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Generalized Inverse of Two Variable Polynomial Matrices and Applications

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ABSTRACT

The main contribution of this paper is to present a) an algorithm for the computation of the generalized inverse of a not necessarily square two variable polynomial matrix and its Laurent expansion and b) some applications of the proposed algorithm to the solution of diophantique equations.

I. INTRODUCTION

Consider the 2—D polynomial matrix

$$A(z_1,z_2)=\sum_{i=0}^{q_1}\sum_{j=0}^{q_2} A_{ij}z_1^i z_2^j \in \mathbb{R}[z_1,z_2]^{n\times m} \quad (1.1)$$

where $A_{ij}\in\mathbb{R}^{n\times m}$, $i=0,1,...,q_1$ and $j=0,1,...,q_2$. The problem of the investigation of the generalized inverse of a polynomial matrix with one or more variables has been the concern of many scientists because of the large number of its implications in multivariable system analysis.

The problem of the computation of the inverse of a regular polynomial matrix has been investigated in its general form by [2] while the same problem for regular polynomial matrices with two variables by [4].

In the nonregular case i.e. $n\neq m$ or $n=m$ with $\det[A(z_1,z_2)]\neq 0$, the generalized inverse of constant matrices defined by [7] and a numerical algorithm for the computation of this matrix is later given by [1]. A recent approach for the investigation of the generalised inverse for one variable polynomial matrices has been proposed by [3] and the extension of this attempt in the two variable polynomial matrix case is proposed in this paper.

The interest of this important problem starts from its numerous applications i.e. computation of the transfer function matrix of a system, inverse systems, solution of systems, controllability and observability matrices of general polynomial matrix descriptions (PMDs), solution of diophantique equations which gives rise to numerous applications e.t.c. (see [3]).

The structure of this paper is seperated in four sections. In the first section we give some preliminary results concerning the definition of the generalized inverse, in the second section we present an algorithm for the computation of the generalized inverse of the matrix (1.1), in the third section we compute its Laurent expansion i.e. in case where it is unique, and in the last section we give as an application of these algorithms the solution of the general diophantique equation $AXB=C$ in case where A, B, C are known two variable polynomial matrices and X is the unknown two variable rational matrix which satisfies the prementioned diophantique equation.

2. PRELIMINARY RESULTS

The definition of the generalized inverse of a constant matrix was originally defined by [7] :

Definition 1. For every matrix $A\in\mathbb{R}^{n\times m}$, a unique matrix $A^{\dagger}\in\mathbb{R}^{m\times n}$ exists which is called generalised inverse satisfying the following :

(i) $A\times A^{\dagger}\times A=A$, (ii) $A^{\dagger}\times A\times A^{\dagger}=A^{\dagger}$,

(iii) $(A\times A^{\dagger})^T=A\times A^{\dagger}$, (iv) $(A^{\dagger}\times A)^T=A^{\dagger}\times A$

where A^T denotes the transpose of A. In the special case that the matrix A is a square and nonsingular matrix, the generalised inverse of A is simply its inverse i.e. $A^{\dagger}=A^{-1}$. In cases where now there exist a matrix A^{g_1} which satisfies only the first condition is called {1}—inverse. {1}—inverses are not unique and play an important role in the solution of diophantique equations as we shall see in section 5. □

In analogous way we define the generalized inverse $A(z_1,z_2)^{\dagger}\in\mathbb{R}(z_1,z_2)^{m\times n}$ of $A(z_1,z_2)\in\mathbb{R}[z_1,z_2]^{n\times m}$ as the matrix which satisfies the properties (i)—(iv) of the Definition 1. The uniqueness of this matrix is obvious because the proof of the uniqueness in [7] is independent of the form of the matrix $A(z_1,z_2)$. Consider now the following :

Theorem 2. [1] Let $A\in\mathbb{R}^{n\times m}$ and

$$a(s):=\det[sI_n-C]=a_0s^n+a_1s^{n-1}+\cdots+a_n$$

with $a_0=1$, be the characteristic polynomial of the product of A and its transpose A^T i.e. $C=A\times A^T$. If $k\neq 0$ is the largest integer such that $a_k\neq 0$, then the generalized inverse of A is given by

$$A^{\dagger} := -a_k^{-1} \times A^T \times [C^{k-1} + a_1 C^{k-2} + \cdots + a_{k-1} I_n]$$

else $(k=0)$ $A^{\dagger}=0$. □

A numerical algorithm for the implementation of the above theorem is given by [1] :

