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ON THE INFORMATION CARRIED BY COLUMN (ROW) REDUCED MFDs

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Abstract. Column reduced matrix fraction descriptions (MFDs) are shown to possess important structural properties in their ability to display the infinite pole (resp. zero) structure of a rational transfer function matrix.

1. Introduction.

If $G(s) \in \mathbb{R}(s)^{m\times l}$ then the simplest internal representations of this input/output behaviour are the matrix fraction descriptions (MFDs)

$$G(s) = N(s)D(s)^{-1} = D_1(s)^{-1}N_1(s) \quad (1)$$

where $N(s) \in \mathbb{R}[s]^{m\times l}, D(s) \in \mathbb{R}[s]^{l\times l}$ (resp. $D_1(s) \in \mathbb{R}[s]^{m\times m}$, $N_1(s) \in \mathbb{R}[s]^{m\times l}$) are not necessarily right (resp. left) coprime. The extent to which coprime MFDs reflect the finite frequency structure of the transfer function matrix is well catalogued [9], [5]. For the infinite frequency structure a second factorisation is apparently required, this time of $G(\frac{1}{w})$.

$$G(\frac{1}{w}) = \tilde{N}(w)\tilde{D}(w)^{-1} = \tilde{D}_1(w)^{-1}\tilde{N}_1(w) \quad (2)$$

In fact this is not necessary, and [6] has described how by taking a specific form of the factorizations (1), both the finite and infinite frequency structure can be deduced from just one factorisation of $G(s) \in \mathbb{R}(s)^{m\times l}$. The specific forms of (1) which carry the complete pole-zero structure of $G(s) \in \mathbb{R}(s)^{m\times l}$ in this ready manner are those where the compound matrices

$$M(s) = \begin{bmatrix} D(s) \\ N(s) \end{bmatrix} ; \; M_1(s) = [N_1(s), \; D_1(s)] \quad (3)$$

are required to be minimal bases [1], i.e., where $M(s)$, (resp. $M_1(s)$) is column (resp. row) reduced and has no finite zeros, and such MFDs will be called *minimal*. Thus the additional requirement of column or row reduceness of (3) has great implications for the MFD's ability to display the complete pole-zero structure of the rational matrix it represents. It will be shown however in the sequel that minimal MFD's are not required for many aspects concerning the infinite frequency information to be detected and that the simple requirement of column or row reduceness of the MFD is sufficient. This paper considers ways in which information about the infinite frequency structure (in particular, the infinite pole-zero multiplicities) of $G(s) \in \mathbb{R}(s)^{m\times l}$ may be directly determined from such MFDs, which can of course be easily computed

2. Preliminaries.

Let $g(s) = \frac{n(s)}{d(s)} \in \mathbb{R}(s)$ where $n(s), d(s) \in \mathbb{R}[s]$, $d(s) \not\equiv 0$ and define $V_\infty(\cdot) : \mathbb{R}(s) \to \mathbb{Z} \cup \{+\infty\}$ via

$$V_\infty(g(s)) := \begin{cases} \delta(d(s)) - \delta(n(s)) & g(s) \not\equiv 0 \\ +\infty & g(s) \equiv 0 \end{cases} \quad (4)$$

where $\delta(\cdot)$ denotes the degree of the indicated polynomial. $V_\infty(\cdot)$is *the valuation at $s = \infty$ of $g(s) \in \mathbb{R}(s)$*. Given $G(s) \in \mathbb{R}(s)^{m\times l}$, $\text{rank}_{\mathbb{R}(s)}G(s) = r \leq \min(m, l)$ we can define its "valuation at $s = \infty$" also via (4)

$$V_\infty(G) \equiv \min \{V_\infty(\cdot) \text{ among the } V_\infty(\cdot) \text{ of all } r \times r \text{ minors of } G(s)\} \quad (5)$$

Definition 1.[7] Let $G(s) \in \mathbb{R}(s)^{m\times l}$, $\text{rank}_{\mathbb{R}(s)}G(s) = r > 0$. Let $\xi_i(G)$ denote the least $V_\infty(\cdot)$ among the $V_\infty(\cdot)$ of all minors of $G(s)$ of order i, $i = 1, \ldots, r$. Then define

$$\begin{aligned} q_1 &:= -\xi_1(G) \\ q_2 &:= \xi_1(G) - \xi_2(G) \\ q_3 &:= \xi_2(G) - \xi_3(G) \\ &\vdots \\ q_r &:= \xi_{r-1}(G) - \xi_r(G). \end{aligned} \quad (6)$$

