FORWARD, BACKWARD AND SYMMETRIC SOLUTIONS OF DISCRETE TIME ARMA-REPRESENTATIONS*

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Abstract

The main objective of this paper is to determine a closed formula for the forward, backward and symmetric solution of a general discrete time AutoRegressive Moving Average (ARMA) Representation. The importance of the above formula is that it is easily implemented in a computer algorithm and gives rise to the solution of analysis, synthesis and design problems.

Keywords : Linear systems, discrete time, numerical methods.

1 Introduction

Consider a nonhomogeneous system of linear difference and algebraic equations described in matrix form by

$$A(\sigma)y(k) = B(\sigma)u(k) \tag{1}$$

where σ denotes the backwards shift operator *i.e.* $\sigma^i y(k) = y(k+i)$,

$$\begin{array}{c} A(\sigma) = A_0 + A_1\sigma + \cdots + A_q\sigma^q \in \mathbf{R}[\sigma]^{r\times r} \\ B(\sigma) = B_0 + B_1\sigma + \cdots + B_q\sigma^q \in \mathbf{R}[\sigma]^{r\times m} \end{array} \tag{2}$$

where $\text{rank}_{\mathbf{R}(\sigma)}A(\sigma) = r$, at least one of A_q, B_q is nonzero, $y(k) : \mathbf{Z}^+ \to \mathbf{R}^r$ be the *output* of the system and $u(k) : \mathbf{Z}^+ \to \mathbf{R}^m$ be the *input* of the system. Following the terminology of [16] we call the set of equations (1) an ARMA representation of B, where B is the solution space of equations (1) defined by

$$B = \pi_y(B_f) \tag{3}$$

with

$$\begin{array}{c} B_f := \{\begin{pmatrix} y(k) & u(k) \end{pmatrix} : \mathbf{Z}^+ \to \mathbf{R}^r \times \mathbf{R}^m \mid \\ \mid \text{(ARMA) is satisfied } \forall k \in Z^+\} \\ \pi_y : \mathbf{R}^r \times \mathbf{R}^m \to \mathbf{R}^r \ \pi_y\begin{pmatrix} y(k) & u(k) \end{pmatrix} = y(k) \end{array} \tag{4}$$

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In case where $A(\sigma) = \sigma E - A \in \mathbf{R}[\sigma]^{r\times r}$ and $B(\sigma) = B \in \mathbf{R}^{r\times m}$ then the ARMA representation (1) is the known generalised state space representation

$$Ex(k+1) = Ax(k) + Bu(k) \tag{5}$$

while in case where $\det[E] \neq 0$, (5) is the known state space representation. For a survey of singular systems of the form (5) see [7].

ARMA representations of the form (1) find numerous applications in analysis of circuits [12], neural networks [2], economics (Leontieff model [9]), power systems [14].

In case where $\det[A_q] \neq 0$ then the solution of the ARMA-representation (5) is trivial while in the more general case where A_q is a singular matrix many different techniques ([1], [6], [8], [10], [13], [15], [11]), have been applied for the solution of (5) and among them we distinguish [8], [11]. This technique gives a solution of (5) in terms of the forward fundamental matrix ϕ_k and the backward fundamental matrix τ_k of $(zE-A)^{-1}$. Following similar lines with [8], [11] we produce in Section 3 a closed formula for the forward, backward and symmetric solution of the general ARMA-representation (1) in terms now of the fundamental matrix H_k and the backward fundamental matrix V_k of $A(s)^{-1}$. A generalized Leverrier technique for computing the forward fundamental matrix H_k is available [3], so that we may assume that this fundamental matrix is given. We shall show in Section 2 that the backward fundamental matrix is the forward fundamental matrix of the dual polynomial matrix $\tilde{A}(w) = A_0w^q + A_1w^{q-1} + \cdots + A_q$ of $A(s)$ and thus we may assume that V_k is also given.

