A FUNDAMENTAL NOTION OF EQUIVALENCE FOR AR-REPRESENTATIONS*

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Abstract

A fundamental definition of equivalence between nonregular AutoRegressive — Representations (AR - Representations) is presented. The proposed definition is reduced to a full equivalence relation between the polynomial matrices which describes the equivalent AR-representations.

Keywords : AR-Representations, equivalence, linear systems.

1 Introduction.

Consider the linear homogeneous system of differential equation with coefficient matrices

$$A(\rho)\beta(t) = 0 \tag{1}$$

where $\rho = d/dt$ is the differential operator, $A(\rho) = A_0 + A_1\rho + \cdots + A_q\rho^q \in \mathbf{R}[\rho]^{p\times m}$ with $rank_{\mathbf{R}} A(\rho) = r$ and $\beta(t) : (0-, +\infty) \rightarrow \mathbf{R}^m$ is not restricted to be infinitely continuously differentiable but instead can be impulsive. Following the terminology of Willems [14], [15] we call the set of equations (1) an *AR representation* (*AutoRegressive representation*) of $\mathcal{B}$ (*behaviour*), where $\mathcal{B}$ is the solution set of equations (1).

In case where $p = m = r$ then $A(\rho)$ is a square and nonsigular (*regular*) polynomial matrix and (1) is a known representation (see [2], [3], [12], [13]) which exhibits both a finite and infinite frequency behaviour. The action of a linear mapping on the finite and infinite solution space of such systems has been considered by many authors, among them [1], [5], [8], [9] and [11].

In system theory however we need sometimes descriptions of dynamical systems where there is no distinction between inputs and outputs i.e. interconnection of systems. In such a case the model (1) in its general form ($p \neq m$ or $p = m$ with $\det[A(s)] = 0$) is very useful [7], [14] and [15]. Such kind of systems, exhibits both finite and infinite frequency behaviour which is connected with the finite and infinite zero structure of $A(s)$ and its right null space [6]. In contrast to the regular AR-representations *i.e.* $p = m = r$, the nonregular AR-representations (i) may not exhibit a solution under certain initial conditions, because of the left null space of $A(s)$ [6] and (ii) may exhibit many solutions under the same initial conditions, because of the right null space of $A(s)$ [6]. For the above reasons in case where a solution exists we correspond all the possible solutions under certain initial conditions of (1) to one equivalence class which define as the *equivalence solution class* of the AR-representation. This strange behaviour of the nonregular AR-representations leads us to investigate (i) the action of a linear map between the solutions of nonregular AR-representations, (ii) the properties like injectiveness and surjectiveness of such a kind of map and (iii) how this map between the solutions induces to a certain linear map between the equivalence solution classes of (1). An answer to the above questions we give with the definition of the fundamental equivalence between nonregular AR-representations in Section 2 and the reduction of this definition to a full equivalence relation between the polynomial matrices which describes the equivalent AR-representations in Section 3.

*The work is supported by the Greek General Secreteriat of Industry, Research and Technology.

2 Preliminary Results.

Consider the AR-representation (1) and let

$$B := \{\beta(t) : (0-, +\infty) \rightarrow \mathbf{R}[\rho]^{p\times m} / \ A(\rho)\beta(t) = 0\} \tag{2}$$

be the smooth and impulsive solution set of equations (1). The solution space of (1) under the following *initial conditions*

$$\begin{bmatrix} \beta(0-) \\ \beta^{(1)}(0-) \\ \vdots \\ \beta^{(q-1)}(0-) \end{bmatrix} \in Kernel \begin{bmatrix} A_q & 0 & \cdots & 0 \\ A_{q-1} & A_q & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_1 & A_2 & \cdots & A_q \end{bmatrix} \tag{3}$$

is denoted [6] by

$$Z := x(t) / \; x(t) := \int_0^t \bar{u}_{r+1}\,(\tau) z_2(t-\tau) d\tau + \int_0^t \bar{u}_{r+2}\,(\tau) z_2(t-\tau) d\tau + \cdots + \int_0^t \bar{u}_m\,(\tau) z_{m-r}(t-\tau) d\tau \quad (4)$$

where $\{u_{r+1}(s),\ u_{r+2}(s), ..., u_m(s)\}$ is a right minimal basis of $A(s)$, $z_1(t)$, $z_2(t), \cdots, z_{m-r}(t)$ are arbitrary functions and $\bar{u}_{r+1}(t) = L^{-1}[u_{r+1}(s)]$, $\bar{u}_{r+2}(t) = L^{-1}[u_{r+2}(s)]$, ..., $\bar{u}_m(t) = L^{-1}[u_m(s)]$.