The *Smith-MacMillan form at $s = \infty$ of $G(s)$* is

$$S^{\infty}_{G(s)}(s) = \text{diag } [s^{q_1}, s^{q_2}, \ldots, s^{q_r}, 0_{m-r,l-r}] \quad (7)$$

with $q_1 \geq q_2 \geq \ldots \geq q_k \geq 0 \geq q_{k+1} \geq \ldots \geq q_r$. □

Definition 2. If ϱ (resp. ζ) is the number of q_i's in (7) satisfying $q_i > 0$ ($q_i < 0$) then $G(s)$ has *ϱ (ζ) poles (zeros) at infinity*, each having *degree* q_i ($|q_i|$). ϱ (resp. ζ) is the *multiplicity* of infinite poles (resp. zeros). □

Let $\mathbb{R}_{pr}(s)^{m\times l}$ denote the set of $m \times l$, proper rational matrices. Then

Definition 3. [7] $N_1(s) \in \mathbb{R}_{pr}(s)^{m\times l}$, $D_1(s) \in \mathbb{R}_{pr}(s)^{l\times l}$ are called *right coprime at* $s=\infty$ iff

$$\lim_{s\to\infty}\begin{bmatrix} D_1(s) \\ N_1(s)\end{bmatrix} = E_1 \quad (8)$$

with $\mathrm{rank}_{\mathbb{R}} E_1 = l$. □

Equivalently $N_1(s)$, $D_1(s)$ are *right coprime at* $s=\infty$ iff the proper rational matrix $[D_1(s)^T \quad N_1(s)^T]^T$ has no zeros at $s=\infty$. □

In what follows dual statements for left coprimeness and the like will not be given explicitly.

Definition 4. Let $G(s)\in\mathbb{R}^{m\times l}(s)$. Then

$$G(s) = N_1(s)D_1(s)^{-1} \quad (9)$$

where $N_1(s)\in\mathbb{R}^{m\times l}_{pr}(s)$, $D_1(s)\in\mathbb{R}^{l\times l}_{pr}(s)$ are right coprime at $s=\infty$ is called a *right proper matrix fraction description* (PMFD) of $G(s)$. □

Lemma 1. [6]. Let $G(s)$ possess a right PMFD of the form (9), then the zero (resp. pole) structure at $s=\infty$ of $G(s)$ is given by the zero structure at $s=\infty$ of $N_1(s)$ (resp. $D_1(s)$). □

Definition 5. [7]. Let $M(s)\in\mathbb{R}(s)^{p\times n}$ with column vectors $m_j(s)$. Then $M(s)$ can be written

$$M(s) = [M]^c_\ell \begin{pmatrix} (\frac{1}{s})^{q_{1\infty}} & & 0 \\ & \ddots & \\ 0 & & (\frac{1}{s})^{q_{n\infty}}\end{pmatrix} + \bar{M}(s) \quad (10)$$

where $q_{j\infty} = V_\infty(m_j), j=1,\dots,n$, and $\bar{M}(s)\in\mathbb{R}(s)^{p\times n}$ has column vectors $\bar{m}_j(s), j=1,\dots,n$, such that $V_\infty(\bar{m}_j) > V_\infty(m_j), j=1,\dots,n$. $[M]^c_\ell\in\mathbb{R}^{p\times n}\neq 0_{p,n}$ is the *least column valuation (at $s=\infty$) coefficient matrix* of $M(s)$. □

Definition 6. [6]. $M(s)\in\mathbb{R}(s)^{p\times n}$ is *column reduced at* $s=\infty$ if $\mathrm{rank}_{\mathbb{R}}[M]^c_\ell = n$ □

3. Infinite Frequency Structure and CRMFDs.

Consider now $G(s)\in\mathbb{R}(s)^{m\times l}$ with $m\geq l$ and $\mathrm{rank}_{\mathbb{R}(s)} G = r \leq l$ and

$$G(s) = N(s)D(s)^{-1} \quad (11a)$$

where $N(s)\in\mathbb{R}[s]^{m\times l}$, $D(s)\in\mathbb{R}[s]^{l\times l}$ (not necessarily coprime) are such that

