Solutions of Matrix Diophantique Equations over Rings via Maple

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Abstract

Solutions of matrix diophantique equations over the rings of polynomial, constant, proper, proper and Hurwitz (Shur) stable matrices are determined via symbolic computational algorithms in the symbolic package MAPLE.

1 Introduction

Tedious, repetive and complex calculations are involved in many problems in linear algebra. It is in general beneficial to implement the corresponding problem computationally rather than solving the problem by hand in terms of certain governing factors. Such factors may include the time, accuracy and cost of obtaining such a solution.

The advent of complete sybolic computational packages, such as Maple [2], Mathematica, MACSYMA, Reduce, MuMath and Scratchpad, have made the computational implication of such problems particularly attractive. The main advantages, in regards to linear algebra, lie in the following.

Symbolic Storage - Variables can be stored in an "exact" form (i.e. 1/3 as opposed to 0.33333...) resulting in no loss of accuracy during a calculation. The main disadvantage of uses "exact numbers" are : a) the large size of memory they use, and b) the slow speed they have, such kind of programmes. Thus we can easily see that symbolic computational programmes may be used in case where accuracy is more important than speed and memory. Additionally variables can be left "unassigned" (i.e.) without holding any numerical values) which enables polynomial operations to be defined, say, in an indeterminate s.

Inbuilt Procedures - The existence of hundrends of inbuilt procedures covering both general and certain specialezed areas of mathematics. In Maple, for example, the package *linalg* consist of numerous procedures used in the solution of problems in linear algebra such as smith form and the inverse of a polynomial matrix.

Programming Language - The existence of unique high-level programming languages allowing specific procedures to be written. In Maple over 90% of these inbuilt procedures are written in its own procedural programming language and therefore any of them can be implemented in any new procedures written. They can therefore be viewed upon as *building blocks* for subsequently more specified and advanced procedures.

Perhaps understandably, however, neither one of these symbolic computer packages contain existing procedures or indeed packages for specialized work in the solution of matrix diophantique equations over rings. However they are clearly flexible enough for the implementation of such work to be carried out. This is the motivation of this paper and we present a package of procedures, developed for use in Maple, for solving matrix diophantique equation problems.

2 The Matrix Diophantique Equation Package

Let $\mathbf{R}$ be the field of reals, $\mathbf{R}(s)$ be the set of rational functions in the indeterminate s and coefficients in $\mathbf{R}$, $\mathbf{R}(s)^{p\times m}$ the set of $p\times m$ rational matrices i.e. matrices with elements in $\mathbf{R}(s)$ and consider the right matrix diophantique equation:

$$A(s)X(s)+B(s)Y(s)=C(s) \tag{1}$$

where $A(s)\in\mathbf{R}(s)^{p\times m}$, $B(s)\in\mathbf{R}(s)^{p\times n}$ and $C(s)\in\mathbf{R}(s)^{p\times l}$.

The mathematical synthesis of a control system having a desired property usually leads to the solution of the

above linear matrix diophantique equation where the solution pair $X(s), Y(s)$ is required to have elements in a specific subring of $\mathbf{R}(s)$. Specifically it can be shown that the choice of the ring can be matched with the ultimate goal of the investigation [7], e.g. depending on the type of the problem and whether the setting is continuous or discrete time the solution might be required to have elements in one of the rings:
$\mathbf{R}_p(s)$, $\mathbf{R}_p(z)$ the ring of proper rational matrices in the Laplace operator s or the z transform useful for the study of the impulse free behavior of continuous systems or the causality of decrete time systems,
$\mathbf{R}_H(s)$ and $\mathbf{R}_s(z)$, the ring a) of the proper and Hurwitz-stable rational matrices in the and b) of the proper and Shur-stable rational functions, useful for the BIBO-stability of continuous time and discrete time systems.
$\mathbf{R}_f(z)$, the ring of the proper and finite-expansion rational matrices analytic for every $s \neq 0$, useful for the study of the finite impulse response in discrete time systems and
$\mathbf{R}[s]$ and $\mathbf{R}[z]$ the ring of polynomials in s, analytic for every $s \neq \infty$, useful for the study of the modal properties of continuous-time and discrete-time systems.

