Solution of an ARMA-Representation via its Boundary Mapping Equation.

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ABSTRACT

Background

Consider the regular, discrete–time, AutoRegressive Moving Average (ARMA)–Representation described by the matrix equation

$$A(\sigma)y(k) = B(\sigma)u(k) \tag{1}$$

where σ denotes the backwards shift operator $\big($i.e. $\sigma^i y(k) = y(k+i)\big)$, $A(\sigma) = A_0 + A_1\sigma + \ldots + A_q\sigma^q \in \mathbb{R}[\sigma]^{r\times r}$, $\big(\text{rank } A(\sigma) = r\big)$, $B(\sigma) = B_0 + B_1\sigma + \ldots + B_q\sigma^q \in \mathbb{R}[\sigma]^{r\times m}$ where at least one of A_q, B_q is non–zero, $y(k) : \mathbb{Z}^+ \to \mathbb{R}^r$ defines the output and $u(k) : \mathbb{Z}^+ \to \mathbb{R}^m$ defines the input of the system.

In the case where $A(\sigma) = \sigma E - A \in \mathbb{R}[\sigma]^{r\times r}$ and $B(\sigma) = B \in \mathbb{R}^{r\times m}$ the ARMA-Representation (1) reduces to the Generalized State Space(GSS)–Representation

$$Ey(k+1) = Ay(k) + Bu(k) \tag{2}$$

Further if E is non–singular (i.e. $|E| \neq 0$) then (2) reduces to the State Space Representation.

Consider a solution to the system (1) (resp. (2)), over the index set $[0, N]$. Such a solution is represented in the form $[y(0), y(1), \ldots, y(N-1), y(N)]$.

The solution of (2) has been considered by many authors [3]–[7]. In Luenberger[6] the conditions under which (2) possessed a solution were discussed for both the time-invariant and time-variant cases and the notions of *solvability* and *conditionability* were introduced. Following [6] we say,

Definition 1 *(Solvability)*
The system (1) (resp.(2)) is *solvable* in case for any input sequence $[u(0), u(1), \ldots, u(N)]$ there exists *at least one* sequence $[y(0), y(1) \ldots, y(N)]$ which solves the system (1) (resp.(2)) □

In general if a system is solvable its solution will not be unique i.e. there will exist a set of solutions. To specify a unique solution additional constraints must be specified and applied to the system. It is natural to apply these additional constraints at the boundary points of the interval concerned. If these additional constraints are specified at intermediate points in the interval, and not at the boundary points, then they can be regarded as being independent of the conditions at either end. In this respect they can be considered as partly redundant. We therefore have the following definition[6].

Definition 2 *(Conditionability)*
The system (1) (resp.(2)) is *conditionable* in case a unique solution $[y(0), y(1) \dots, y(N)]$ exists for any input sequence $[u(0), u(1) \dots, u(N)]$ by the specification of additional restrictions at the boundary points in the interval. $\square$

It is noted in [6] that for the regular case of (2) (i.e. $|sE - A| \neq 0$) the system (2) is always solvable and conditionable. This can be shown to be true for the ARMA-Representation (1) if $A(\sigma)$ is regular and this will be shown in the paper.

To obtain a solution $[y(0), y(1) \dots, y(N)]$ to (2) Luenberger [6] noted that the system can be completely characterized by all the pairs $\big(y(0),\ y(N)\big)$ which satisfy (2). These can be regarded as the *admissible boundary conditions* of (2) and are represented in the form

$$Z_0 y(0) + Z_N y(N) = C \tag{3}$$

for some matrices Z_0, Z_N, C. (3) is referred to as the *boundary mapping equation* for (2).

In cases where (3) does not give a *unique* boundary pair $\big(y(0),\ y(N)\big)$, and hence not a unique solution, additional constraints can be applied to the system in the form of an *auxiliary equation*

$$\hat{Z}_0 y(0) + \hat{Z}_N y(N) = \hat{C} \tag{4}$$

Combining (3) and (4) results in

$$\underbrace{\begin{pmatrix} Z_0 & Z_N \\ \hat{Z}_0 & \hat{Z}_N \end{pmatrix}}_{\mathcal{Z}} \underbrace{\begin{pmatrix} y(0) \\ y(N) \end{pmatrix}}_{\mathcal{Y}} = \underbrace{\begin{pmatrix} C \\ \hat{C} \end{pmatrix}}_{\mathcal{C}} \tag{5}$$

Hence, a unique boundary pair $\big(y(0),\ y(N)\big)$, and hence a unique solution, will be produced iff $\mathcal{Z}\mathcal{Z}^{\dagger}\mathcal{C} = \mathcal{C}$ where $\mathcal{Z}^{\dagger}$ denotes the generalized inverse of Z.