We define the following relation :

$$\begin{array}{c} R(\beta_1(t), \beta_2(t)) = \{(\beta_1(t), \beta_2(t)) / \\ \beta_1(t) - \beta_2(t) \in Z \text{ with } \beta_1(t), \beta_2(t) \in B\} \end{array} \quad (5)$$

The relation (5) is actually [6] an equivalence relation. An equivalence class of the element $\beta(t) \in B$ denoted by $[\beta(t)]$, is the set of all the elements of $\beta(t)$ which are equivalent to $\beta(t)$ modulo equivalent in Z or equivalently

$$\begin{array}{c} [\beta(t)] := \{\beta_1(t) \in B \; / \; (\beta(t), \beta_1(t)) \in R\} = \beta(t) \oplus Z = \\ = \{\beta(t) + x(t) \text{ where } \beta(t) \in B \text{ and } x(t) \in Z\} \end{array} \quad (6)$$

Any **equivalence class** of an element $\beta(t)$ gives the complete solution of (1) under certain initial conditions *i.e.* $\beta^{(i)}(0-)$ $i = 0, 1, .., q-1$. We shall refer in the sequel to the equivalence class of an element $\beta(t)$ as the *equivalence solution class* so as to distinguish it from the solution vector $\beta(t)$. It is obvious from the above that $[0] = Z$. Thus we view Z as the "zero solution class" *i.e.* the solution class corresponding to the initial conditions specified in (3). In the case where $A(s)$ has no right kernel then every equivalence class contains just one element, while in the nonregular case each equivalence class contains an arbitrary number of elements of B. We conclude therefore that the whole solution space B of the AR-representation (1), can be divided into equivalence classes, which are defined by (6). The resulting space is denoted by B/Z

Figure 1. B is divided into equivalence classes

Define now the following "**sum**" between the equivalence classes of the form

$$[\beta_1(t)] + [\beta_2(t)] := [\beta_1(t) + \beta_2(t)] \quad (7)$$

and the scalar "**product**"

$$\lambda[\beta(t)] := [\lambda\ \beta(t)] \text{ where } \lambda \in \mathbf{R} \quad (8)$$

It was shown [6] that

Theorem 1 *[6] The equivalence solution class space*

$$\hat{B} = \{[\beta(t)] \;\; / \;\; \beta(t) \in B\} = B/Z$$

is a finite dimensional vector space with dimension

$$\dim \hat{B} = f = n + q + \epsilon \quad (9)$$

where n, q, ϵ are respectively the total number of the finite zeros, infinite zeros and right minimal indices (order accounted for) of $A(s)$. f is called the **generalised order** *of the AR-representation (1).* ∎

Thus while the solution space B has infinity dimension because of the right kernel of $A(s)$, the equivalence solution class space B/Z has a finite dimension. It is obvious from Theorem 1 that in case where $A(s)$ has no finite nor infinite zeros and full column rank then the equivalence solution class space of the AR-representation (1) is $\{0\}$.

An extension of the known fundamental equivalence [9], [11] of regular AR-representations is given by the following

Definition 1 *Let*

$$\Sigma_i \; : \; A_i(\rho)\beta_i(t) = 0 \quad i = 1, 2 \quad (10)$$

Σ_1, Σ_2 are said to be **equivalent** *if and only if there exist an isomorphism*

$$N(\rho) : \beta_1(t) \in B_1 \rightarrow \beta_2(t) = N(\rho)\beta_1(t) \in B_2 \quad (11)$$

which induces to a unique isomorphism

$$\begin{array}{c} \bar{N}(\rho) : \beta_1(t) + Z_1 \in B_1/Z_1 \rightarrow \\ \rightarrow \beta_2(t) = N(\rho)\beta_1(t) + Z_2 \in B_2/Z_2 \end{array} \quad (12)$$

Figure 2. An isomorphism between B_i and B_i/Z_i , $i = 1, 2$

Some interesting questions arising from Definition 1 are the following :

a) When (11) is a mapping in the formal sense (many-one relation) ?

b) In case where (11) is a map, then when this map induces to a map between the equivalence class solution spaces B_1/Z_1 and B_2/Z_2 ?

c) When the codomain of the solution space B_2 via (11) belongs into the solution space B_2 ?

d) When (11) and thus (12) map is injective and surjective ?