Therefore (1) is useful to solve synthesis and design control problems such as stabilization, pole placement, dead-beat control, model matching, disturbance rejection, minimum variance control, LQG or H_2 optimal control, H_∞ optimization, tracking problems etc. [7]. Control problems from the area of time invariant systems, time variant systems, infinite dimensional linear systems, n-D systems and generally nonlinear systems can also be reduced to the problem of the solution of the diophantique equation (1) over a specific ring.

In the following subsections we study the solution of (1) over the ring of rational, polynomial (and constant), proper and proper and Hurwitz-stable (Shur-stable) matrices via the symbolic package MAPLE. The symbolic algorithms used in MAPLE for the investigation of the solution space of (1) over a specific ring are based on the already known algorithms proposed by [1], [4]-[11] etc.. Note also that any left matrix diophantique equation can be reduced to a right one by taking transposes of the right and left term of the equation. Thus we restrict our research to the right diophantique equations. One example from control theory is presented at the end of this paper such that to illustrate one of the proposed procedures. The name of the linear algebra package presented in the following subsections is : *lindioph*.

2.1 General Solutions of Diophantique Equations

To compute the solutions of (1), i.e. X,Y with all its elements in $\mathbf{R}(s)$, we present three procedures, detailed below :

gensolve(A,B,C,X,Y) - general solution of matrix diophantique equations

Parameters : A,B,C are known rational matrices which are involved in the diophantique equation (1) while X and Y are unknown rational matrices which satisfy (1) and are computed through this procedure. The output of this procedure is true if the diophantique equation (1) has a rational solution and false if not.

ginverse(A) - generalised inverse of the rational matrix A(s)

Parameters : A is a rational matrix. The output of this procedure is the generalised inverse of the rational matrix A [3].

iden(n) - construct an n $\times$ n identity matrix.

Parameters : n is a constant. The output of this procedure is an $n \times n$ identity matrix.

2.1.1 Implementation via MAPLE

(i) Read in the linear algebra package contained within MAPLE via >with(linalg):
(ii) Read in the general solve procedures **gensolve**, **ginverse** and **iden(n)** via >with (lindioph)::
(iii) Implement the procedure via >gensolve(A,B,C, X,Y); where A,B,C are the rational matrices defined in equation (1) and X, Y are the unknown matrices to be found in (1).

2.2 Constant Solutions of Diophantine Equations

Some design problems such as the pole assignment problem [6] requires constant solutions for Diophantique equations of the form (1). To find the constant solutions of (1) i.e. X, Y with all its elements in $\mathbf{R}$, we define the procedure **consolve**. This procedure subsequently calls the procedures **arrdeg**, **arrlcm** and **ginverse** (as defined above).

consolve(A,B,C,X,Y) - constant solution of matrix diophantique equations

Parameters : A,B,C are known rational matrices which are involved in the diophantique equation (1) while X and Y are unknown constant matrices which satisfy (1) and are computed through this procedure. The output of this procedure is true if the diophantique equation (1) has a constant solution and false if not.

arrdeg(A) - greatest degree of all the elements of the polynomial matrix A.

Parameters : A is a polynomial matrix. The output of this procedure is the greatest degree of all the elements of A.

arrlcm(A) - least common multiple of all the denominators of the elements of A.

Parameters : A is a rational matrix. The output of this procedure is the the least common multiple of the denominators of all the elements of A.

2.2.1 Implementation via MAPLE

(i) Read in the linear algebra package contained within MAPLE via >with(linalg):
(ii) Read in the constant solve procedures **consolve**, **arrdeg**, **arrlcm** and **ginverse** via >with(lindioph):
(iii) Implement the procedure via >consolve(A,B,C, X,Y); where A,B,C are known rational matrices defined in equation (1) and X, Y are the unknown constant matrices to be found in (1).

2.3 Polynomial Solutions of Diophantique Equations

An algorithm for the polynomial solution of (1) i.e. X, Y with all its elements in $\mathbf{R}[s]$, is given by the procedure pseudocode **polsolve** which subsequently calls the procedures **pansmith** and **arrlcm**.

polsolve(A,B,C,X,Y) - polynomial solutions of matrix diophantique equations

Parameters : A,B and C are known rational matrices embedded in (1) while X and Y are unknown polynomial matrices which satisfy (1) and are computed through this procedure. The output of this procedure is true if the diophantine equation (1) has a polynomial solution and false if not.

pansmith(A,U,V) - the finite smith form of the polynomial matrix A(s).