$$M(s) = \begin{bmatrix} D(s) \\ N(s)\end{bmatrix} \equiv \underbrace{\begin{bmatrix} D_h \\ N_h\end{bmatrix}_h}_{[M]_h} \underbrace{\mathrm{diag}[s^{\delta(m_1)},\dots s^{\delta(m_l)}}_{\Lambda(s)} + \begin{bmatrix}\bar{D}(s) \\ \bar{N}(s)\end{bmatrix} \quad (11b)$$

is column reduced, i.e. $[M]_h = [D_h^T \quad N_h^T]_h^T \in \mathbb{R}^{(l+m)\times l}$ has $\mathrm{rank}_{\mathbb{R}} = l$ where $\delta(m_i)$, $i=1,2,..,l$ are the column degrees of $M(s)$. (11a) will be called a *column reduced matrix fraction description* (CRMFD). The connection between PMFDs and CRMFDs is

Theorem 1. Consider the CRMFD (11) then

$$G(s) = [N(s)\Lambda(1/s)][D(s)\Lambda(1/s)]^{-1} \quad (12)$$

is (a coprime at $s=\infty$) PMFD.

Proof. Consider (12) with

$$\hat{M}(s) = \begin{bmatrix} D(s)\Lambda(1/s) \\ N(s)\Lambda(1/s)\end{bmatrix} \in \mathbb{R}_{pr}(s)^{(l+m)\times l}. \quad (13)$$

By the definition of $\Lambda(s)$, $N(s)\Lambda(1/s)$ and $D(s)\Lambda(1/s)$ are proper rational matrices. Also from the (polynomial) CRMFD (11b),

$$\lim_{s\to\infty}\hat{M}(s) = \lim_{s\to\infty}\left\{\begin{bmatrix} D_\ell \\ N_\ell\end{bmatrix}_\ell + \begin{bmatrix}\bar{D}(s) \\ \bar{N}(s)\end{bmatrix}\Lambda(1/s)\right\} = \begin{bmatrix} D_\ell \\ N_\ell\end{bmatrix}_\ell = \begin{bmatrix} D_h \\ N_h\end{bmatrix}_h \quad (14)$$

where $\mathrm{rank}_{\mathbb{R}}[D_h^T \quad N_h^T]_h^T = l$. Thus (12) is a right coprime at $s=\infty$ PMFD. □

Corollary 1. The pole (resp. zero) structure at $s=\infty$ of $G(s)$ in (12) is given by the zero structure at $s=\infty$ of $D(s)\Lambda(1/s)$ (resp. $N(s)\Lambda(1/s)$). □

Thus to obtain the infinite frequency structure of $G(s)$ the sole requirement of column reduceness of a polynomial MFD suffices. Such MFDs are readily obtained since there is no requirement for coprimeness. We now show how to extract this information from a CRMFD either by inspection or by simple rank calculations. Let the Laurent expansion at $s=\infty$ of $G(s)$ be

$$G(s) = \sum_{i=-\infty}^{k} G_i s^i \quad (15)$$

The *i-th Toeplitz matrix* of $G(s)$ is

$$T_i^\infty(G) \triangleq \begin{pmatrix} G_k & \cdots & G_{-i} \\ \vdots & \cdots & \vdots \\ 0 & \cdots & G_k\end{pmatrix},\ i\geq -k \quad (16)$$

The *rank indices at infinity* of $G(s)$ are

$$\rho_i^\infty = \mathrm{rank}_{\mathbb{R}}\,(T_i^\infty(G)) - \mathrm{rank}_{\mathbb{R}}\,(T_{i-1}^\infty(G)) \quad (17)$$

for $i=-k,-k+1,\dots$ where it is assumed that $\mathrm{rank}_{\mathbb{R}}\, T_{-k-1}^\infty(G) = 0$. The ρ_i^∞ determine the infinite frequency structure of $G(s)$ as follows

Lemma 2. [11, 5]. If in (17) $\rho_i^\infty - \rho_{i-1}^\infty \neq 0$ then $G(s)$ has

$$\rho_i^\infty - \rho_{i-1}^\infty, \qquad i=-k,-k+1,\dots \quad (18)$$

zeros (resp. poles) at infinity of degree i (resp. $|i|$) if $i>0$ (resp. $i<0$), $i=-k,-k+1,\dots$. □

This provides a computational algorithm to obtain the infinite frequency structure directly from $G(s)$. This structure can be obtained from the (polynomial) CRMFD (11a) as follows.

