A Characterisation of Admissibility of the Initial Conditions of Non-Regular AR-Representations

N. P. Karampetakis[†], S. Mahmood[†], A. C. Pugh[†] & G. E. Hayton[♮]

[†]Mathematical Sciences Department
Loughborough University, Loughborough, LE11 3TU, U.K.
A.C.Pugh@Lboro.ac.uk

[♮]Faculty of Information & Engineering Systems
Leeds Metropolitan University, Leeds, LS1 3HE, U.K.
g.e.taylor@lmu.ac.uk

Abstract

Admissibility of the initial conditions for AR-representations is considered. It is shown to exist as a concept provided a specific view of what constitutes the solution space is taken.

1. Introduction

Consider a system of linear homogeneous differential and algebraic equations described in matrix form by

$$\mathcal{T}(\rho)\xi(t) = 0 \qquad (1)$$

where $\mathcal{T}(\rho) = \mathcal{T}_0 + \mathcal{T}_1\rho + \cdots + \mathcal{T}_q\rho^q \in \mathbb{R}[\rho]^{p\times m}$ with $\mathrm{rank}_{\mathbb{R}(s)}\mathcal{T}(s) = r$, $\rho := d/dt$, and $\xi(t) : [0-, +\infty) \to \mathbb{R}^m$. Following the terminology of [5] (1) is called an *Auto Regressive (AR-) representation* of (behaviour) $\mathcal{B}(\mathcal{T})$, where $\mathcal{B}(\mathcal{T})$ is the distribution (smooth and impulsive) solution space of (1) defined by

$$\mathcal{B}(\mathcal{T}) := \{\xi(t) : (1) \text{ is satisfied } \forall\, t \in [0-, +\infty)\} \qquad (2)$$

In the regular case where $p = m = r$, i.e., $\mathcal{T}(\rho)$ is square and nonsingular, the initial conditions of the AR-represenation (1) which can be viewed as "admissible", from the point of view that they uniquely characterize a solution, have been defined by [9]. An extension of these results to non-regular AR-represenations, i.e., where $\mathcal{T}(\rho) \in \mathbb{R}[\rho]^{p\times m}$ is square (i.e., $p = m$) and $\det|\mathcal{T}(\rho)| \neq 0$ or where $\mathcal{T}(\rho) \in \mathbb{R}[\rho]^{p\times m}$ is non-square (i.e., $p \neq m$) is proposed here. It will be seen that "admissibility" is to be interpreted in the sense that there is a one-to-one correspondence between the admissible initial conditions and the *solution class space* rather than $\mathcal{B}(\mathcal{T})$.

2. Preliminaries

Taking Laplace transforms of (1) gives

$$\mathcal{T}(s)\hat{\xi}(s) = \mathcal{S}_{q-1}\mathcal{X}_{\mathcal{T}}\hat{\xi}(0-) := [s^{q-1}I_p, \ldots, I_p]\begin{bmatrix} \mathcal{T}_q & \ldots & 0 \\ \vdots & \ddots & \vdots \\ \mathcal{T}_1 & \ldots & \mathcal{T}_q \end{bmatrix}\begin{bmatrix} \xi(0-) \\ \vdots \\ \xi^{(q-1)}(0-) \end{bmatrix} \qquad (3)$$

We call $\hat{\xi}(0-)$ the *initial value* vector and $\mathcal{X}_{\mathcal{T}}\hat{\xi}(0-)$ the *initial condition* vector of the AR-representation (1). It is easily seen from (3) that the initial condition vector $\mathcal{X}_{\mathcal{T}}\hat{\xi}(0-)$ uniquely specifies the right hand side of equation (3) because $[s^{q-1}I_p, \ldots, I_p]$ is always an isomorphism. Thus the solution of the system (3) is in some sense determined by the initial condition vector $\mathcal{X}_{\mathcal{T}}\hat{\xi}(0-)$, and the precise manner in which this occurs is considered in the sequel.

In the regular case, i.e., where $\mathcal{T}(s)$ is square and nonsingular, (3) and thus (1) always has a solution for the given initial condition. However in the non-regular case (3) and thus (1) does not necessarily have a solution even in a distributional sense. A necessary and sufficient condition for solvability is

Theorem 1. ([2]). Let $v_i(s) := v_{i,0} + \cdots + v_{i,\eta_i}s^{\eta_i}$ $i = r = 1, \ldots, p$ be the polynomial vectors in a left minimal basis of $\mathcal{T}(s)$. Then (1) has a solution iff the initial condition vector $\mathcal{X}_{\mathcal{T}}\hat{\xi}(0-)$ (1) satisfies

$$V\mathcal{X}_{\mathcal{T}}\hat{\xi}(0-) = 0, \;\; V = [V_{r+1}^t, \ldots, V_p^t]^t$$
$$V_i = \begin{bmatrix} v_{i,0} & \cdots & v_{i,q-1} \\ \vdots & \ddots & \vdots \\ 0 & \cdots & v_{i,0} \end{bmatrix} \qquad (4)\ \square$$