2 Preliminary Results

We are concerned with the discrete time ARMA-representation (1) where $y(k) \in \mathbf{R}^r$, $u(k) \in \mathbf{R}^m$, $k \in [0,N]$ and $u(k)$ is nonzero for $k = 0, 1, ..., N-q$. We assume that $A(z)$ is *regular i.e.* $\det[A(z)] \neq 0$. Given regularity the Laurent series expansion about infinity for the resolvent matrix exists and is given by

$$A(z)^{-1} = H_{\hat{q}_r} z^{\hat{q}_r} + H_{\hat{q}_r - 1} z^{\hat{q}_r - 1} + \cdots \tag{6}$$

where $\hat{q}_r$ is the greatest order of the zeros of $A(z)$ at $z = \infty$ and the sequence $\{H_k\}$ is known as the *forward fundamental matrix* [3]. The Laurent expansion about zero of $A(z)^{-1}$ is

$$A(z)^{-1} = V_{-\ell} z^{-\ell} + V_{-\ell+1} z^{-\ell+1} + \cdots \tag{7}$$

where the sequence $\{V_k\}$ is known [8] as the *backward fundamental matrix.*

The Laurent expansion about zero of $A(z)^{-1}$ given in (7) is related with the Laurent expansion about infinity given in (6) of the inverse of the dual matrix $\tilde{A}(w) = A_0 w^q + A_1 w^{q-1} + \cdots + A_q$ of $A(z)$ as we can see in the following

Lemma 1 *Let the Laurent expansion about infinity of* $\tilde{A}(w)^{-1}$ *is*

$$\tilde{A}(w)^{-1} = \tilde{H}_f w^f + \tilde{H}_{f-1} w^{f-1} + \cdots \tag{8}$$

and (7) is the Laurent expansion about zero of $A(z)^{-1}$. *Then*

$$q + f = \ell \text{ and } \quad V_{-i} = \tilde{H}_{-\ell+f+i} \;\; i = \ell, \ell-1, .., -1, .. \tag{9}$$

Proof. We have that

$$\begin{array}{l} A(z) = z^q \tilde{A}\left(\frac{1}{z}\right) \Leftrightarrow A(z)^{-1} = z^{-q} \tilde{A}\left(\frac{1}{z}\right)^{-1} \overset{(8)}{\Leftrightarrow} \\ A(z)^{-1} = z^{-q}\left[\tilde{H}_f z^{-f} + \tilde{H}_{f-1} z^{-f+1} + \cdots\right] \\ \quad = \tilde{H}_f z^{-q-f} + \tilde{H}_{f-1} z^{-q-f+1} + \cdots \\ \quad \equiv V_{-\ell} z^{-\mu} + V_{-\ell+1} z^{-\mu+1} + \cdots \end{array} \tag{10}$$

Equating the coefficients of the powers of z we obtain the proof of Lemma 1. ∎

A direct result from Lemma 1 is that the Leverrier algorithm in [3] may be used for the computation both of the forward and backward fundamental matrix. An interesting result which connects the solutions of the ARMA-representation (1) and the ones of the dual discrete time ARMA-representation :

$$\begin{array}{c} A_q \tilde{y}(k) + A_{q-1} \tilde{y}(k+1) + \cdots + A_0 \tilde{y}(k+q) = \\ B_q \tilde{u}(k) + B_{q-1} \tilde{u}(k+1) + \cdots + B_0 \tilde{u}(k+q) \end{array} \tag{11}$$

in the closed interval $[0, N]$ is given by

Theorem 2 (a) *If* $\tilde{y}(k)$ *is a solution of equation (11) for the nonzero input* $\tilde{u}(k)$ *then the sequence* $y(k) = \tilde{y}(N-k)$ *is a solution of the dual equation (1) for the nonzero input* $u(k) = \tilde{u}(N-k)$.

(b) *If* $y(k)$ *is a solution of equation (1) for the nonzero input* $u(k)$ *then the sequence* $\tilde{y}(k) = y(N-k)$ *is a solution of the dual equation (11) for the nonzero input* $\tilde{u}(k) = u(N-k)$.