Algorithm 3. (Computation of the generalized inverse of a constant matrix $A \in \mathbb{R}^{n\times m}$)

Step 1. Let $A \in \mathbb{R}^{n\times m}$ and $C = A \times A^T$. Consider the sequences $\{a_0, a_1, \ldots, a_n\}$, $\{B_0, B_1, \ldots, B_n\}$ constructed by the following way

$$B_0 = I_n$$

$$B_1 = C \times B_0 + a_1 I_n \qquad a_1 = -\frac{tr[C \times B_0]}{1}$$

$$B_2 = C \times B_1 + a_2 I_n \qquad a_2 = -\frac{tr[C \times B_1]}{2}$$

$$\cdots\cdots\cdots\cdots$$

$$B_n = C \times B_{n-1} + a_n I_n \qquad a_n = -\frac{tr[C \times B_{n-1}]}{n}$$

Step 2. If $k \neq 0$ is the largest integer such that $a_k \neq 0$, then the generalized inverse of A is given by

$$A^{\dagger} := -a_k^{-1} \times A^T \times B_{k-1}$$

else $(k=0)$ $A^{\dagger}=0$. □

3. GENERALIZED INVERSE OF A TWO VARIABLE POLYNOMIAL MATRIX.

Consider the two variable polynomial matrix (1.1) and its transpose

$$A(z_1,z_2)^T = \sum_{i=0}^{q_1} \sum_{i=0}^{q_2} A^T_{ij} z_1^i z_2^j \in \mathbb{R}[z_1,z_2]^{n\times m} \quad (3.1)$$

Following similar lines with Theorem 2 we can easily show the following :

Theorem 4. Let $A(z_1,z_2) \in \mathbb{R}[z_1,z_2]^{n\times m}$ and

$$a(s,z_1,z_2) = \det[sI_n - A(z_1,z_2) \times A(z_1,z_2)^T] = \quad (3.2)$$
$$= a_0(z_1,z_2)s^n + a_1(z_1,z_2)s^{n-1} + \cdots + a_n(z_1,z_2)$$

with $a_0(z_1,z_2)=1$, be the characteristic polynomial of $A(z_1,z_2) \times A(z_1,z_2)^T$. If $k \neq 0$ is the largest integer such that $a_k(z_1,z_2) \neq 0$ for $(z_1,z_2) \in \Lambda (\neq \{\emptyset\}) \subseteq \mathbb{R}^2$, then the generalized inverse of $A(z_1,z_2)$ for $(z_1,z_2) \in \Lambda$ is

$$A(z_1,z_2)^{\dagger} := -a_k(z_1,z_2)^{-1} \times A(z_1,z_2)^T \times$$
$$[(A(z_1,z_2) \times A(z_1,z_2)^T)^{k-1} + \cdots + a_{k-1}(z_1,z_2) I_n]$$

If $k=0$ is the largest integer such that $a_k(z_1,z_2) \neq 0$, then $A(z_1,z_2)^{\dagger}=0$. For those $(z_1,z_2) \in \mathbb{R}^2 - \Lambda$ we use the same algorithm.

Proof. It can be readily seen that the proof is the same with this of Theorem 2 because it is independent of the variables of the matrix $A(z_1,z_2)$. □

A numerical algorithm for the implementation of the above theorem is given similar to the one presented in Algorithm 3 (the proofs are exactly the same).

Algorithm 5. (Computation of the generalized inverse of $A(z_1,z_2) \in \mathbb{R}[z_1,z_2]^{n\times m}$)

Step 1. Let $C(z_1,z_2) = A(z_1,z_2) \times A(z_1,z_2)^T$. Consider the sequences $\{a_0(z_1,z_2), \ldots, a_n(z_1,z_2)\}$, $\{B_0(z_1,z_2), \ldots, B_n(z_1,z_2)\}$ constructed by the following way

$$B_0(z_1,z_2) = I_n$$
$$B_1(z_1,z_2) = C(z_1,z_2) \times B_0(z_1,z_2) + a_1(z_1,z_2) I_n$$
$$B_2(z_1,z_2) = C(z_1,z_2) B_1(z_1,z_2) + a_2(z_1,z_2) I_n$$
$$\cdots\cdots\cdots\cdots \quad (3.3a)$$
$$B_n(z_1,z_2) = C(z_1,z_2) B_{n-1}(z_1,z_2) + a_n(z_1,z_2) I_n$$