Parameters : A(s) is a polynomial matrix while U(s) and V(s) are unimodular matrices computed from the mentioned procedure, such that $U(s)A(s)V(s) = S^{C}_{A(s)}(s)$. The output of this procedure is the Smith-form of A(s) at C. It returns also the unimodular matrices $U(s)$ and $V(s)$ which satisfy the following $U(s)A(s)V(s) = S^{C}_{A(s)}(s)$.

arrlcm(A) - least common multiple of all the denominators of the elements of A.

Parameters : A is a rational matrix. The output of this procedure is the the least common multiple of the denominators of all the elements of A.

2.3.1 Implementation via MAPLE

(i) Read in the linear algebra package contained within MAPLE via >with(linalg):
(ii) Read in the constant solve procedures **polsolve**, **pansmith**, and **arrlcm** via >with(lindioph):
(iii) Implement the procedure via >polsolve(A,B,C, X,Y); where A,B,C are the rational matrices defined in equation (1) and X, Y are the unknown polynomial matrices to be found in (1).

2.4 Proper Solutions of Diophantique Equations

An algorithm for the investigation of the proper solutions of (1) i.e. X, Y with all its elements in $\mathbf{R}_p(s)$, is given by a procedure called **prosolve** which subsequently calls the procedures **degr**, **arrmindeg**, **isproper** and **ratsmith** :

prosolve(A,B,C,X,Y) - proper solutions of matrix diophantique equations.

Parameters : A,B and C are known rational matrices embedded in (1) while X and Y are unknown proper matrices which satisfy (1) and are computed through this procedure. The output of this procedure is true if the diophantine equation (1) has a proper solution and false if not.

degr(q) - degree of a rational function

Parameters : q is a rational function. The output of this procedure is the degree of q i.e. if $q(s) = a(s)/b(s)$ where $a(s), b(s) \in R[s]$ then

$$\deg[q(s)] := \left\{ \begin{array}{cc} \deg[b(s)] - \deg[a(s)] & q(s) \neq 0 \\ +\infty & q(s) = 0 \end{array} \right\}$$

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arrmindeg(A) - minimum degree of all the elements of the rational matrix A(s)

Parameters : A(s) is a rational matrix. The output of this procedure is the minimum degree of all elements of A(s).

isproper(A) - check if the rational matrix A(s) is proper

Parameters : A(s) is a rational matrix. The output of this procedure is true if all the elements of A(s) are proper functions and false otherwise.

ratsmith(A,U,V) - smith form at infinity of A(s)

Parameters : A(s) is a rational matrix while U(s) and V(s) are biproper matrices computed from the mentioned procedure, such that $U(s)A(s)V(s) = S^{\infty}_{A(s)}(s)$. The output of this procedure is the Smith-McMillan form of A(s) at $s = \infty$. It returns also the biproper matrices $U(s)$ and $V(s)$ which satisfy the following $U(s)A(s)V(s) = S^{\infty}_{A(s)}(s)$. A theoretical algorithm proposed in [9], [11] was used for the construction of the mentioned procedure.

2.4.1 Implementation via MAPLE

(i) Read in the linear algebra package contained within MAPLE via >with(linalg):
(ii) Read in the constant solve procedures **prosolve**, **ratsmith**, **degr**, and **arrmindeg** via >with(lindioph):
(iii) Implement the procedure via >prosolve(A,B,C, X,Y); where A,B,C are the rational matrices defined in equation (1) and X, Y are the unknown proper matrices to be found in (1).

2.5 Proper and Hurwitz (Shur) Stable Solutions of Diophantique Equations

An algorithm for the investigation of the proper and Hurwitz (Shur) stable solutions of (1) i.e. X, Y with all its elements in $\mathbf{R}_H(s)$ ($\mathbf{R}_S(s)$) is given by a procedure called **stbsolve** which subsequently calls the procedures **degrs**, **makestable**, **is_proper_stable** and **stbsmith**.

stbsolve(A,B,C,X,Y,WhatIs,s) - proper and Hurwitz (Shur) stable solutions of matrix diophantique equations.