Proof. a) Let $\tilde{y}(k)$ be a solution of (11). This implies that (11) is satisfied. Now consider the equation (1). If we set $y(k) = \tilde{y}(N-k)$ and $u(k) = \tilde{u}(N-k)$ and take into account that $y(k+j) = \tilde{y}(N-(k+j))$, $u(k+j) = \tilde{u}(N-(k+j))$, $j = 0, 1, .., q$ we have

$$\begin{array}{c} A(\sigma)\tilde{y}(N-k) = \sum\limits_{i=0}^{q} A_i \tilde{y}(N-k-i) \overset{(11)}{=} \\ = \sum\limits_{i=0}^{q} B_i \tilde{u}(N-k-i) \overset{u(k)=\tilde{u}(N-k)}{=} B(\sigma)\tilde{u}(N-k) \end{array} \tag{12}$$

b) Following the same way we can show the second part of the Theorem. ∎

A direct result from the above theorem is that the backward solution of the ARMA representation (1) comes directly from the forward solution of the dual ARMA representation (11).

3 Solutions of ARMA - Representations

There are three different interpretations of (1) [8] :

(i) We may consider that the initial conditions $\{y(0), y(1), ..., y(q-1)\}$ are given and that is desired to determine $y(k)$ in a *forward* fashion from the input sequence and the previous values of the output.

(ii) We may consider that the final conditions $\{y(N), y(N-1), ..., y(N-q+1)\}$ are given and that is desired to determine $y(k)$ in a *backward* fashion from the input sequence and the future values of the output.

(iii) We may consider (1) as a relationship between the inputs and outputs *i.e.* economics and thus no causality is assumed. It is desired to determine $y(k)$ for the values of $k \in [q, N-q]$, in terms of the input sequence and the initial and final conditions. We could call this the *symmetric* solution of (1).

3.1 The Forward Solution of ARMA-Representations

Consider the discrete time ARMA- representation (1) where $A(z)$ is *regular i.e.* $\det[A(z)] \neq 0$ and the Laurent series expansion about infinity for the resolvent matrix exists and is given by (6). Then we have

Theorem 3 *The forward solution of (1) will be :*

$$\begin{array}{c} y(k) = \sum\limits_{i=1}^{q} \sum\limits_{j=1}^{q} H_{-k-i} A_j y(j-i) + \sum\limits_{i=0}^{k+\hat{q}_r} \sum\limits_{j=0}^{q} H_{-k+i} B_j u(i+j) \\ k = q, q+1, .. \end{array} \tag{13}$$

Proof. Equating the coefficients of the powers of z in the relation $A(z) \times A(z)^{-1} = I_r$ we have that :

$$\sum_{n=0}^{q} A_n H_{i-n} = \delta_i I_r \text{ or } \sum_{n=0}^{q} H_{i-n} A_n = \delta_i I_r \tag{14}$$

where $\delta_i = 0$ for $i \neq 0$ and $\delta_0 = 1$. Now substituting $y(k)$ from (13) in (1) and using (14) we observe that (1) is satisfied. ∎

From the above formula we can see that the discrete time ARMA-representations have no impulsive terms in their responses in contrast to the continuous time ARMA-representations. Another difference is that the discrete time ARMA-representations do not always have a solution. A necessary and sufficient condition such that the ARMA-representation (1) has a solution is that the initial conditions $\{y(0), y(1), ..., y(q-1)\}$ satisfies the relation (1) for $k = 0, 1, .., q-1$. Therefore we define :