Theorem 2. If $G(s)$ has the CRMFD (11) write

$$N(s) = N_0\Lambda(s) + \cdots + N_\sigma\Lambda_{-\sigma}(s) \qquad (19)$$

$$(D(s) = D_0\Lambda(s) + \cdots + D_\sigma\Lambda_{-\sigma}(s))$$

where $\sigma = \max\{\delta(m_i)\}$, $i = 1\ldots,l$ and $\Lambda_{-\ell}(s)$, is the diagonal matrix formed from $\Lambda(s)$ by reducing each power of s by the amount $\ell \in \mathbb{N}$, where within $\Lambda_{-\ell}(s)$ $s^j = 1$ $\forall$ $j < 0$ and the corresponding column in N_ℓ (resp. D_j) is zero. Then the rank indices of the Toeplitz matrices sociated with (19) determine the zero (resp. pole) structure at $s = \infty$ of $G(s)$.

Proof. From Corollary 1, the zero structure at $s = \infty$ of $G(s)$ is isomorphic to the zero structure at $s = \infty$ of the proper matrix $N(s)\Lambda(1/s)$. The Laurent expansion of $N(s)\Lambda(1/s)$ has the form

$$N(s)\Lambda(1/s) = N'_0 + N'_1 1/s + \cdots + N'_\sigma 1/s^\sigma \quad (20)$$

where $\sigma = \max\{\delta(m_i)\}$, $i = 1\ldots,l$ and N'_i are exactly the constant matrices to be used to construct the Toeplitz matrices (16). Post-multiplying (20) by $\Lambda(s) = \Lambda(1/s)^{-1}$ gives

$$N(s) = N'_0\Lambda(s) + \cdots + N'_\sigma\Lambda_{-\sigma}(s) \qquad (21)$$

and comparison with (19) gives

$$N'_i = N_i, \qquad i = 1,\ldots,\sigma. \qquad (22)$$

Thus the N'_i of (20) can be identified directly from the expansion (19). Similarly for the infinite pole ructure of $G(s)$. □

Example 1. Let

$$G(s) = \begin{pmatrix} 1/s & 0 & 0 \\ s & 1/s^3 & s^2 \\ 0 & 0 & 1/s^5 \end{pmatrix} = \underbrace{\begin{pmatrix} s^4 & 0 & 0 \\ s^6 & s^2 & s^7 \\ 0 & 0 & 1 \end{pmatrix}}_{N'(s)} \underbrace{\left(s^5 \;\; I_3 \right)^{-1}}_{[d(s)\; I_3]} \qquad (23)$$

A CRMFD is obtained by postmultiplying the compound matrix $[[d(s)\; I_3]^T\; N'(s)^T]^T$ by the unimodular matrix

$$\begin{pmatrix} 1 & 0 & -s \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad (24)$$

which gives the following CRMFD

$$M(s) = \left[\begin{array}{ccc} s^5 & 0 & -s^6 \\ 0 & s^5 & 0 \\ 0 & 0 & s^5 \\ \hline s^4 & 0 & -s^5 \\ s^6 & s^2 & 0 \\ 0 & 0 & 1 \end{array}\right] = \begin{bmatrix} D(s) \\ N(s) \end{bmatrix}$$

with $\Lambda(s) = \mathrm{diag}[s^6, s^5, s^6]$. The expansion of $N(s)$ in the form (19) is

$$N(s) = \underbrace{\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{N_0} \Lambda(s)$$

$$+ \underbrace{\begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{N_1} \Lambda_{-1} + \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{N_2} \Lambda_{-2}$$

$$+ \underbrace{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{N_3} \Lambda_{-3} + \underbrace{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{N_6} \Lambda_{-6}$$

where $N_i = 0_{3,3}$, $i = 4, 5$. Then

$$\left.\begin{array}{ll} \text{rank } (T_{-1}(N)) = 0, & \\ \text{rank } (T_0(N)) = 1, & \rho_0^\zeta = 1 \\ \text{rank } (T_i(N)) = 2i + 1, & \rho_i^\zeta = 2, \\ \quad \forall i = 1,\ldots,9 & \\ \text{rank } (T_{10}(N)) = 22, & \rho_{10}^\zeta = 3. \end{array}\right\} \qquad (25)$$

It follows from Lemma 3 that G(s) possesses two infinite zeros, one of degree 1 and one of degree 10 (as indicated by the changes in the rank indices at ρ_1^ζ and ρ_{10}^ζ). In a similar way $G(s)$ can be seen to possess an infinite pole of degree 2. This can be confirmed directly by computing $S^\infty_{G(s)}(s)$. □

This example illustrates the simplicity of what is proposed above. That is, column reduceness is all that is required of an MFD to compute the infinite frequency structure of a rational matrix $G(s)$. Such MFDs are of course fairly readily found. The example also illustrates the point that it is common with the Toeplitz method to continue determining ranks after $\mathrm{rank}_{\mathbf{R}}$ $(T_\sigma(N))$ has been found.