and

$$a_1(z_1,z_2) = -\frac{tr[C(z_1,z_2) B_0(z_1,z_2)]}{1}$$
$$a_2(z_1,z_2) = -\frac{tr[C(z_1,z_2) B_1(z_1,z_2)]}{2}$$
$$\cdots\cdots\cdots \quad (3.3b)$$
$$a_n(z_1,z_2) = -\frac{tr[C(z_1,z_2) B_{n-1}(z_1,z_2)]}{n}$$

Step 2. If $k \neq 0$ is the largest integer such that $a_k(z_1,z_2) \neq 0$ for $(z_1,z_2) \in \Lambda (\neq \{\emptyset\}) \subseteq \mathbb{R}^2$, then the generalized inverse of $A(z_1,z_2)$ for $(z_1,z_2) \in \Lambda$ is

$$A(z_1,z_2)^{\dagger} := -a_k(z_1,z_2)^{-1} A(z_1,z_2)^T B_{k-1}(z_1,z_2) \quad (3.4)$$

If $k=0$ is the largest integer such that $a_k(z_1,z_2) \neq 0$, then $A(z_1,z_2)^{\dagger}=0$. For those $(z_1,z_2) \in \mathbb{R}^2 - \Lambda$ we use the same algorithm. □

It is seen from (3.3) that :

$$a_i(z_1,z_2) = \sum_{j_1=0}^{2iq_1} \sum_{j_2=0}^{2iq_2} \hat{a}_{i,j_1,j_2} z_1^{j_1} z_2^{j_2} \quad (3.5)$$

for $i=0,1,\ldots,n$ and

$$B_i(z_1,z_2) = \sum_{j_1=0}^{2iq_1} \sum_{j_2=0}^{2iq_2} \hat{B}_{i,j_1,j_2} z_1^{j_1} z_2^{j_2} \quad (3.6)$$

for $i=0,1,\ldots,n-1$, where $\hat{B}_{i,j_1,j_2}$, $\hat{a}_{i,j_1,j_2}$ are constant coefficient matrix and scalar of the powers $z_1^{j_1} z_2^{j_2}$. It is seen from Algorithm 5 that for the computation of the generalized inverse of $A(z_1,z_2)$ we need the integer k, the quantities $a_k(z_1,z_2)$ and $B_{k-1}(z_1,z_2)$ i.e. the coefficients

$\hat{a}_{k,j_1,j_2}$ and the coefficient matrices $\hat{B}_{k-1,j_1,j_2}$, defined by :

$$a_k(z_1,z_2) = \sum_{j_1=0}^{2kq_1} \sum_{j_2=0}^{2kq_2} \hat{a}_{k,j_1,j_2} z_1^{j_1} z_2^{j_2} \quad (3.7)$$

and

$$B_{k-1}(z_1,z_2) = \sum_{j_1=0}^{2(k-1)q_1} \sum_{j_2=0}^{2(k-1)q_2} \hat{B}_{k-1,j_1,j_2} z_1^{j_1} z_2^{j_2} \quad (3.8)$$

Now taking into account that :

$$A(z_1,z_2) \times A(z_1,z_2)^T \times B_i(z_1,z_2) = \quad (3.9)$$

$$\sum_{i_1=0}^{2(i+1)q_1} \sum_{i_2=0}^{2(i+1)q_2} \left[\sum_{m_1=0}^{i_1} \sum_{m_2=0}^{i_2} \left[\sum_{n_1=0}^{i_1-m_1} \sum_{n_2=0}^{i_2-m_2} A_{i_1-m_1-n_1,i_2-m_2-n_2} A^T_{n_1,n_2} \right] \hat{B}_{i,m_1,m_2} \right] z_1^{i_1} z_2^{i_2}$$

and substituting (3.5), (3.6) and (3.9) in the recursive relations (3.3), we obtain the following recursive relations that determine $\hat{a}_{i+1,j_1,j_2}$ and $\hat{B}_{i+1,j_1,j_2}$ for $j_z=0,1,\dots,2(i+1)q_z$ and $z=1,2$.