Definition 1 *We define as*

$$H_{iu} := \{y(i),\ u(i)\ (i = 0, 1, ..., q-1) :$$

$$\tilde{A}_1 \begin{bmatrix} H_0 & \cdots & H_{q-1} \\ H_{-1} & \cdots & H_{q-2} \\ \vdots & \ddots & \vdots \\ H_{-q+1} & \cdots & H_0 \end{bmatrix} \tilde{A}_1 \begin{bmatrix} y(0) \\ y(1) \\ \vdots \\ y(q-1) \end{bmatrix} =$$

$$= \tilde{A}_1 \begin{bmatrix} H_0 & \cdots & H_{\hat{q}_r} & 0 & \cdots & 0 \\ H_{-1} & \cdots & H_{\hat{q}_r-1} & H_{\hat{q}_r} & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ H_{-q+1} & \cdots & H_{\hat{q}_r-q+1} & H_{\hat{q}_r-q+2} & \cdots & H_{\hat{q}_r} \end{bmatrix}$$

$$\begin{bmatrix} B_0 & \cdots & B_q & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & B_0 & B_1 & \cdots & B_q \end{bmatrix} \begin{bmatrix} u(0) \\ u(1) \\ \vdots \\ u\left(2q+\hat{q}_r-1\right) \end{bmatrix} \} \tag{15}$$

where

$$\tilde{A}_1 = \begin{bmatrix} A_0 & 0 & \cdots & 0 \\ A_1 & A_0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_{q-1} & A_{q-2} & \cdots & A_0 \end{bmatrix} \tag{16}$$

the *admissible initial condition space* of (1) under nonzero inputs.

Proof. Consider the relation (1) for $k = 0, 1, ..., q-1$ and substitute the values $y(q)$, $y(q+1)$, ..., $y(2q-1)$ with the respective formula of (13) and use of (14) give us that the initial conditions $\{y(0), y(1), ..., y(q-1)\}$ satisfy the system iff the relation (15) is satisfied. ∎

As we can see in (13) the solution of (1) is determined in terms of the initial conditions $\{y(0), y(1), ..., y(q-1)\}$ and the input sequences of the system. An obvious disadvantage is that for each successive output $y(k)$, specified by $k = q, q+1, ..$, the coefficient matrices H_j comprising each specific solution change. Therefore if the solution is required over a comparatively large range, say $[y(q), y(q+1), ..., y(100)]$ corresponding to $k = q, q+1, ..., 100$, we would require the coefficient matrices H_{-101}, $H_{-100}, ..., H_{\hat{q}_r}$. An equivalent forward solution is presented in what follows for the general solution $y(k)$ depends on the *previous* q outputs $\{y(k-1), y(k-2), ..., y(k-q)\}$ and not on the q fixed initial conditions $\{y(0), y(1), ..., y(q-1)\}$. In this case the coefficient matrices required over a solution range is fixed, (i.e. independent of k), namely $H_{-q}, H_{-q+1}, ..., H_{\hat{q}_r}$.

Corollary 4 *Equation (13) is equivalent to the following forward recursion :*

$$\begin{aligned} y(k) = &- \sum_{i=1}^{q} \sum_{j=0}^{i-1} H_{-i} A_j y(k-i+j) + \\ &+ \sum_{i=0}^{q+\hat{q}_r} \sum_{j=0}^{q} H_{-q+i} B_j u(k-q+j+i) \end{aligned} \tag{17}$$

Proof. The system is time invariant and thus the relation which connect $y(k)$ with the previous vectors $\{y(k-1), y(k-2), ..., y(k-q)\}$ will also connect the vector $y(q)$ with the vectors $\{y(q-1), y(q-2), ..., y(0)\}$. Thus if we replace $\{y(q), y(q-1), ..., y(0)\}$ with $\{y(k), y(k-1), ..., y(k-q)\}$ respectively and $\{u(0), u(1), ..., u(2q+\hat{q}_r)\}$ with $\{u(k-q), u(k-q+1), ..., u(k+q+\hat{q}_r)\}$ in (13) respectively we get the relation (17). ∎