We next consider the problem of directly computing the multiplicity of the infinite poles and zeros. It would appear that to determine this one would require the complete information of the Smith-MacMillan form at $s = \infty$. However the following Theorem shows how to obtain these multiplicities directly from a CRMFD by performing a simple calculation.

Theorem 3. In the above notation, let $N_h \in \mathbb{R}^{m \times l}$ (resp. $D_h \in \mathbb{R}^{l \times l}$) be the constant matrix extracted from $[M]_h \in \mathbb{R}^{(l+m)\times l}$, the high order coefficient matrix of $M(s) \in \mathbb{R}[s]^{(l+m)\times l}$ derived from the CRMFD (11). Then, the multiplicity of the zeros (poles) at $s = \infty$ of $G(s)$ of rank r is

$$\zeta = r - \text{rank}_{\mathbf{R}}\,(N_h),\ (\varrho = l - \text{rank}_{\mathbf{R}}\,(D_h)) \quad (26)$$

Proof. Consider the infinite zero multiplicity of $G(s)$. From Lemma 3, this is given by

$$\begin{aligned} \sum_{i=0}^{\alpha_\zeta} \rho_i^\zeta - \rho_{i-1}^\zeta &= \rho_{\alpha_\zeta}^\zeta - \rho_0^\zeta \\ &= \text{rank}_{\mathbf{R}(s)}\,(N(s)) - \text{rank}_{\mathbf{R}}\,(N_h) \\ &= r - \text{rank}_{\mathbf{R}}\,(N_h) \end{aligned} \quad (27)$$

where α_ζ is the maximum infinite zero degree. Similarly for the infinite poles. □

Corollary 2. $G(s)$ possesses no zeros at $s = \infty$ iff $\text{rank}_{\mathbf{R}}(N_h) = r$. Similarly $G(s)$ possesses no poles at $s = \infty$ iff $\text{rank}_{\mathbf{R}}(D_h) = l$. □

[3] gave a test for the absence of infinite zeros of a rational matrix $G(s)$ which involved the calculation of the ranks of the corresponding Teoplitz matrices associated with its Laurent expansion at $s = \infty$. Corollary 2 simply requires the computation of the rank of the high order column coefficient matrix of the CRMFD $M(s) \in \mathbb{R}[s]^{(l+m)\times l}$, which is generally easier to find.

4. Infinite Frequency Structure Bounds.

In this section we investigate the extent to which the infinite frequency structure of $G(s)$ can be deduced from direct inspection of a CRMFD. This parallels the scalar case where it is known that a rational function possesses an infinite zero if degree the degree of its numerator is strictly less than that of its denominator, and an infinite pole if its numerator degree exceeds its denominator degree.

Unless otherwise stated, it will be assumed that the CRMFD (11) has been transformed by appropriate unimodular transformations to the form

$$M_1(s) = \begin{bmatrix} D_1'(s) & D_2(s) \\ 0_{m,\mu} & N_2(s) \end{bmatrix} \in \mathbb{R}[s]^{(l+m)\times l} \quad (28)$$

and $\mu = l - r$ and $N_2(s) \in \mathbb{R}[s]^{m \times r}$ has full column rank.

Definition 7. Let (28) be a CRMFD and $m_{2,i}, d_{2,i}, n_{2,i}$, $i = 1, \ldots, r$, denote the ith columns of $M_2(s) = [D_2(s)^T, N_2(s)^T]^T$, $D_2(s)$, $N_2(s)$ respectively, and $\delta(m_{2,i}), \delta(d_{2,i}), \delta(n_{2,i})$ their respective degrees. The *minimal differences* of the CRMFD (28) are $\hat{k}_{2,i} = \delta(m_{2,i}) - \delta(n_{2,i}) (\geq 0)$ and $\check{k}_{2i} = \delta(m_{2i}) - \delta(d_{2,i}) (\geq 0)$. □

It follows that for each $i = 1, \ldots, r$, either $\hat{k}_{2,i}$ or $\check{k}_{2,i}$, or possibly both, are zero. If one is nonzero then certain infinite frequency information may be obtained. Re-order the columns of $M_2(s)$ so that $\hat{k}_{2,i} > 0$, occur in ascending order in columns $i = 1, \ldots, \theta$ of $M_2(s)$ and $\check{k}_{2,i} > 0$ occur in ascending order in columns $i = r - \phi + 1, \ldots, r$, and so that all other $\hat{k}_{2,i}$, $\check{k}_{2,i}$, $i = \theta + 1, \ldots, r - \phi$, are zero.