Algorithm 6. (Generalized inverse $A(z_1,z_2)^\dagger$ of $A(z_1,z_2)$)

<u>Initial Conditions</u>

$$B_{0,0,0} = I_n \quad (3.10)$$

<u>Boundary Conditions</u>

$$B_{0,j_1,j_2} = 0 \quad \forall\ j_z>0\ z=1,2 \quad (3.11)$$

$$B_{i,j_1,j_2} = 0 \quad j_z=2iq_z+1,\dots,2(n-1)q_z \quad z=1,2 \quad (3.12)$$

<u>Recursive Relations for $a_i(z_1,z_2)$</u>

$$\hat{a}_{i+1,j_1,j_2} = -\frac{1}{i+1} \times \mathrm{tr}\left\{ \left[\sum_{m_1=0}^{i_1} \sum_{m_2=0}^{i_2} \left[\sum_{n_1=0}^{i_1-m_1} \sum_{n_2=0}^{i_2-m_2} A_{i_1-m_1-n_1,i_2-m_2-n_2} A^T_{n_1,n_2} \right] \hat{B}_{i,m_1,m_2} \right] \right\}$$

$$i_z=0,1,\dots,2(i+1)q_z,\ z=1,2$$

$$i=0,1,\dots,n-1 \quad (3.13)$$

<u>Recursive Relation for $B_i(z_1,z_2)$</u>

$$\hat{B}_{i+1,j_1,j_2} = \left[\sum_{m_1=0}^{i_1} \sum_{m_2=0}^{i_2} \left[\sum_{n_1=0}^{i_1-m_1} \sum_{n_2=0}^{i_2-m_2} A_{i_1-m_1-n_1,i_2-m_2-n_2} A^T_{n_1,n_2} \right] \hat{B}_{i,m_1,m_2} \right] +$$

$$+ \hat{a}_{i+1,i_1,i_2} I_n$$

$$i_z=0,1,\dots,2(i+1)q_z,\ z=1,2$$

$$i=0,1,\dots,n-2 \quad (3.14)$$

<u>Terminate</u>

FIND k : $a_{k+1}(z_1,z_2)=\cdots=a_n(z_1,z_2)=0$

or $\hat{a}_{k+1,i_1,i_2}=\cdots=\hat{a}_{n,i_1,i_2}=0\ \forall i_z \in \mathbb{N}$

then

$$B_{i_1,i_2} = \hat{B}_{k-1,i_1,i_2} \quad i_z=0,1,\dots,2(k-1)q_z\ z=1,2$$

$$a_{i_1,i_2} = a_{k,i_1,i_2} \quad i_z=0,1,\dots,2kq_z\ z=1,2 \quad (3.15)$$

<u>OUTPUT</u>

$$A(z_1,z_2)^\dagger = -\left[\sum_{i_1=0}^{2kq_1} \sum_{i_2=0}^{2kq_2} a_{i_1,i_2} z_1^{i_1} z_2^{i_2} \right]^{-1} \times$$

$$\times \left[\sum_{i_1=0}^{(2k-1)q_1} \sum_{i_2=0}^{(2k-1)q_2} \left[\sum_{m_1=0}^{i_1} \sum_{m_2=0}^{i_2} A^T_{i_1-m_1,i_2-m_2} B_{m_1,m_2} \right] z_1^{i_1} z_2^{i_2} \right] \quad (3.16) \quad \square$$

It's readily seen that the generalized inversion algorithm is a three-dimensional algorithm since it depends of three independent variables i, i_1, i_2. Note also that a) we use the same algorithm for $(z_1,z_2) \in \mathbb{R}^2 - \Lambda$ i.e. $a_k(z_1,z_2)=0$, by finding another i such that $a_i(z_1,z_2) \neq 0$ for those $(z_1,z_2) \in \mathbb{R}^2 - \Lambda$ and b) in case where $k=0$ is the largest integer such that $a_k(z_1,z_2) \neq 0$ then $A(z_1,z_2)^\dagger = 0$.

4. EVALUATION OF THE LAURENT EXPANSION

It is known that the Laurent expansion of a rational matrix of two variables is not always unique [8]. For this reason we separate this section into two parts. In the first part we give a necessary condition such that a unique Laurent expansion about infinity of $A(z_1,z_2)^\dagger$ exists, while in the second part we give an algorithm for the computation of this expansion when this condition is satisfied.