The advantage of the formula (17) is, as we have already mentioned, is that it depends only on the $q+\hat{q}_r+1$ Laurent expansion terms $\{H_{-q},\ H_{-q+1}, ..., H_0, ..., H_{\hat{q}_r}\}$. The above formula is very usefull when we need to determine $y(k)$ in the closed interval $[q, +\infty]$, because we always have to start to compute from $y(q)$, $y(q+1)$, ... in contrast to the solution formula (13) where only the q first initial conditions are enough for the determination of the solution in the interval $[n, +\infty]$ where $n > q$. Another advantage of (17) is that the round-of errors for the determination of the $q+\hat{q}_r+1$ Laurent expansion terms $\{H_{-q},\ H_{-q+1}, ..., H_0, ..., H_{\hat{q}_r}\}$ are less than the ones for the determination of $\{H_{-k}, ..., H_{\hat{q}_r}\}$ in (13).

3.2 The Backward Solution of ARMA-Representations

Consider the discrete time ARMA-representation (1). The Laurent series expansion about zero for the resolvent matrix is given by (7). The sequence V_k is the *backward fundamental matrix* and is easily implemented according to Lemma 1 and [3]. Then we have

Theorem 5 *The backward solution of (1) will be :*

$$\begin{aligned} y(k) = &\sum_{i=0}^{q-1} \sum_{j=0}^{i} V_{N-k-i} A_j y(N-i+j) + \\ &+ \sum_{i=0}^{q+k-N-\ell} \sum_{j=0}^{q} V_{N-k-q-i} B_j u(N+j-i-q) \end{aligned} \tag{18}$$

Proof. Determine the forward solution of the dual ARMA-representation (11) and make use of Theorem 2. ∎

A necessary and sufficient condition such that the ARMA-representation (1) has a solution is that the final conditions $\{y(N), y(N-1), ..., y(N-q+1)\}$ satisfies the relation (1) for $k = N, N-1, ..., N-q+1$. Therefore we define :

Definition 2 *We define as*

$$\bar{H}_{iu} := \{y(i),\, u(i)\ (i = N, N-1, ..., N-q+1) :$$
$$\tilde{A}_2 \begin{bmatrix} V_{-q} & \cdots & V_{-2q+1} \\ V_{-q+1} & \cdots & V_{-2q+2} \\ \vdots & \ddots & \vdots \\ V_{-1} & \cdots & V_{-q} \end{bmatrix} \tilde{A}_2 \begin{bmatrix} y(N) \\ y(N-1) \\ \vdots \\ y(N-q+1) \end{bmatrix} =$$
$$= \tilde{A}_2 \begin{bmatrix} V_{-q} & \cdots & V_{-\ell} & 0 & \cdots & 0 \\ V_{-q+1} & \cdots & V_{-\ell+1} & V_{-\ell} & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ V_{-1} & \cdots & V_{-\ell+q-1} & V_{-\ell+q-2} & \cdots & V_{-\ell} \end{bmatrix}$$
$$\begin{bmatrix} B_0 & \cdots & B_q & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & B_0 & B_1 & \cdots & B_q \end{bmatrix} \begin{bmatrix} u(N) \\ u(N-1) \\ \vdots \\ u(N-q-\ell+1) \end{bmatrix} \} \tag{19}$$

where

$$\tilde{A}_2 = \begin{bmatrix} A_q & 0 & \cdots & 0 \\ A_{q-1} & A_q & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_1 & A_2 & \cdots & A_q \end{bmatrix} \tag{20}$$

the *admissible final condition space* of (1) under nonzero inputs.

Proof. Consider the relation (1) for $k = N-q, N-q-1, ..., N-2q+1$. Then substituting the values $y(N-q)$, $y(N-q-1)$, ..., $y(N-2q+1)$ with the respective formula of (18) and using (14) we get that the final conditions $\{y(N), y(N-1), ..., y(N-q+1)\}$ satisfy the system iff the relation (19) is satisfied. ∎

A backward solution formula in terms of the following q terms and the input sequence of the system is provided by the following

Corollary 6 *Equation (18) is equivalent to the backward recursion :*

$$y(k) = \sum_{i=0}^{q-1} \sum_{j=0}^{i} V_{q-i} A_j y(k+q-i+j) + \sum_{i=0}^{-\ell} \sum_{j=0}^{q} V_{-i} B_j u(k+j-i) \tag{21}$$

Proof. Following similar lines with the proof of Corollary 4 we obtain the result. ∎

The advantage of the formula (21) is, that depends only from the $q+\ell+1$ Laurent expansion terms $[V_q, V_{q-1}, ...V_{-\ell}]$ and thus we don't need the continuous computation of the Laurent expansion terms which gives rise to numerical errors.