Theorem 4. Given the CRMFD (28) $G(s)$ has at least θ infinite zeros of degrees $\nu_1, \ldots, \nu_\theta$ satisfying $\nu_i \geq \hat{k}_{2,i}$, $i = 1, \ldots, \theta$, and at least ϕ infinite poles of degrees $\gamma_1, \ldots, \gamma_\phi$ satisfying $\gamma_i \geq \check{k}_{2,l-\phi+i}$, $i = 1, \ldots, \phi$.

Proof. The infinite zero structure of $G(s)$ is given by the infinite zero structure of $N(s)_2\Lambda(1/s)$. Let ξ_i denote the least $V_\infty(\cdot)$ among the $V_\infty(\cdot)$ of all minors of $N_2(s)\Lambda(1/s)$ of order i, $i \in \mathbf{r}$. Now $\xi_{r-\theta+1}$ is at least equal to $\hat{k}_{2,1}$ since all the $(r - \theta + 1)$th order minors of $N_2(s)\Lambda(1/s)$ contain the factor $1/s^{\hat{k}_{2,1}}$, i.e., at least one of the first θ columns are used in all $(r - \theta + 1)$th minors of $N_2(s)\Lambda(1/s)$. Similarly, $\xi_{r-\theta+2}$ is at least equal to $\hat{k}_{2,1} + \hat{k}_{2,2}$ since all the $(r - \theta + 2)$th order minors of $N_2(s)\Lambda(1/s)$ contain the factor $1/s^{\hat{k}_{2,1}} 1/s^{\hat{k}_{2,2}}$. It then follows that ξ_r is at least equal to $\sum_{i=1}^{\theta} \hat{k}_{2i}$ since all the rth order minors of $N(s)\Lambda(1/s)$ contain the factor $\prod_{i=1}^{\theta} 1/s^{\hat{k}_{2i}}$. Thus

$$\begin{pmatrix} \xi_{r-\theta+1} \geq & \hat{k}_{2,1} \\ \xi_{r-\theta+2} \geq & \hat{k}_{2,1} + \hat{k}_{2,2} \\ \vdots & \vdots \\ \xi_r \geq & \hat{k}_{2,1} + \ldots + \hat{k}_{2,\theta} \end{pmatrix} \Longrightarrow$$

$$\begin{pmatrix} q_{r-\theta+1} := & \xi_{r-\theta} - \xi_{r-\theta+1} \leq & \hat{k}_{2,1} \\ q_{r-\theta+2} := & \xi_{r-\theta+1} - \xi_{r-\theta+2} \leq & \hat{k}_{2,2} \\ \vdots & \vdots & \\ q_r := & \xi_{r-1} - \xi_r \leq & \hat{k}_{2,\theta} \end{pmatrix} \quad (36)$$

since ξ_i, $i \in r$ is always positive. The result for the zeros then follows by Lemma 1 and Corollary 1. Similarly for the infinite poles. □

Theorem 4 thus enables some lower bounds on the infinite frequency structure of $G(s)$ to be determined virtually by inspection. This connects to the well-known result [2] in the strictly proper case.

Corollary 3. Consider the CRMFD (28). $G(s)$ is strictly proper iff each column of $N_2(s)$ has degree strictly less than the degree of the corresponding column of $D_2(s)$.

Proof. If $G(s)$ is strictly proper then it possesses no infinite poles. Suppose that $\delta(n_{2i}) > \delta(d_{2i})$, for some $i = 1, \dots, r$, then by Theorem 4, $G(s)$ possesses infinite poles which is a contradiction. On the other hand if $\delta(n_{2,i}) < \delta(d_{2,i})$ $\forall$ i then by (11b) the compound matrix (28) is column reduced and its column degrees $\delta(m_{2,i})$ are dictated by $\delta(d_{2,i})$ which by Theorem 4 proves the corollary. □