For a two variable polynomial $p(z_1,z_2)$ define $\deg_{z_1}[p(z_1,z_2)]$ as the degree of $p(z_1,z_2)$ in z_1 and similarly define $\deg_{z_2}[p(z_1,z_2)]$. Denote also $\deg_z[p(z_1,z_2)] := \deg[p(z,z)]$. Then define :

$$r_i := \deg_{z_i}[a_k(z_1,z_2)] \quad i=1,2$$

$$f_i := \deg_{z_i}[B_{k-1}(z_1,z_2)] \quad i=1,2$$

$$r := \deg_z[a_k(z,z)] \quad (4.1)$$

Then following similar lines with [6, Theorem 1] we give the following

Theorem 7. Suppose that $r = r_1 + r_2$, i.e. $a_{r_1,r_2} \neq 0$. Then the Laurent expansion about infinity, convergent for large enough $|z_1|$ and $|z_2|$, of $A(z_1,z_2)^{\dagger}$ is unique and is given by

$$A(z_1,z_2)^{\dagger} = \sum_{i_1=v_1}^{-\infty} \sum_{i_2=v_2}^{-\infty} H_{i_1,i_2} z_1^{i_1} z_2^{i_2} \quad (4.2)$$

where $v_i := \deg_{z_i}[A(z_1,z_2)^{\dagger}]$, $i=1,2$ are finite and have some minimum value defined by

$$v_i = f_i + q_i - r_i \quad i=1,2 \qquad (4.3) \; \square$$

In case where $a_{r_1,r_2}=0$, the Laurent expansion of $A(z_1,z_2)^{\dagger}$ may not be unique as it has been shown in [4], [8]. The computation of the matrix coefficient terms H_{i_1,i_2} in (4.2), for the case where $a_{r_1,r_2} \neq 0$ is presented in the sequel. From (3.4)

$$a_k(z_1,z_2)A(z_1,z_2)^{\dagger} = -A(z_1,z_2)^T B_{k-1}(z_1,z_2) \quad (4.4)$$

Substituting $a_k(z_1,z_2)$, $B_{k-1}(z_1,z_2)$, $A(z_1,z_2)^{\dagger}$ and $A(z_1,z_2)^T$ from (3.7), (3.8), (4.2) and (3.1) respectively in (4.4) we obtain

$$\left[\sum_{i_1=0}^{2kq_1} \sum_{i_2=0}^{2kq_2} a_{i_1,i_2} z_1^{i_1} z_2^{i_2}\right] \times \left[\sum_{i_1=v_1}^{-\infty} \sum_{i_2=v_2}^{-\infty} H_{i_1,i_2} z_1^{i_1} z_2^{i_2}\right] = \sum_{i_1=0}^{2(k-1)q_1} \sum_{i_2=0}^{2(k-1)q_2} \left[-\sum_{j_1=0}^{q_1} \sum_{j_2=0}^{q_2} A^T_{i_1-j_1,i_2-j_2} B_{j_1,j_2}\right] z_1^{i_1} z_2^{i_2} \equiv \quad (4.5)$$

$$\equiv \sum_{i_1=0}^{2(k-1)q_1} \sum_{i_2=0}^{2(k-1)q_2} \bar{B}_{i_1,i_2} z_1^{i_1} z_2^{i_2}$$

Equating the coefficient matrices of the corresponding power of $z_1^{i_1} z_2^{i_2}$, on both sides of the resulting equation yields

$$\bar{B}_{i_1,i_2} = \sum_{j_1=0}^{r_1} \sum_{j_2=0}^{r_2} a_{j_1,j_2} H_{i_1-j_1,i_2-j_2} \quad (4.6a)$$

for $(0,0) \le (i_1,i_2) \le (f_1,f_2)$ and

$$0 = \sum_{j_1=0}^{r_1} \sum_{j_2=0}^{r_2} a_{j_1,j_2} H_{i_1-j_1,i_2-j_2} \quad (4.6b)$$

for $\{i_1 < -r_1 \wedge i_2 \le v_2\} \vee \{i_1 \le v_1 \wedge i_2 < -r_2\}$.

An algorithm which summarizes how to find the solution H_{i_1,i_2} for $(-\infty,-\infty) < (i_1,i_2) \le (v_1,v_2)$ of (4.6) is given by the following

Algorithm 8. (Laurent expansion of $A(z_1,z_2)^{\dagger}$)

Step 1. Compute $f_1, f_2, r_1, r_2, r, v_1, v_2$.