3.3 The Symmetric Solution

In this section we consider (1) as a relation between the $y(k)$ and $u(k)$ over an interval $[0, N]$, with k not necessarily the time index. Such an interpretation is used in economics and elsewhere [7], [9]. Consider the discrete time ARMA-representation (1) and the Laurent series expansion about infinity for its resolvent matrix in (6).

Theorem 7 *The symmetric solution of the ARMA-representation (1) is given by the following formula :*

$$\begin{aligned} y(k) = & \sum_{i=1}^{q} \sum_{j=1}^{q} H_{-k-i} A_j\, y(j-i) + \\ & + \sum_{i=0}^{q-1} \sum_{j=0}^{i} H_{N-k-i} A_j y(N-i+j) + \\ & + \sum_{i=0}^{N-q} \sum_{j=0}^{q} V_{N-k-q-i} B_j u(N+j-i-q) \end{aligned} \tag{22}$$

under the following restrictions between the initial conditions, final conditions and input sequences :

$$\begin{bmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{bmatrix} \begin{bmatrix} X_A y_{N-q+1,N} \\ X_{\tilde{A}} y_{0,q-1} \end{bmatrix} = \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix} B_N u_{0,N} \tag{23}$$

where

$$W_{11} = \begin{bmatrix} H_{-q} & H_{-q-1} & \cdots & H_{-2q+1} \\ H_{-q+1} & H_{-q} & \cdots & H_{-2q+2} \\ \vdots & \vdots & \ddots & \vdots \\ H_{-1} & H_{-2} & \cdots & H_{-q} \end{bmatrix}$$
$$W_{12} = \begin{bmatrix} H_{-N+q-1} & H_{-N+q-2} & \cdots & H_{-N} \\ H_{-N+q} & H_{-N+q-1} & \cdots & H_{-N+1} \\ \vdots & \vdots & \ddots & \vdots \\ H_{-N+2q-2} & H_{-N+2q-3} & \cdots & H_{-N+q-1} \end{bmatrix}$$
$$W_{21} = \begin{bmatrix} H_{N-2q+1} & H_{N-2q} & \cdots & H_{N-3q+2} \\ H_{N-2q+2} & H_{N-2q+1} & \cdots & H_{N-3q+3} \\ \vdots & \vdots & \ddots & \vdots \\ H_{N-q} & H_{N-q-1} & \cdots & H_{N-2q+1} \end{bmatrix}$$
$$W_{22} = \begin{bmatrix} H_0 & H_{-1} & \cdots & H_{-q+1} \\ H_1 & H_0 & \cdots & H_{-q+2} \\ \vdots & \vdots & \ddots & \vdots \\ H_{q-1} & H_{q-2} & \cdots & H_0 \end{bmatrix} \tag{24}$$

$$X_A = \begin{bmatrix} A_q & A_{q-1} & \cdots & A_1 \\ 0 & A_q & \cdots & A_2 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_q \end{bmatrix}$$

$$X_{\bar{A}} = \begin{bmatrix} A_0 & 0 & \cdots & 0 \\ A_1 & A_0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_{q-1} & A_{q-2} & \cdots & A_0 \end{bmatrix}; \; u_{0,N} = \begin{bmatrix} u(N) \\ u(N-1) \\ \vdots \\ u(0) \end{bmatrix}$$

$$B_N = \begin{bmatrix} B_0 & B_1 & \cdots & B_q & 0 & \cdots & 0 \\ 0 & B_0 & \cdots & B_{q-1} & B_q & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & B_0 & B_1 & \cdots & B_q \end{bmatrix}$$