5. CRMFDs Displaying Complete Infinite Pole-Zero Multiplicity Information

We define the following

Definition 8. If $G(s) \in \mathbb{R}(s)^{m \times l}$ has ζ (resp. ϱ) finite zeros (resp. poles) with degrees $\alpha_1, \dots, \alpha_\zeta$ (resp. $\beta_1, \dots, \beta_\varrho$) then the CRMFD (28) is said to display the *complete infinite zero* (resp. *pole*) *multiplicity information* in case $\theta = \zeta$ (resp. $\phi = \varrho$). If in addition $\hat{k}_{2,i} = \alpha_i$, $i = 1, \dots, \zeta$ (resp. $\hat{k}_{2,l-\varrho+i} = \beta_i$, $i = 1, \dots, \varrho$) then it will be said to display the *complete infinite zero* (resp. *pole*) *degree* information. □

Thus if (28) satisfies Definition 8 then the lower bounds for the infinite poles and zeros proposed in Theorem 4 coincide with the infinite pole-zero multiplicity and degree information.

A natural question concerns the existence of a CRMFD (28) which simultaneously displays the complete infinite zero and pole multiplicity information. There is a positive answer to this question which is established in an algorithmic manner.

Theorem 5. Given $G(s) \in \mathbb{R}^{m \times l}(s)$ there exists a CRMFD of the form (35) which displays complete infinite pole and zero multiplicity information.

Proof. We will show that any CRMFD can be transformed into another, say $M_{\varrho\zeta}(s)$, satisfying $\hat{k}_{\varrho\zeta,i} > 0$ $\forall$ $i = \{l - (\zeta + \varrho) + 1\}, \dots, (l - \zeta)$ and $\hat{k}_{\varrho\zeta,i} > 0$ $\forall$ $i = (l - \zeta + 1), \dots, l$. Let $M_1(s)$ be the CRMFD (28) and $[M_1]_h$ its high order coefficient matrix.

Step (1) Consider $M_2(s) = [D_2(s)^T,\ N_2(s)^T]^T$ as in (28). Reorder the columns of $M_2(s)$ so that

$$[M_2]_h = \begin{bmatrix} D_{2h} \\ N_{2h} \end{bmatrix}_h = \begin{bmatrix} \overbrace{D'_h}^{k} & D''_h & 0_{l,\phi} \\ N'_h & 0_{m,\theta} & N'''_h \end{bmatrix} \quad (30)$$

where $k + \theta + \phi = r$ and (30) is column reduced. Consider first the infinite pole multiplicity of $G(s)$.

Step (2) Find a non-zero constant vector

$$a = (a_1, a_2, \dots, a_{r-\phi}, 0_{r-\phi+1}, \dots, 0_r)^T$$

such that $D_{2h}a = 0$. Let $\delta(d_{2,i})$, $\delta(n_{2,i})$, $i = 1, \dots, r$, be the degree of the ith column of $D_2(s)$, $N_2(s)$ respectively, and let $\delta_{d_2} = \max\ \{\delta(d_{2,i})\}$ and $\delta_{n_2} = \max\ \{\delta(n_{2,i})\}$. Define

$$a(s) := [a_1 s^{\delta_{d_2} - \delta(d_{2,1})}, \dots, a_{r-\varrho} s^{\delta_{d_2} - \delta(d_{2,r-\phi})}, 0_{1,\phi}]^T \quad (31)$$

Let a_j, $j = 1, \dots, r-\phi$, be the first non-zero entry in (31). Then replace $m_{2,j}$ by $m'_{2,j}$ in which

$$\begin{aligned} d'_{2,j}(s) &:= D_2(s)a(s) \\ &= [D_{2h}\ \text{diag}\ \{s^{\delta(d_{2,i})}\} + \hat{D}(s)]a(s) \\ &= \underbrace{D_{2h}as^{\delta_{d_2}}}_{=0} + \hat{D}(s)a(s) = \hat{D}(s)a(s) \end{aligned} \quad (32)$$

Note that the leading term in $n'_{2,j}$ is not zero since there exists no a such that $[M_2]_h a = 0$ since $M_2(s)$ is column reduced. Thus we will have

$$\deg d'_{2,j}(s) < \deg n'_{2,j}(s) \Rightarrow \hat{k}_{2,j0} > 0 \quad (33)$$

since $D_{2h}a = 0$ but $N_{2h}a \neq 0$. The new column $m'_{2,j}$, displays an infinite pole of degree $\geq \hat{k}'_{2,j}$.