(4) specifies necessary and sufficient constraints (due to the left minimal indices of $\mathcal{T}(s)$) on the initial condition vectors for the existence of a solution of the AR-representation (1). The initial conditions $\mathcal{X}_{\mathcal{T}}\hat{\xi}(0-)$ satisfying (4) will be called *consistent.*

Corollary 1. The dimension of the space of consistent initial conditions is

$$\delta_{\mathrm{M}}(\mathcal{T}) - \varpi$$

where $\delta_{\mathrm{M}}(\mathcal{T})$ is the MacMillan degree, and ϖ the sum of the left minimal indices, of $\mathcal{T}(s)$. $\square$

We choose to designate initial conditions as being *admissible* in case to each such initial condition there corresponds a unique solution of the AR-representation (1). Because of the non-regularity of (1) such unique "solutions" cannot be elements of $\mathcal{B}(\mathcal{T})$. We must therefore determine excatly in what sense initial conditions can be admissible. Now if (4) holds and $\hat{\xi}(s)$ satisfies (3) then $\xi(t) =: \mathcal{L}^{-1}[\hat{\xi}(s)] \in \mathcal{B}(\mathcal{T})$ is given by

$$\xi(t) := \xi_0(t) + z(t) = \mathcal{L}^{-1}\left[\mathcal{T}(s)^+\mathcal{S}_{q-1}\mathcal{X}_{\mathcal{T}}\hat{\xi}(0-)\right] + \mathcal{L}^{-1}\left[\left(I_m - \mathcal{T}(s)^+\mathcal{T}(s)\right)\hat{z}(s)\right] \qquad (5)$$

where $\mathcal{T}(s)^+$ denotes the generalised inverse of $\mathcal{T}(s)$, and $\hat{z}(s)$ is an arbitrary rational vector. Thus for specific consistent initial condition we may have an arbitrary number of vectors $\xi(t)$ which satisfy (1) due to the appearance of the arbitrary function $\hat{z}(s)$. A second candidate for the solution space of (1) is a partition of $\mathcal{B}(\mathcal{T})$ which is constructed as follows

From [2] the solutions of (1) under zero initial conditions define a space of the following form:

$$Z := \left\{ z(t) : z(t) := \sum_{i=1}^{m-r} \int_0^t \hat{u}_{r+i}(\tau) z_i(t-\tau)\mathrm{d}\tau \right\} \qquad (6)$$

where $\{\hat{u}_{r+1}(s), \hat{u}_{r+2}(s), \ldots, \hat{u}_m(s)\}$ is a right minimal basis of $\mathcal{T}(s)$, $z_1(t), z_2(t), \ldots, z_{m-r}(t)$ are arbitrary functions and $\hat{u}_{r+i}(s) = \mathcal{L}[\hat{u}_{r+i}(t)]$ for $i = 1, \ldots, m-r$. Define the equivalence relation $R \subseteq \mathcal{B}(\mathcal{T}) \times \mathcal{B}(\mathcal{T})$ between the solutions of (1) as

$$R = \{(\xi_1(t), \xi_2(t)) \mid \xi_1(t) - \xi_2(t) \in Z\} \qquad (7)$$

The *equivalence class* of the element $\xi(t) \in \mathcal{B}(\mathcal{T})$ is

$$\begin{aligned} [\xi(t)] : &= \{\xi_1(t) \in \mathcal{B}(\mathcal{T}) \mid (\xi(t), \xi_1(t)) \in R\} \\ &= \{\xi(t)\} \oplus Z \qquad (8) \\ &:= \{\xi(t) + z(t) \mid \xi(t) \in \mathcal{B}(\mathcal{T}),\ z(t) \in Z\} \end{aligned}$$

It is obvious that $[0] = Z$. Thus we view Z as the "zero solution", i.e., the equivalence class corresponding to the zero initial conditions. In the case where $\mathcal{T}(s)$ is regular then every equivalence class contains just one element, while in the non-regular case each equivalence class contains an arbitrary number of elements of $\mathcal{B}(\mathcal{T})$.Thus $\mathcal{B}(\mathcal{T})$ can be divided into equivalence classes defined by (8). The resulting space is denoted $\mathcal{B}/Z$and termed the *solution class* space of (1).