Parameters : A,B and C are known rational matrices of s embedded in (1). If WhatIs=0 (WhatIs=1) then X and Y are unknown proper and Hurwitz (Shur) stable matrices which satisfy (1) and are computed through this procedure. The output of this procedure is true if the diophantique equation (1) has a proper and Hurwitz (Shur) stable solution and false if not.

degr(q) - the degree of the rational function q(s).

Parameters : q is a rational function of s. The output of this procedure is the degree of q(s) as defined above.

degrs(q,WhatIs,s) - $\boldsymbol{\delta_\Omega}$[q(s)]

Parameters : q is a known rational function of s. Whatis takes the values 0 and 1 such that to define that Ω is the ring of proper and Hurwitz stable matrices or the ring of proper and Shur stable matrices respectively. The output of the procedure is the degree $\delta_\Omega[q(s)]$ i.e.

$$q(s) = q_\Omega(s)q'(s) \tag{2}$$

where $q_\Omega(s) = n_\Omega(s)/d_\Omega(s)$, $n_\Omega(s), d_\Omega(s) \in R[s]$ are coprime with all their zeros not outside Ω and $q'(s) = n'(s)/d'(s)$, $n'(s), d'(s) \in R[s]$ are coprime with all their zeros outside Ω, then

$$\delta_\Omega[q(s)] := \left\{ \begin{array}{cc} \deg[d'(s)] - \deg[n'(s)] & q(s) \neq 0 \\ +\infty & q(s) = 0 \end{array} \right\} \tag{3}$$

breakf(q,$n_\Omega(s), d_\Omega(s)$,WhatIs) - factorize q as defined in (2)

Parameters : q is a known rational function. $n_\Omega(s)$ and $d_\Omega(s)$ are the numerator and the denominator of $q_\Omega(s)$ if we factorize q(s) in the form of (2). Ω is defined according to values of WhatIs as defined above. The output of this procedure are the rational functions $n_\Omega(s)$ and $d_\Omega(s)$.

divis(pa1,pb1,s,WhatIs) - Euclidian division in the ring of proper and Hurwitz (Shur) stable functions.

Parameters : pa1 and pb1 are known rational functions of s. Ω is defined according to values of WhatIs as defined above. The output of this procedure is the quotient of the Euclidian division over the ring of proper and Hurwitz (Shur) stable functions, of the rational functions pa1(s) and pb1(s).

TheFunc(a,WhatIs) - check if a belongs to Ω^C.

Parameters : a is a complex number.WhatIs has been defined above. The output of this procedure is true if a belongs to Ω^C and false otherwise.

arrmindegs(A,WhatIs) - minimum degree δ_Ω of all the elements of A

Parameters : A is a rational matrix. WhatIs has been defined above. The output of this procedure is the minimum degree $\boldsymbol{\delta_\Omega}$ of all the elements of A.

is_proper_stable(A,WhatIs) - check if A is proper and Hurwitz (Shur) stable

Parameters : A is a rational matrix. WhatIs has been defined above. The output of this procedure is true if A is proper and Hurwitz (Shur) stable and false otherwise.

stbsmith(A,U,V, WhatIs,s)

Parameters : A(s) is a rational matrix while U(s) and V(s) are units of the ring of proper and Hurwitz (Shur) stable matrices, computed from the mentioned procedure and such that $U(s)A(s)V(s) = S^\Omega_{A(s)}(s)$. WhatIs has been defined above. The output of this procedure is the Smith-McMillan form of A(s) at Ω. It returns also the matrices $U(s)$ and $V(s)$ which satisfy the following $U(s)A(s)V(s) = S^\Omega_{A(s)}(s)$. A theoretical algorithm proposed in [8], [11] was used for the construction of the mentioned procedure.

arrdenomstable(A,WhatIs) - least common multiple of the stable parts of the denominators of all the elements of the matrix A.

Parameters : A is a rational matrix of s. WhatIs has been defined above. The output of this procedure is the least common multiple of the stable parts of the denominators of all the elements of the matrix A(s).

