time equation

\[ x_{k+1} = Jx_k + Bu_k \]

where

\[ J = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \]

Then for \( x_0 = [1, 1, 1]^T \) and \( u_k = 0 \) the response of the system will be

![Graph showing the response of the system over time]
Now consider the ‘backward’ system

\[ Jx_{k+1} = x_k + Bu_k \]

Then for final condition \( x_5 = [1,1,1]^T \) and \( u_k = 0 \) we have the ‘backward response’

If we compose the above two systems we obtain a singular descriptor system having the form

\[
\begin{bmatrix}
I & 0 \\
0 & J
\end{bmatrix} x_{k+1} = \begin{bmatrix}
J & 0 \\
0 & I
\end{bmatrix} x_k + \begin{bmatrix}
B
\end{bmatrix} u_k
\]

or in general

\[ Ex_{k+1} = Ax_k + Bu_k \]

The response of such a system given \( x_0 = [1,1,1]^T \), \( x_5 = [1,1,1]^T \) and \( u_k = 0 \) will be
Such systems are described by

$$(\sigma E - A)x_k = Bu_k$$

where $\sigma x_k = x_{k+1}$ is the forward shift operator.

A more general model examined here has the form

$$A(\sigma)y(k) = B(\sigma)u(k)$$

or

$$A_qy(k + q) + ... + A_0y(k) = B_qu(k + q) + ... + B_0u(k)$$

where

$$A(\sigma) = A_q\sigma^q + A_{q-1}\sigma^{q-1} + ... + A_0 \in \mathbb{R}^{r \times r}[\sigma]$$

$$B(\sigma) = B_q\sigma^q + B_{q-1}\sigma^{q-1} + ... + B_0 \in \mathbb{R}^{r \times m}[\sigma]$$

$y(k) \in \mathbb{R}^r$ is the output vector

$u(k) \in \mathbb{R}^m$ is the input vector

There are three kinds of solutions

- Forward
- Symmetric
- Backward
2. Forward Solution

We solve the equation for $k=0,1,2,...$ given the initial values of $y(k)$

$$y(0), y(1), ..., y(q-1)$$

and the input

$$u(k)$$ for $k=0,1,2,...$

Let

$$A^{-1}(\sigma) = H_{\mu} \sigma^{\mu} + H_{\mu-1} \sigma^{\mu-1} + ... + H_0 + H_{-1} \frac{1}{\sigma} + H_{-2} \frac{1}{\sigma^2} + ...$$

be the Laurent expansion of $A^{-1}(\sigma)$ at $\sigma=\infty$.

The Forward solution is

$$y(k) = [H_{-k-q} \ldots H_{-k-1}] \begin{bmatrix} A_q & 0 & \cdots & 0 \\ A_{q-1} & A_q & \cdots & \vdots \\ \vdots & \vdots & \ddots & 0 \\ A_1 & \cdots & A_{q-1} & A_q \end{bmatrix} \begin{bmatrix} y(0) \\ y(1) \\ \vdots \\ y(q-1) \end{bmatrix} +$$

$$+ [H_{-k} \ldots H_{\mu}] \begin{bmatrix} B_0 & \cdots & B_q & 0 & \cdots & 0 \\ 0 & B_0 & \cdots & \cdots & \vdots & \vdots \\ \vdots & \vdots & \ddots & B_q & 0 & \vdots \\ 0 & \cdots & 0 & B_0 & \cdots & B_q \end{bmatrix} \begin{bmatrix} u(0) \\ u(1) \\ \vdots \\ u(k + \mu + q) \end{bmatrix}$$
A recursive Forward solution is given by

\[
y(k) = - \begin{bmatrix} H_{-1} & \cdots & H_{-q} \end{bmatrix} \begin{bmatrix} A_0 & 0 & \cdots & 0 \\ A_1 & A_0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ A_{q-1} & \cdots & A_1 & A_0 \end{bmatrix} \begin{bmatrix} y(k-1) \\ y(k-2) \\ \vdots \\ y(k-q) \end{bmatrix} + \begin{bmatrix} B_0 & \cdots & B_q & 0 & \cdots & 0 \\ 0 & B_0 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & B_q & 0 & \vdots \\ 0 & \cdots & 0 & B_0 & \cdots & B_q \end{bmatrix} \begin{bmatrix} u(k-q) \\ u(k-q+1) \\ \vdots \\ u(k+\mu+q) \end{bmatrix}
\]