Step 2. If $r \neq r_1+r_2$ then Stop.

Step 3. $d_0 = \dfrac{1}{a_{r_1,r_2}}$

For $i=1$ to f_1

$$d_i = -\frac{1}{a_{r_1,r_2}} \sum_{j=0}^{i-1} [a_{r_1-i+j,r_2} \times d_j]$$

Next i

Step 4. For $i=-r_1$ to v_1

$$H_{i,v_2} = -\sum_{j=0}^{v_1-i} \left[\left[\sum_{j_1=0}^{r_1+i+j} \sum_{j_2=0}^{f_2} A^T_{r_1+i+j-j_1,f_2-j_2} B_{j_1,j_2}\right] \times d_j\right]$$

Next i

Step 5.
$$D_0 = \begin{bmatrix} d_0 I_n & 0 & \cdots & 0 \\ d_1 I_n & d_0 I_n & \cdots & 0 \\ \vdots & \vdots & \ddots & \\ d_{f_1} I_n & d_{f_1-1} I_n & \cdots & d_0 I_n \end{bmatrix}$$

$$P_i = \begin{bmatrix} a_{k,r_1,i} I_n & 0 & & & \\ a_{k,r_1-1,i} I_n & a_{k,r_1,i} I & & & 0 \\ \vdots & \vdots & \ddots & & \\ a_{k,0,i} I_n & a_{k,1,i} I_n & & \ddots & \\ 0 & a_{k,0,i} I_n & & & \\ & & \ddots & & \\ & 0 & & a_{k,0,i} \cdots & a_{k,r_1,i} I_n \end{bmatrix}$$

For $i=1$ to f_2

$$D_i = -\left[\sum_{j=0}^{i-1} D_j P_{r_2-i+j}\right] D_0$$

Next i

Step 6. $\bar{B}_{i_1,i_2} := -\sum_{j_1=0}^{q_1} \sum_{j_2=0}^{q_2} A^T_{i_1-j_1,i_2-j_2} B_{j_1,j_2}$

For $i=-r_2$ to v_2

$$\begin{bmatrix} H_{v_1,i} \\ H_{v_1-1,i} \\ \vdots \\ H_{-r_1,i} \end{bmatrix} = \sum_{j=0}^{v_2-i} D_j \begin{bmatrix} \bar{B}_{f_1,r_2+i-j} \\ \bar{B}_{f_1-1,r_2+i-j} \\ \vdots \\ \bar{B}_{0,r_2+i-j} \end{bmatrix}$$

Next i

Step 7. $\{i_1 < -r_1 \wedge i_2 \leq v_2\} \vee \{i_1 \leq v_1 \wedge i_2 < -r_2\}$

$$H_{i_1,i_2} = -\frac{1}{a_{r_1,r_2}} \left[\sum_{j_1=0}^{r_1} \sum_{j_2=0}^{r_2} a_{j_1,j_2} H_{i_1+r_1-j_1,\, i_2+r_2-j_2} \right]_{(j_1,j_2)\neq(r_1,r_2)}$$

Step 8. $$A(z_1,z_2)^{\dagger} = \sum_{i_1=v_1}^{-\infty} \sum_{i_2=v_2}^{-\infty} H_{i_1,i_2} z_1^{i_1} z_2^{i_2}$$

END

5. IMPLICATIONS OF THE GENERALIZED INVERSE IN LINEAR SYSTEM THEORY.

One of the important applications of the computation of the generalized inverse is in the solution of the equation $AXB=C$ presented by [7]. The above problem may be extended to the case of two variable polynomial matrices as we shall see in the sequel.

Theorem 9. [7] A necessary and sufficient condition for the matrix equation $AXB=C$ to have a solution is that $AA^{\dagger}CB^{\dagger}B=C$, in which case the general solution is

$$X=A^{\dagger}CB^{\dagger}+Y-A^{\dagger}AYBB^{\dagger} \qquad (5.1)$$

where $A^{\dagger}$, $B^{\dagger}$ are the generalized inverses of A and B respectively and Y is arbitrary to within having the dimension of X. □

It can be readily seen from [7] that the proof of the above theorem is independent of the variable existence in the matrix A and thus we can state the following