2.5.1 Implementation via MAPLE

(i) Read in the linear algebra package contained within MAPLE via >with(linalg):

(ii) Read in the constant solve procedures **stbsolve**, **stbsmith**, e.t.c. via >with(lindioph):
(iii) Implement the procedure via >stbsolve(A,B,C, X,Y,W); where A,B,C are the rational matrices defined in equation (1) and X, Y are the unknown matrices to be found in (1) and W=0 if we are interested for proper and Hurwitz stable solutions while W=1 if we are interested for proper and Shur stable solutions.

3 Illustrated Example

Let $\Sigma(P)$ be a given linear, time invariant whose input-output relation is described by

$$y(s) = \underbrace{\begin{bmatrix} \frac{s+2}{s+1} & \frac{s+3}{s+2} \\ \frac{1}{s+1} & 0 \end{bmatrix}}_{P(s)} u(s)$$

where $u(s), y(s)$ are the Laplace transforms of the input and output vectors respectively and $P(s)$ is the transfer function matrix of $\Sigma(P)$. Consider now a compensation scheme which is described by

$$u(s) = C_p(s)r(s) - C_f(s)y(s)$$

where $C_p(s) \in \mathbf{R}_{pr}(s)^{2\times 1}$ and $C_f(s) \in \mathbf{R}_{pr}(s)^{2\times 2}$ are the transfer functions of the pre and feedback compensators respectively as follows

$r(s)$ is the Laplace transform of an external vector input and let $H(s)$ be the closed loop transfer function matrix $H(s) : r(s) \to y(s)$

$$H(s) = \begin{bmatrix} \frac{s+1}{s+4} \\ -2\frac{1}{(s+2)(s+4)} \end{bmatrix} = P(s)\left(I_2 + C_f(s)P(s)\right)^{-1} C_p(s)$$

Let

$$X(s) = \left(I_2 + C_f(s)P(s)\right)^{-1} C_p(s) \qquad (4)$$

We examine the existence of a proper and Hurwitz stable solution $X(s)$ to the proper rational matrix equation

$$P(s)X(s) + 0Y(s) = H(s)$$

Using the MAPLE procedure **stbsolve** we compute (see Appendix) such $X(s)$

$$X(s) = \begin{pmatrix} -2\frac{s+1}{(s+2)(s+4)} \\ \frac{s+2}{s+4} \end{pmatrix}$$

Solving (4) we have the diophantique equation

$$-C_f(s)[P(s)X(s)] + C_p(s)I_2 = X(s)$$

or equivalently

$$[-X(s)^T P(s)^T]C_f(s)^T + I_1 C_p^T(s) = X(s)^T$$

which can also be solved using the MAPLE procedure **stbsolve** (see Appendix) and gives rise to the proper and Hurwitz stable solutions :

$$C_f(s) = \begin{pmatrix} \frac{2}{s+2} + \frac{Q_{21}s^2+6Q_{21}s+8Q_{21}+2Q_{11}}{(s+1)(s+2)} & Q_{11} \\ -\frac{s+2}{s+1} + \frac{Q_{22}s^2+6Q_{22}s+8Q_{22}+2Q_{12}}{(s+1)(s+2)} & Q_{12} \end{pmatrix}$$

$$C_p(s) = \begin{pmatrix} Q_{21} \\ Q_{22} \end{pmatrix}$$

where Q_{ij} with i=1,2 and j=1,2 are arbitrary proper and Hurwitz stable rational functions.

4 Conclusions.

In this paper we have implemented five algorithms in the symbolic package MAPLE for the construction of the solution space of a right diophantique equation over the ring of rational, constant, polynomial, proper and proper and Hurwitz (Shur) stable matrices. The MAPLE codes can be supplied free by P.Tzekis in the email address Tzekis@olymp.ccf.auth.gr under request. One example arising from control theory has also been considered to show the implementation of one of these procedures in MAPLE. Control synthesis and design problems such as stabilization, pole placement, dead-beat control, model matching e.t.c. will be discussed in further work, using the proposed procedures.

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6 Appendix

Below we illustrate how the package **lindioph** is implemented in a Maple session and call the procedure **stbsolve** to solve the problem in example above.