Admissible Initial Condition Space

\[ H_\nu := \{y(i), \ i = 0, 1, \ldots, q-1:\]
3. Backward Solution

We solve the equation for \( k=N,N-1,N-2, \ldots \), given the final values of \( y(k) \)

\[
y(N), y(N-1), \ldots, y(N-q+1)
\]

and the input

\[
u(k) \text{ for } k=N,N-1,N-2, \ldots
\]

Let

\[
A^{-1}(\sigma) = V^{-1}_l \sigma^{-l} + V^{-1}_{l+1} \sigma^{-l+1} + \ldots + V_0 + V_1 \sigma + V_2 \sigma^2 + \ldots
\]

be the Laurent expansion of \( A^{-1}(\sigma) \) at \( \sigma=0 \).

The Backward solution is

\[
y(k) = \begin{bmatrix} V_{N-k} & \cdots & V_{N-k-q+1} \end{bmatrix} \begin{bmatrix} A_0 & 0 & \cdots & 0 \\ A_1 & A_0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ A_{q-1} & \cdots & A_1 & A_0 \end{bmatrix} \begin{bmatrix} y(N) \\ y(N-1) \\ \vdots \\ y(N-q+1) \end{bmatrix} +
\]

\[
+ \begin{bmatrix} B_q & \cdots & B_0 & 0 & \cdots & 0 \\ 0 & B_q & \ddots & \cdots & \vdots & \vdots \\ \vdots & \ddots & \ddots & B_0 & 0 & \vdots \\ 0 & \cdots & 0 & B_q & \cdots & B_0 \end{bmatrix} \begin{bmatrix} u(N) \\ u(N-1) \\ \vdots \\ u(k-l) \end{bmatrix}
\]
A recursive Backward solution is given by

\[
y(k) = \begin{bmatrix} V_q & \cdots & V_1 \end{bmatrix} \begin{bmatrix} A_0 & 0 & \cdots & 0 \\ A_1 & A_0 & \cdots & \vdots \\ \vdots & \vdots & \ddots & 0 \\ A_{q-1} & \cdots & A_1 & A_0 \end{bmatrix} \begin{bmatrix} y(k+q) \\ y(k+q-1) \\ \vdots \\ y(k+1) \end{bmatrix} + \begin{bmatrix} B_q & \cdots & B_0 & 0 & \cdots & 0 \\ 0 & B_q & \cdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \ddots & B_0 & 0 \\ 0 & \cdots & 0 & B_q & \cdots & B_0 \end{bmatrix} \begin{bmatrix} u(k+q) \\ u(k+q-1) \\ \vdots \\ u(k-l) \end{bmatrix}
\]

Admissible Final Condition Space

\[\bar{H}_{lu} = \{y(i), i = N, N-1, \ldots, N-q+1:\}
\]

\[
\tilde{A}_2 \begin{bmatrix} V_{-q} & V_{-q-1} & \cdots & V_{-2q+1} \\ V_{-q+1} & V_{-q} & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ V_{-1} & \cdots & V_{-q} \end{bmatrix} \begin{bmatrix} y(N) \\ y(N-1) \\ \vdots \\ y(N-q+1) \end{bmatrix} = \begin{bmatrix} V_{-q} & \cdots & V_{-1} & 0 \\ V_{-q+1} & \vdots & \vdots & \vdots \\ \vdots & \vdots & \ddots & 0 \\ V_{-1} & \cdots & V_{-l+1} & V_{-l} \end{bmatrix} \times \begin{bmatrix} B_q & \cdots & B_0 & 0 & \cdots & 0 \\ 0 & B_q & B_0 & \cdots & \vdots & \vdots \\ \vdots & \vdots & \ddots & 0 & \vdots \\ 0 & \cdots & 0 & B_q & \cdots & B_0 \end{bmatrix} \begin{bmatrix} u(N) \\ u(N-1) \\ \vdots \\ u(N-q-l+1) \end{bmatrix}
\]