Luenberger [7] gave a recursive procedure to solve (3) from which a solution to (2) was directly found. Another approach to form the boundary mapping equation, and hence obtain a solution, was presented by Karamancioglu [3]. This method can be seen to be superior to that of [7] in that it easily extends to the case of 2-D systems.

Contribution

In this paper we extend the method of [3] to the case of regular ARMA-Representations as given in (1). For this case we compute a boundary mapping equation of the form

$$Z_0 \underline{Y}_{in} + Z_N \underline{Y}_{fin} = C \tag{6}$$

where $\underline{Y}_{in}^T = \big(y(0)^T, \dots, y(q-1)^T\big)^T \in \mathbb{R}^{qr}$, $\underline{Y}_{fin}^T = \big(y(N-q+1)^T, \dots, y(N)^T\big)^T \in \mathbb{R}^{qr}$.

This is done as follows. Consider the system (1) where we assume the input is known throughout the range $[0, N]$. Taking $\mathcal{Z}$-transforms of (1), assuming $\hat{y}(z) = \mathcal{Z}\big[y(k)\big] = \sum_{i=0}^{N} y(i) z^{-i}$, $\hat{u}(z) = \mathcal{Z}\big[u(k)\big] = \sum_{i=0}^{N} u(i) z^{-i}$, it can be shown that the following relation holds

$$
\begin{aligned}
A(z)\hat{y}(z) &= [z^q I_r \dots z I_r] \underbrace{\begin{pmatrix} A_q & \cdots & 0 \\ \vdots & \ddots & \vdots \\ A_1 & \cdots & A_q \end{pmatrix}}_{\hat{A}} \underbrace{\begin{pmatrix} y(0) \\ \vdots \\ y(q-1) \end{pmatrix}}_{\hat{Y}} \\
&+ \left[z^{-N} I_r \dots z^{-N+q-1} I_r\right] \underbrace{\begin{pmatrix} A_0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ A_{q-1} & \cdots & A_0 \end{pmatrix}}_{\tilde{A}} \underbrace{\begin{pmatrix} y(N) \\ \vdots \\ y(N-q+1) \end{pmatrix}}_{\tilde{Y}} \\
&+ \left[I_r \dots z^{-N+q} I_r\right] \underbrace{\begin{pmatrix} B_0 & \cdots & B_q & & 0 \\ & \ddots & & \ddots & \\ 0 & & B_0 & \cdots & B_q \end{pmatrix}}_{B_u} \underbrace{\begin{pmatrix} u(0) \\ \vdots \\ u(N) \end{pmatrix}}_{\underline{U}} \stackrel{\text{def}}{=} Q(z)
\end{aligned}
\tag{7}
$$

where I_r denotes the $(r \times r)$ identity matrix

From [3]

$$
A(z)^{-1} = \frac{H(z)}{d(z)} \tag{8}
$$

where

$$
H(z) = \sum_{i=0}^{n-1} H_i z^i \in \mathbb{R}[z]^{r \times r}, \quad d(z) = \sum_{i=0}^{n} d_i z^i \in \mathbb{R} \tag{9}
$$

where at least one of H_{n-1}, d_n is non–zero. This inversion can be easily implemented through the numerical algorithm proposed by [2] and consequently the coefficient matrices H_i and scalars d_i may be readily obtained. Combining (7) and (8)

$$
\hat{y}(z) = \frac{H(z)}{d(z)} Q(z) \Rightarrow \left[d(z) I_r\right] \hat{y}(z) = H(z) Q(z) \tag{10}
$$

Expanding both sides of (10) and equating powers of z we can obtain the matrix equation

$$
\left[D \otimes I_r\right] \underline{Y} = \left(\hat{R}\hat{A}\hat{Y} + \tilde{R}\tilde{A}\tilde{Y} + R_u B_u \underline{U}\right) \tag{11}
$$

where $\otimes$ denotes the Kronecker product, the matrices $\hat{A}, \tilde{A}, B_u$ and vectors $\hat{Y}, \tilde{Y}, \underline{U}$ are as defined in (7) and

$$
\tilde{R} = \begin{pmatrix} & O_{(N+1)r,qr} & \\ H_{n-1} & & \\ \vdots & \ddots & \\ H_0 & & H_{n-1} \\ & \ddots & \vdots \\ & & H_0 \end{pmatrix}, \quad R_u = \begin{pmatrix} & O_{qr,(N-q+1)r} & \\ H_{n-1} & & \\ \vdots & \ddots & \\ H_0 & & H_{n-1} \\ & \ddots & \vdots \\ & & H_0 \\ & 0_{qr,(N-q+1)r} & \end{pmatrix}
$$