Theorem 10. A necessary and sufficient condition for the matrix equation $A(z_1,z_2)X(z_1,z_2)B(z_1,z_2)=C(z_1,z_2)$ to have a solution is that $A(z_1,z_2)\times A(z_1,z_2)^{\dagger}C(z_1,z_2)B(z_1,z_2)^{\dagger}B(z_1,z_2)=C(z_1,z_2)$ $\forall(z_1,z_2)\in\mathbb{R}^2$, in which case the general solution is

$$X(z_1,z_2)=A(z_1,z_2)^{\dagger}C(z_1,z_2)B(z_1,z_2)^{\dagger}+Y(z_1,z_2)- A(z_1,z_2)^{\dagger}A(z_1,z_2)Y(z_1,z_2)B(z_1,z_2)B(z_1,z_2)^{\dagger} \quad (5.2)$$

where $A(z_1,z_2)^{\dagger}$, $B(z_1,z_2)^{\dagger}$ are the generalized inverses of $A(z_1,z_2)$ and $B(z_1,z_2)$ respecti[vel]y and $Y(z_1,z_2)$ is arbitrary to within having the dimension of $X(z_1,z_2)$. □

We have to note here that the above theorem remains the same if we substitute the generalized inverses $A(z_1,z_2)^{\dagger}$, $B(z_1,z_2)^{\dagger}$ with the $\{1\}$-inverses of $A(z_1,z_2)$ and $B(z_1,z_2)$ respectively.

An interesting application of Theorem 10 is the investigation of the solution space of the diophantique equation

$$A(z_1,z_2)X(z_1,z_2)+B(z_1,z_2)Y(z_1,z_2)=C(z_1,z_2) \quad (5.3)$$

where $A(z_1,z_2)$, $B(z_1,z_2)$ and $C(z_1,z_2)$ are known two variable polynomial matrices and $X(z_1,z_2)$, $Y(z_1,z_2)$ are unknown two variable rational matrices.

Theorem 11. A necessary and sufficient condition for (5.3) to have a solution is that

$$[A(z_1,z_2)\ B(z_1,z_2)]\times[A(z_1,z_2)\ B(z_1,z_2)]^{\dagger}\times C(z_1,z_2) = C(z_1,z_2) \qquad (5.4)$$

$\forall(z_1,z_2)\in\mathbb{R}^2$ in which case the general solution is

$$\begin{bmatrix} X(z_1,z_2) \\ Y(z_1,z_2) \end{bmatrix} = [A(z_1,z_2)\ B(z_1,z_2)]^{\dagger}C(z_1,z_2) + \begin{bmatrix} X_0(z_1,z_2) \\ Y_0(z_1,z_2) \end{bmatrix} - [A(z_1,z_2)\ B(z_1,z_2)]^{\dagger}\times$$
$$\times[A(z_1,z_2)\ B(z_1,z_2)]\begin{bmatrix} X_0(z_1,z_2) \\ Y_0(z_1,z_2) \end{bmatrix} \qquad (5.5)$$

where $[X_0(z_1,z_2)^T\ Y_0(z_1,z_2)^T]^T$ is an arbitrary rational matrix of two variables and $[A(z_1,z_2)\ B(z_1,z_2)]^{\dagger}$ is the generalized inverse of the compound matrix $[A(z_1,z_2)\ B(z_1,z_2)]$.

Proof. The proof is a direct application of Theorem 10 if we take into account that relation (5.3) may be rewritten as

$$[A(z_1,z_2)\ B(z_1,z_2)]\begin{bmatrix} X(z_1,z_2) \\ Y(z_1,z_2) \end{bmatrix} = C(z_1,z_2) \quad (5.6)$$

which is a special case of the relation $AXB=C$. □

The investigation of the solution space of (5.3) plays an important role in problems of 1—D linear systems as like as parameterization of stabilizing controllers, robust stabilization, disturbance rejection, reference tracking, model matching, H_2—optimal control e.t.c. (see [3]). Here we examine the model matching problem in 2—D case.

Consider an open loop system S_1 with transfer function matrix $G(z_1,z_2)\in\mathbb{R}(z_1,z_2)^{n\times m}$

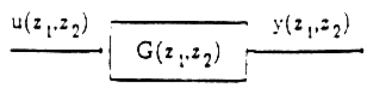

Diagram 1. Open loop system.