where

\[\tilde{A}_2 = \begin{bmatrix} A_q & 0 & \cdots & 0 \\ A_{q-1} & A_q & \cdots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ A_1 & \cdots & A_q \end{bmatrix}\]
4. Symmetric Solution

We give the solution for a finite time interval

\[ k=0,1,2,...,N-2,N-1,N \]

given boundary conditions

\[ y(0), y(1),..., y(q-1) \quad \text{and} \quad y(N), y(N-1),..., y(N-q+1) \]

and the input vector

\[ u(0), u(1),..., u(N-1), u(N) \]

The symmetric solution is given by

\[ y(k) = \begin{bmatrix} H_{k-1} & \cdots & H_{k-q} \end{bmatrix} \begin{bmatrix} A_g & A_{g-1} & \cdots & A_1 & y(q-1) \\ 0 & A_g & \cdots & \vdots & y(q-2) \\ \vdots & \vdots & \ddots & A_{g-1} & \vdots \\ 0 & \cdots & 0 & A_g & y(0) \end{bmatrix} + \]

\[ + \begin{bmatrix} H_{N-k} & \cdots & H_{N-k-q+1} \end{bmatrix} \begin{bmatrix} A_0 & 0 & \cdots & 0 & y(N) \\ A_1 & A_0 & \cdots & \vdots & y(N-1) \\ \vdots & \vdots & \ddots & 0 & \vdots \\ A_{q-1} & \cdots & A_1 & A_0 & y(N-q+1) \end{bmatrix} + \]

\[ + \begin{bmatrix} H_{N-k-q} & \cdots & H_k \end{bmatrix} \begin{bmatrix} B_g & \cdots & B_0 & 0 & 0 & u(N) \\ 0 & B_g & B_0 & \cdots & \vdots & u(N-1) \\ \vdots & \vdots & \vdots & \ddots & 0 & \vdots \\ 0 & \cdots & 0 & B_g & \cdots & B_0 \end{bmatrix} u(0) \]
Forward-Symmetric solution

\[
y(k) = \left[ H_{-1} \ldots H_{-q} \right] \begin{bmatrix} A_q & A_{q-1} & \cdots & A_1 & y(k-1) \\ 0 & A_q & \ddots & \vdots & y(k-2) \\ \vdots & \vdots & \ddots & A_{q-1} & \vdots \\ 0 & \cdots & 0 & A_q & y(k-q) \end{bmatrix} + \\
+ \left[ H_{N-k} \ldots H_{N-k-q+1} \right] \begin{bmatrix} A_0 & 0 & \cdots & 0 & y(N) \\ A_1 & A_0 & \ddots & \vdots & y(N-1) \\ \vdots & \vdots & \ddots & 0 & \vdots \\ A_{q-1} & \cdots & A_1 & A_0 & y(N-q+1) \end{bmatrix} + \\
+ \left[ H_{N-k-q} \ldots H_{-q} \right] \begin{bmatrix} B_q & \cdots & B_0 & 0 & \cdots & 0 & u(N) \\ 0 & B_q & B_0 & \cdots & \vdots & u(N-1) \\ \vdots & \vdots & \ddots & 0 & \vdots \\ 0 & \cdots & 0 & B_q & \cdots & B_0 & u(k-q) \end{bmatrix}
\]