$$Z_1 = \begin{bmatrix} H_{-q} & H_{-q-1} & \cdots & H_{-N} \\ H_{-q+1} & H_{-q} & \cdots & H_{-N+1} \\ \vdots & \vdots & \ddots & \vdots \\ H_{-1} & H_{-2} & \cdots & H_{-N+q-1} \end{bmatrix}$$

$$Z_2 = \begin{bmatrix} H_{N-2q+1} & H_{N-2q} & \cdots & H_{-q+1} \\ H_{N-2q+2} & H_{N-2q+1} & \cdots & H_{-q+2} \\ \vdots & \vdots & \ddots & \vdots \\ H_{N-q} & H_{N-q-1} & \cdots & H_0 \end{bmatrix}$$

$$y_{N-q+1,N} = \begin{bmatrix} y(N) \\ y(N-1) \\ \vdots \\ y(N-q+1) \end{bmatrix}; \; y_{0,q-1} = \begin{bmatrix} y(q-1) \\ y(q-2) \\ \vdots \\ y(0) \end{bmatrix}$$

We call the equations (23) the *boundary mapping equations* of (1).

Proof. Rewritting (1) in the form

$$\underbrace{\begin{bmatrix} A_q & \cdots & A_0 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & A_q & A_{q-1} & \cdots & A_0 \end{bmatrix}}_{A_N} \underbrace{\begin{bmatrix} y(N) \\ y(N-1) \\ \vdots \\ y(0) \end{bmatrix}}_{y_{0,N}}$$

$$\underbrace{\begin{bmatrix} B_q & \cdots & B_0 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & B_q & B_{q-1} & \cdots & B_0 \end{bmatrix}}_{B_N} \underbrace{\begin{bmatrix} u(N) \\ u(N-1) \\ \vdots \\ u(0) \end{bmatrix}}_{u_{0,N}} \Leftrightarrow$$

$$\begin{bmatrix} X_A y_{N-q+1,N} \\ 0 \\ X_{\bar{A}} y_{0,q-1} \end{bmatrix} = \begin{bmatrix} -A_0 & \cdots & 0 & \\ \vdots & \ddots & \vdots & \\ -A_q & \ddots & -A_0 & \mid B_N \\ \vdots & \ddots & \vdots & \\ 0 & \cdots & -A_q & \end{bmatrix} \begin{bmatrix} y_{q,N-q} \\ u_{0,N} \end{bmatrix} \tag{25}$$

where $y_{q,N-q} = [y(N-q)^T, ..., y(q)^T]^T$. Premultiply both sides of (25) by

$$A_N^r = \begin{bmatrix} H_{-q} & H_{-q-1} & \cdots & H_{-N} \\ H_{-q+1} & H_{-q} & \cdots & H_{-N+1} \\ \vdots & \vdots & \ddots & \vdots \\ H_{-q+N} & H_{-q+N-1} & \cdots & H_0 \end{bmatrix} \tag{26}$$

we obtain from the first q and the last q equations the relations (23), while from the middle $N-2q$ equations we obtain (22). ∎

A necessary and sufficient condition such that the ARMA-representation (1) has a solution is that the initial, final conditions and input sequences satisfies the relation (23). Therefore we give the following :

Definition 3 *We define as*

$$\tilde{H}_{iu} := \{y_{0,q-1},\, y_{N-q+1,N} : \begin{bmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{bmatrix} \begin{bmatrix} X_A y_{N-q+1,N} \\ X_{\bar{A}} y_{0,q-1} \end{bmatrix} = \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix} B_N u_{0,N}\} \tag{27}$$

the symmetric boundary condition space of (1) *under nonzero inputs.* ∎

The boundary mapping equation (23) represents the restrictions that the system places on the boundary variables $y_{0,q-1}$, $y_{N-q+1,N}$ in order for the system to be solvable. Addition restrictions on the variables can be applied to the system in the form of an auxiliary equation