An answer to the above questions we propose in the following section.

3 Fundamental equivalence of AR-representations.

A necessary and sufficient condition for (11) to be a map is given by the following

Theorem 2 *[10] The relation (11) is a mapping in the formal sense (many-one relation) if and only if*

$$\delta_M \left(\begin{array}{c} A_1(\rho) \\ N(\rho) \end{array} \right) = \delta_M \left(A_1(\rho) \right)$$

A second question is when this map from B_1 to B_2 induces to a map from the equivalence class solution space B_1/Z_1 into the equivalence class solution space B_2/Z_2 ? Two necessary tools for the investigation of this specific property are given by the following two lemmata.

Lemma 3 *Let $f : V_1 \rightarrow V_2$ be an homorphism between two vector spaces. If Z_1, Z_2 are submodules of V_1, V_2 then f induces to the* **unique** *homorphism*

$$\bar{f} : x + Z_1 \in V_1/Z_1 \rightarrow f(x) + Z_2 \in V_2/Z_2$$

if and only if $f(Z_1) \subseteq Z_2$.

Diagram 3. The above diagram is commutative.

The above lemma may also be applied if $f : V_1 \rightarrow V_2$ is an epimorphism, a monomorphism or an isomorphism.

Lemma 4 *([7], p.43) Let $P(s) \in \mathbf{R}[s]^{g\times q}$ and $M(s) \in \mathbf{R}[s]^{k\times q}$. Then*

$$Ker\left[P(s)\right] \subset Ker\left[M(s)\right] \tag{13}$$

if and only if there exists a polynomial matrix $A(s) \in \mathbf{R}[s]^{k\times g}$ such that

$$M(s) = A(s)P(s) \quad \blacksquare \tag{14}$$

Let

$$\begin{array}{c} \Sigma_i : A_i(\rho)\beta_i(t) = 0 \quad i = 1, 2 \\ A_i(s) = A_{0,i} + A_{1,i}s + \cdots + A_{q,i}s^q \ i = 1, 2 \end{array} \tag{15}$$

be two AR-representations with solution spaces

$$B_i := \{\beta_i(t) / \ A_i(\rho)\beta_i(t) = 0\} \ \ i = 1, 2 \tag{16}$$

and

$$Z_i := \left\{ \beta_i(t) / \ A_i(s)\tilde{\beta}_i(s) = 0 \right\} \ \ i = 1, 2 \tag{17}$$

where $\tilde{\beta}_i(s)$ denotes the Laplace transform of $\beta_i(t)$ i.e. $\tilde{\beta}_i(s) = L[\beta_i(t)]$. We define the following relations

$$\begin{array}{c} R_i\left(\beta_i^1(t), \beta_i^2(t)\right) := \left\{\left(\beta_i^1(t), \beta_i^2(t)\right) / \right. \\ \left. \beta_i^1(t) - \beta_i^2(t) \in Z_i \ , \ \ \beta_i^1(t), \beta_i^2(t) \in B_i \right\} \ i = 1, 2 \end{array} \tag{18}$$

Then an exact application of the above lemma is the following

Theorem 5 *Let Σ_1,Σ_2 defined in (15) with solution spaces B_1, B_2 respectively. Let also R_1, R_2 be the equivalence relations and Z_1, Z_2 the solution spaces defined respectively in (18) and (17). If*

$$\begin{array}{c} N(\rho) : \beta_1(t) \in B_1 \rightarrow \beta_2(t) = N(\rho)\beta_1(t) \in B_2 \\ N(s) = N_0 + N_1 s + \cdots + N_q s^q \end{array} \tag{19}$$

be a homorphism then $N(\rho)$ induces to the unique homorphism (see Fugure 2)

$$\begin{array}{c} \bar{N}(\rho) : \beta_1(t) + Z_1 \in B_1/Z_1 \rightarrow \\ \rightarrow \beta_2(t) = N(\rho)\beta_1(t) + Z_2 \in B_2/Z_2 \end{array} \tag{20}$$