Backward-Symmetric solution

\[
y(k) = \left[ H_{-k-1} \ldots H_{-k-q} \right] \begin{bmatrix} A_q & A_{q-1} & \cdots & A_1 & y(q-1) \\ 0 & A_q & \ddots & \vdots & y(q-2) \\ \vdots & \vdots & \ddots & A_{q-1} & \vdots \\ 0 & \cdots & 0 & A_q & y(0) \end{bmatrix} - \\
- \left[ H_{-q} \ldots H_{-q+1} \right] \begin{bmatrix} A_0 & 0 & \cdots & 0 & y(k+q) \\ A_1 & A_0 & \ddots & \vdots & y(k+q-1) \\ \vdots & \vdots & \ddots & 0 & \vdots \\ A_{q-1} & \cdots & A_1 & A_0 & y(k+1) \end{bmatrix} + \\
+ \left[ H_0 \ldots H_{-k} \right] \begin{bmatrix} B_q & \cdots & B_0 & 0 & \cdots & 0 & u(k+q) \\ 0 & B_q & B_0 & \cdots & \vdots & u(k+q-1) \\ \vdots & \vdots & \ddots & 0 & \vdots \\ 0 & \cdots & 0 & B_q & \cdots & B_0 & u(0) \end{bmatrix}
\]
Admissible Boundary condition space

\[ \hat{H}_{iu} = \{ \mathbf{y}_{0,q-1}, \mathbf{y}_{N-q+1,N} \} \begin{bmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{bmatrix} \begin{bmatrix} \mathbf{X}_\mathbf{A} \mathbf{y}_{N-q+1,N} \\ \mathbf{X}_\mathbf{A} \mathbf{y}_{0,q-1} \end{bmatrix} = \begin{bmatrix} \mathbf{Z}_1 \\ \mathbf{Z}_2 \end{bmatrix} \mathbf{B}_N \mathbf{u}_{0,N} \}

where

\[ W_{11} = \begin{bmatrix} H_{-q} & \cdots & H_{-2q+1} \\ \vdots & \ddots & \vdots \\ H_{-1} & \cdots & H_{-q} \end{bmatrix}, \quad W_{12} = \begin{bmatrix} H_{-N+q-1} & \cdots & H_{-N} \\ \vdots & \ddots & \vdots \\ H_{-N+2q-2} & \cdots & H_{-N+q-1} \end{bmatrix} \]

\[ W_{21} = \begin{bmatrix} H_{N-2q+1} & \cdots & H_{N-3q+2} \\ \vdots & \ddots & \vdots \\ H_{N-q} & \cdots & H_{N-2q+1} \end{bmatrix}, \quad W_{22} = \begin{bmatrix} H_0 & \cdots & H_{-q+1} \\ \vdots & \ddots & \vdots \\ H_{q-1} & \cdots & H_0 \end{bmatrix} \]

\[ \mathbf{X}_\mathbf{A} = \begin{bmatrix} A_q & \cdots & A_1 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & A_q \end{bmatrix}, \quad \mathbf{X}_\mathbf{A}^T = \begin{bmatrix} A_0 & 0 \\ \vdots & \ddots \\ A_{q-1} & \cdots & A_0 \end{bmatrix} \]

\[ \mathbf{Z}_1 = \begin{bmatrix} H_{-q} & \cdots & H_{-N} \\ \vdots & \ddots & \vdots \\ H_{-1} & \cdots & H_{-N+q-1} \end{bmatrix}, \quad \mathbf{Z}_2 = \begin{bmatrix} H_{N-2q+1} & \cdots & H_{-q+1} \\ \vdots & \ddots & \vdots \\ H_{N-q} & \cdots & H_0 \end{bmatrix} \]

\[ \mathbf{B}_N = \begin{bmatrix} B_q & \cdots & B_0 & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & B_q \end{bmatrix} \]

\[ \mathbf{y}_{N-q+1,N} = \begin{bmatrix} y(N) \\ \vdots \\ y(N-q+1) \end{bmatrix}, \quad \mathbf{y}_{0,q-1} = \begin{bmatrix} y(q-1) \\ \vdots \\ y(0) \end{bmatrix}, \quad \mathbf{u}_{0,N} = \begin{bmatrix} u(N) \\ \vdots \\ u(0) \end{bmatrix} \]