Theorem 2. ([2]). The *solution class* space $\mathcal{B}/Z$ is a finite dimensional vector space with dimension

$$\dim \mathcal{B}/Z = \mathrm{f} = \upsilon + \psi + \epsilon$$

where υ, ψ, ϵ are respectively the total number of finite zeros, infinite zeros and right minimal indices (orders accounted for) of $\mathcal{T}(s)$. f is called the *generalized order* of the AR-representation (1). □

Because of the finite dimensionality of the solution class space $\mathcal{B}/Z$ we may take this as a candidate for the solution vector space of the AR-representation (1).

3. The Admissible Initial Conditions

The question of what constitutes the initial condition set for the AR-representation (1) has been covered by [1] in the case $\mathcal{T}(s)$ is in pencil form, and by [3] for the regular case of a general polynomial matrix $\mathcal{T}(s)$. From (3) we can easily see that the polynomial vector $\mathcal{S}_{q-1}\mathcal{X}_{\mathcal{T}}\hat{\xi}(0-)$, contains all the information on the initial conditions $\mathcal{X}_{\mathcal{T}}\hat{\xi}(0-)$, relating to (1), but what actually constitutes the admissible initial condition set necessary to determine a unique "solution" is the subject of the following result.

Theorem 3. The set of consistent initial conditions

$$\Delta = \{\mathcal{X}_{\mathcal{T}}\hat{\xi}(0-) \mid \ V\mathcal{X}_{\mathcal{T}}\hat{\xi}(0-) = 0\}$$

is isomorphic to the solution class space $\mathcal{B}/Z$ of (1).

Proof. Suppose $\mathcal{X}_{\mathcal{T}}\hat{\xi}_1(0-), \mathcal{X}_{\mathcal{T}}\hat{\xi}_2(0-)$ are two consistent initial conditions which give rise to the same solution class $\mathcal{B}/Z$, or equivalently from (7) to solutions $\xi_1(t), \xi_2(t)$ respectively such that $\xi_1(t) - \xi_2(t) \in Z$, i.e.,

$$\mathcal{T}(s)(\hat{\xi}_1(s) - \hat{\xi}_2(s)) = 0$$

We have that

$$\mathcal{T}(s)\hat{\xi}_i(s) = \mathcal{S}_{q-1}\mathcal{X}_{\mathcal{T}}\hat{\xi}_i(0-),\ i = 1, 2$$

Substituting for i and subtracting the resulting equations yields

$$\mathcal{T}(s)(\hat{\xi}_1(s) - \hat{\xi}_2(s)) = \mathcal{S}_{q-1}\mathcal{X}_{\mathcal{T}}\Big(\hat{\xi}_1(0-) - \hat{\xi}_2(0-)\Big)$$

$$\overset{(27)}{\Rightarrow} 0 = \mathcal{S}_{q-1}\mathcal{X}_{\mathcal{T}}\Big(\hat{\xi}_1(0-) - \hat{\xi}_2(0-)\Big)$$

Now the kernel of the polynomial matrix $\mathcal{S}_{q-1}$ is exactly the zero vector, and so

$$\mathcal{X}_{\mathcal{T}}\hat{\xi}_1(0-) = \mathcal{X}_{\mathcal{T}}\hat{\xi}_2(0-)$$

which is contradiction and the theorem follows. □

Thus the admissible initial conditions are precisely the consistent initial conditions and they are admissible in the sense that each such condition corresponds to a unique solution class of $\mathcal{B}/Z$. In this sense $\mathcal{B}/Z$ is established as a candidate for the solution space of (1). Furthermore it can be seen that $\dim \mathcal{B}/Z = \dim \Delta$ and so from Theorems 2 & 3 and Corollary 1 we have

Corollary 2. If $\upsilon, \psi, \epsilon, \varpi$ are the respective sums of the finite zeros, infinite zeros, right minimal indices and left minimal indices (orders accounted for) of $\mathcal{T}(s)$ then

$$\underbrace{\upsilon + \psi + \epsilon}_{\dim \mathcal{B}/Z} = \underbrace{\delta_{\mathrm{M}}(\mathcal{T}) - \varpi}_{\dim \Delta} \ \square$$

The algebraic result in Corollary 2 is attributed to [4] and Theorem 3 provides a nice dynamic interpretation.

4. Conclusions

The question of what constitutes the admissible initial conditions, in the sense of characterising those initial conditions which determine a unique "solution", has been addressed in the case of non-regular AR-representations. It was seen that admissibilty exists as a concept provided we view the solution class space $\mathcal{B}/Z$ as the actual solution space of the AR-representation.

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