Step (3) Repeat step (1) until

$$[M_3]_h = \begin{bmatrix} D_{3h} \\ N_{3h} \end{bmatrix}_h := \begin{bmatrix} D'_{3h} & D''_h & 0_{l,\varrho} \\ N'_{3h} & 0_{m,\theta} & N'''_{3h} \end{bmatrix} \quad (34)$$

is the high order coefficient matrix of

$$M_3(s) = \begin{bmatrix} D_3(s) \\ N_3(s) \end{bmatrix} \quad (35)$$

where $(D'_{3h} D''_h)$ has full column rank, i.e.,

$$\text{rank}_{\mathbf{R}}(D'_{3h} D''_h) = \text{rank}_{\mathbf{R}}(D_{2h}) \quad (36)$$

and $\varrho = r - \text{rank}_{\mathbf{R}}(D_{2h})$. (35) now displays complete infinite pole multiplicity. Consider now the infinite zero multiplicity of $G(s)$.

Step (4) For simplicity, re-arrange the columns of $M_3(s)$ such that

$$[M_3]_h = \begin{bmatrix} D_{3h} \\ N_{3h} \end{bmatrix}_h := \begin{bmatrix} D'_{3h} & 0_{l,\varrho} & D''_h \\ N'_{3h} & N'''_{3h} & 0_{m,\theta} \end{bmatrix} \quad (37)$$

Step (5) Repeat step 2 in respect of $[N'_{3h},\ N''_{3h}]$ to obtain

$$[M_4]_h = \begin{bmatrix} D_{4h} \\ N_{4h} \end{bmatrix}_h := \begin{bmatrix} D'_{4h} & 0_{l,\varrho} & D'''_{4h} \\ N'_{4h} & N'''_{3h} & 0_{m,\zeta} \end{bmatrix} \quad (38)$$

as the high order coefficient matrix of

$$M_4(s) = \begin{bmatrix} D_4(s) \\ N_4(s) \end{bmatrix} \quad (39)$$

such that

$$\begin{aligned} \text{rank}_{\mathbf{R}}(D'_{4h} D'''_{4h}) &= \text{rank}_{\mathbf{R}}(D_{2h}) \\ \text{rank}_{\mathbf{R}}(N'_{4h} N'''_{3h}) &= \text{rank}_{\mathbf{R}}(N_{2h}) \end{aligned} \quad (40)$$

and $\zeta = r - \text{rank}_{\mathbf{R}}(N_{2h})$. (39) now displays both complete infinite pole and zero multiplicity information. □

It should be noted that the operations in the above algorithm are non-singular constant transformations on $[M_2]_h$ and thus preserve the column reduceness of $M_2(s)$.Further we may determine the complete infinite pole-zero information of $G(s)$ from $M_4(s)$ by inspection.

Corollary 4. Let $G(s) \in \mathbb{R}^{m\times l}(s)$ with $\text{rank}_{\mathbb{R}(s)} r \leq \min(m,l)$ be a strictly proper transfer function matrix then all CRMFDs of $G(s)$ display complete infinite zero multiplicity information.

Proof. Let $M_1(s)$ be a CRMFD for the strictly proper $G(s)$. By observing the form (39) we can see that $[M_1]_h$ can only have the form

$$[M_1]_h = \begin{bmatrix} D_{1h} \\ N_{1h} \end{bmatrix}_h = \begin{bmatrix} D'_{1h} & D'''_{4h} \\ 0_{m,\mu} & 0_{m,\zeta} \end{bmatrix} \qquad (41)$$

since $G(s)$ only possesses infinite zeros and thus the Smith-MacMillan form at infinity of $G(s)$ will possess no unit elements and will show no infinite poles. Thus $\hat{k}_{1,i} > 0 \ \forall \ i = l-\zeta+1,\ldots,l$ and since $\text{rank}_{\mathbb{R}}(N_{1h}) = 0$, by Corollary 3, $G(s)$ can be said to possess $r = \theta = \zeta$ infinite zeros. □

Conclusions.

This paper has highlighted a form of MFD, the so called CRMFD which carries a number of interesting properties as regards its ability to display certain aspects of the infinite frequency structure of the associated transfer function matrix. Many of these properties are revealed virtually by inspection and so provide satisfying extensions of the scalar case where again this information is immediately available. Further a structured form of a CRMFD (resp. RRMFD) which conveniently displays the complete infinite pole-zero multiplicity information has been developed. CRMFDs have other attributes of this form which will be discussed elsewhere.

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