We would like to find out when there exists an output feedback of the form

$$u(z_1,z_2) = -F(z_1,z_2)y(z_1,z_2)+v(z_1,z_2) \quad (5.7)$$

with $F(z_1,z_2)\in\mathbb{R}(z_1,z_2)^{m\times n}$ such that the closed loop system

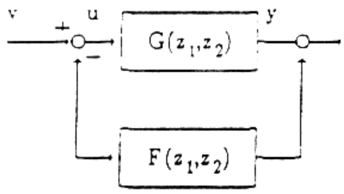

Diagram 2. Closed loop system.

has transfer function $H(z_1,z_2)\in\mathbb{R}(z_1,z_2)^{n\times m}$. We would like therefore to find out the rational matrix $F(z_1,z_2)\in\mathbb{R}(z_1,z_2)^{m\times n}$ which satisfies the following equation

$$H(z_1,z_2)=(I_n+G(z_1,z_2)F(z_1,z_2))^{-1}G(z_1,z_2)\Leftrightarrow$$

$$G(z_1,z_2)F(z_1,z_2)H(z_1,z_2)=G(z_1,z_2)-H(z_1,z_2) \quad (5.8)$$

Let $G(z_1,z_2)=\tilde{G}(z_1,z_2)/g(z_1,z_2)$ where $\tilde{G}(z_1,z_2)\in \mathbb{R}[z_1,z_2]^{n\times m}$ and $g(z_1,z_2)$ is the least common multiple (lcm) of all the denominators of the matrix $G(z_1,z_2)$. In the same way, let $H(z_1,z_2)=\tilde{H}(z_1,z_2)/h(z_1,z_2)$. Then (5.8) may be rewritten as

$$\tilde{G}(z_1,z_2)F(z_1,z_2)\tilde{H}(z_1,z_2)= \quad (5.9)$$

$$=\tilde{G}(z_1,z_2)h(z_1,z_2)-\tilde{H}(z_1,z_2)g(z_1,z_2)$$

In the light of Theorem 10 we can easily obtain the following

Theorem 12. A necessary and sufficient condition for the equation (5.9) to have a solution is that

$$\tilde{G}(z_1,z_2)\tilde{G}(z_1,z_2)^{\dagger}[\tilde{G}(z_1,z_2)h(z_1,z_2)-$$

$$-\tilde{H}(z_1,z_2)g(z_1,z_2)]\tilde{H}(z_1,z_2)^{\dagger}\tilde{H}(z_1,z_2)=$$

$$=\tilde{G}(z_1,z_2)h(z_1,z_2)-\tilde{H}(z_1,z_2)g(z_1,z_2) \quad (5.10)$$

in which case the general solution is

$$F(z_1,z_2)=\tilde{G}(z_1,z_2)^{\dagger}[\tilde{G}(z_1,z_2)h(z_1,z_2)-$$

$$-\tilde{H}(z_1,z_2)g(z_1,z_2)]\tilde{H}(z_1,z_2)^{\dagger}+Y(z_1,z_2)-$$

$$-\tilde{G}(z_1,z_2)^{\dagger}\tilde{G}(z_1,z_2)Y(z_1,z_2)\tilde{H}(z_1,z_2)\tilde{H}(z_1,z_2)^{\dagger}$$

where $Y(z_1,z_2)$ is arbitrary to within having the dimension of $F(z_1,z_2)$ and $\tilde{G}(z_1,z_2)^{\dagger}$, $\tilde{H}(z_1,z_2)^{\dagger}$ are the generalized inverses of $\tilde{G}(z_1,z_2)$ and $\tilde{H}(z_1,z_2)$ respectively.

Proof. Let $A(z_1,z_2)=\tilde{G}(z_1,z_2)$, $B(z_1,z_2)=\tilde{H}(z_1,z_2)$, $F(z_1,z_2)=X(z_1,z_2)$ and $C(z_1,z_2)=\tilde{G}(z_1,z_2)h(z_1,z_2)-\tilde{H}(z_1,z_2)g(z_1,z_2)$ in Theorem 10. Then the proof of Theorem 12 follows.

6. CONCLUSIONS

A three dimensional algorithm is determined for the computation of the generalized inverse of a two variable polynomial matrix in terms of its coefficient matrices and its Laurent expansion has also been evaluated under some specified conditions. The whole theory has been illustrated via examples in Multidimensional Systems Theory i.e. solution of diophantique equations and the model matching problem.

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