$$
D = \begin{pmatrix}
 & & O_{q-1,N+1} & & \\
d_n & & & & \\
\vdots & \ddots & & & \\
d_s & & \ddots & & \\
\vdots & \ddots & & \ddots & \\
d_0 & & \ddots & & d_n \\
 & \ddots & & \ddots & \vdots \\
 & & \ddots & & d_s \\
 & & & \ddots & \vdots \\
 & & & & d_0
\end{pmatrix}, \quad
\hat{R} = \begin{pmatrix}
H_{n-1} & & \\
\vdots & \ddots & \\
H_0 & & H_{n-1} \\
 & \ddots & \vdots \\
 & & H_0 \\
 & O_{(N+1)r,qr} &
\end{pmatrix}
$$

$$
\underline{Y}^T = \left(y(0)^T, \ldots, y(N)^T\right)^T \tag{12}
$$

where d_s is defined as the first *non-zero* element in the first column of the matrix D as represented in (12). Consider a non-singular matrix $P \in \mathbb{R}^{(N+n+q)\times(N+n+q)}$ such that

$$
PD = \begin{pmatrix} 0_{(q+n-s-1),(N+1)} \\ I_{(N+1)} \\ 0_{s,(N+1)} \end{pmatrix} \tag{13}
$$

Such a matrix P can be selected to have the form

$$
P = \left(\begin{array}{c|ccccc}
I_{\beta,\beta} & & & O_{\beta,\alpha} & & \\
\hline
 & a_0 & & & & \\
 & a_1 & a_0 & & & \\
0_{\alpha,\beta} & \vdots & & \ddots & & \\
 & \vdots & & & a_0 & \\
 & a_{N+s} & \cdots & \cdots & a_1 & a_0
\end{array}\right) \tag{14}
$$

where $\alpha = (N+s+1), \beta = (n+q-s-1)$ and

$$
a_k = \begin{cases}
\dfrac{1}{d_s} & , \quad k = 0 \\[2mm]
\dfrac{1}{d_s} \displaystyle\sum_{i=1}^{k} \left[-a_{(k-i)} d_{(s-i)}\right] & , \quad 1 \le k \le s \\[2mm]
\dfrac{1}{d_s} \displaystyle\sum_{i=1}^{s} \left[-a_{(k-i)} d_{(s-i)}\right] & , \quad s+1 \le k \le N+s
\end{cases} \tag{15}
$$

By multiplying both sides of (11) by the matrix $\left(P \otimes I_r\right)$ results in

$$
\begin{pmatrix} 0_{(q+n-s-1)r,1} \\ y(0) \\ \vdots \\ y(N) \\ 0_{sr,1} \end{pmatrix} = \left(P \otimes I_r\right)\left(\hat{R}\hat{A}\hat{Y} + \tilde{R}\tilde{A}\tilde{Y} + R_u B_u \underline{U}\right) \in \mathbb{R}^{(N+n+q)} \tag{16}
$$

i.e. the l.h.s of (16) is explicitly in terms of the solution vectors $y(i),\ i = 0, \ldots, N$. The selection of the matrix P in (14) will be futher discussed in the paper.

From (16) the boundary mapping equation can be immediately realized and a admissible set of boundary conditions found as follows. It is clear that the first $(2q + n - s - 1)r$ and last $(s + q)r$ equations only involve the boundary conditions $\underline{Y}_{in}$ and $\underline{Y}_{fin}$. Hence we can extract these $(3q + n - 1)r$ equations to form the boundary mapping equation as given in (6) where $Z_0, Z_N \in \mathbb{R}^{(3q+n-1)r \times qr}$ and $C \in \mathbb{R}^{(N+n+q)}$. To define a unique solution an auxiliary equation, of the form (4), can be appended to the boundary mapping equation formed. This is always the case for a regular ARMA–Representation as it is always conditionable (see paper).

Once an admissible set of boundary conditions have been computed, via (6), the r.h.s of (14) is known. Hence the intermediate states $y(q), \ldots, y(N - q)$ can be immediately determined *in any order*.

Subsequently algorithms for the construction of the boundary mapping equation and solving of the system are presented in the paper. These algorithms are especially attractive for implementation via a language such as MAPLE [1] due to their high dependence on matrix manipulation techniques. MAPLE is a symbolic computational package which contains such inbuilt matrix procedures. These combine to form the linear algebra package `linalg`. Subsequently, in the paper, we present such computer procedures, based on the aforementioned algorithms, for the computation of the boundary mapping equation and the corresponding solution of the system.

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