$$W_{31} y_{N-q+1,N} + W_{32} y_{0,q-1} = C \tag{28}$$

The combined boundary equation formed from (23),(28)

$$\begin{bmatrix} W_{11} X_A & W_{12} X_{\bar{A}} \\ W_{21} X_A & W_{22} X_{\bar{A}} \\ W_{31} & W_{32} \end{bmatrix} \begin{bmatrix} y_{N-q+1,N} \\ y_{0,q-1} \end{bmatrix} = \begin{bmatrix} Z_1 u_{0,N} \\ Z_2 u_{0,N} \\ C \end{bmatrix} \Leftrightarrow$$

$$\Leftrightarrow ZY = \tilde{C} \tag{29}$$

will subsequently define a unique solution iff $ZZ^+ \tilde{C} = \tilde{C}$ and Z has full column rank, where Z^+ denotes the pseudoinverse of Z i.e. $Y = Z^+ \tilde{C}$.

Alternative forms of the solution formula (22) are given by the following

Corollary 8 *The symmetric solution (22) can be written in the alternative forms*

FORWARD-SYMMETRIC

$$\begin{aligned} y(k) = & \sum_{i=1}^{q} \sum_{j=0}^{i-1} H_{-i} A_j \, y(k-j-i) + \\ & + \sum_{i=0}^{q-1} \sum_{j=0}^{i} H_{N-k-i} A_j y(N-i+j) + \\ & + \sum_{i=0}^{N-k} \sum_{j=0}^{q} H_{N-k-q-i} B_j u(N+j-i-q) \end{aligned} \tag{30}$$

BACKWARD-SYMMETRIC

$$\begin{aligned} y(k) = & \sum_{i=1}^{q}\sum_{j=i}^{q} H_{-k-i}A_j\, y(j-i) - \\ & -\sum_{i=0}^{q-1}\sum_{j=i+1}^{q} H_{-i}A_j y(k+j-i) + \\ & + \sum_{i=0}^{k}\sum_{j=0}^{q} H_{-i}B_j u(k+j-i) \end{aligned} \tag{31}$$

Proof. Assuming that $k = \nu q + \upsilon$ ($N - k = \nu q + \upsilon$) and using alternatively either the relation (14) or the relation (1) we get the result. ∎

In the *Forward-Symmetric* case we still solve within the region $[0, N]$ but now the solution depends on the q final conditions $\{y(N), y(N-1), ..., y(N-q+1)\}$ and the *previous* q outputs $\{y(k-1), y(k-2), ..., y(k-q)\}$ and no longer on the q fixed initial conditions $\{y(0), y(1), ..., y(q-1)\}$. Therefore we solve *forwards* in the interval.

In the *Backward-Symmetric* case we again still solve within the region $[0, N]$ but now the solution depends on the q initial conditions $\{y(0), y(1), ..., y(q-1)\}$ and the *future* q outputs $\{y(k+1), y(k+2), ..., y(k+q)\}$ and no longer on the q fixed final conditions $\{y(N), y(N-1), ...,$ $y(N-q+1)\}$. Therefore we solve *backwards* in the interval.

4 Conclusion.

In the case of regular discrete time ARMA- representations exact solutions where proposed in three different forms : a) forward solutions, b) backward solutions and c) symmetric solutions. It is easily seen that the proposed solutions are extensions of the ones proposed by [8] for the less complicated case of discrete time generalized state space systems. A disadvantage of the method proposed above is that the expressions are based on a Leverrier type algorithm to calculate Laurent expansions. Since the numerical properties of these type of algorithms is questionable, we have applied these algorithms on the symbolic computational language MAPLE and presented in [4] without any cost in the accuracy of the above algorithms. However the investigation of better algorithms for the computation of the Laurent expansion of a polynomial matrix will directly improve the efficiency of the proposed expressions in this paper. Certain controllability, reachability and observability criteria based on the proposed solutions are being studied and will be discussed in a future publication.

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