Theorem 6 *if and only if there exist a polynomial matrix*

$$M(s) = M_0 + M_1 s + \cdots + M_q s^q$$

such that :

$$M(s)A_1(s) = A_2(s)N(s) \tag{21}$$

Proof. From Lemma 3, (19) is an induced homorphism iff $N(Z_1) \subseteq Z_2$. Suppose that there exists $\beta_1(t) \in Z_1$ *i.e.*

$$A_1(s)\tilde{\beta}_1(s) = 0 \text{ or } \quad X_{A_1}\bar{\beta}_1(0-) = 0 \tag{22}$$

where

$$X_{A_1} := \left[\begin{array}{cccc} A_{q,1} & 0 & \cdots & 0 \\ A_{q-1,1} & A_{q,1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_{1,1} & A_{2,1} & \cdots & A_{q,1} \end{array} \right] \tag{23}$$

$\bar{\beta}_1(0-) := \left(\beta\left(0-\right)^T, \beta^{(1)}\left(0-\right)^T, ..., \beta^{(q-1)}\left(0-\right)^T\right)^T$ and $\tilde{\beta}_1(s)$ denotes the Laplace transform of $\beta_1(t)$. Then $N(\rho)\beta_1(t) \in Z_2$ iff its Laplace transform

$$N(s)\tilde{\beta}_1(s) - S_{q-1}X_N\bar{\beta}_1(0-) \tag{24}$$

with

$$X_N := \left[\begin{array}{cccc} N_q & 0 & \cdots & 0 \\ N_{q-1} & N_q & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ N_1 & N_2 & \cdots & N_q \end{array} \right] \tag{25}$$

belongs to the right kernel of $A_2(s)$ *i.e.*

$$A_2(s)(N(s)\tilde{\beta}_1(s) - S_{q-1}X_N\tilde{\beta}_1(0-)) = 0 \quad (26)$$

(Note that we have assumed that the highest degrees of $A_i(s)$ $i = 1, 2$ and $N(s)$ are the same (equal to q) without any restriction to the problem.) However relation (19) is a map according to Theorem 2 iff

$$rank_R \begin{pmatrix} X_{A_1} \\ X_N \end{pmatrix} = rank_R (X_{A_1}) \overset{\exists Q}{\Leftrightarrow} X_N = QX_{A_1} \quad (27)$$

From (26) and (27) we have that

$$\begin{gathered} A_2(s)\left(N(s)\tilde{\beta}_1(s) - S_{q-1}QX_{A_1}\tilde{\beta}_1(0-)\right) = 0 \\ \overset{(22)}{\Rightarrow} A_2(s)N(s)\tilde{\beta}_1(s) = 0 \end{gathered} \quad (28)$$

From (22) and (28) we conclude that any element $\tilde{\beta}_1(s)$ which belongs to the right kernel of $A_1(s)$ belongs also to the right kernel of $A_2(s)N(s)$

$$\text{Rig. Kern. of } A_1(s) \subseteq \text{Rig. Kern. of } A_2(s)N(s) \quad (29)$$

and thus from Lemma 4 there exists a polynomial matrix $M(s)$ which satisfies (21) and thus which verifies the Theorem. ∎

A third question is when the codomain of the solution space B_1 through the map (19) belongs to the solution space B_2 and therefore the codomain of the equivalence class solution space B_1/Z_1 via (20) belongs to the equivalence class solution space B_2/Z_2 ; An answer to this question is given by the following

Theorem 7 *Let Σ_1, Σ_2 defined in (15) with solution spaces B_1, B_2 respectively. Consider also the map (19) with the property of inducing to a map between the equivalence class solution spaces B_1/Z_1 and B_2/Z_2. If the codomain of B_1 via the map (19) belongs to B_2 then*

$$\delta_M \begin{pmatrix} M(s) & A_2(s) \end{pmatrix} = \delta_M (A_2(s)) \quad (30)$$

Proof. Suppose that $\beta_1(t) \in B_1$, i.e.

$$A_1(\rho)\beta_1(t) = 0 \Leftrightarrow A_1(s)\tilde{\beta}_1(s) = S_{q-1}X_{A_1}\tilde{\beta}_1(0-) \quad (31)$$

where $\tilde{\beta}_1(s)$, S_{q-1}, X_{A_1} and $\tilde{\beta}_1(0-)$ have been defined above. Consider also the map (19) i.e.

$$\begin{gathered} \beta(t) = N(\rho)\beta_1(t) \Leftrightarrow \tilde{\beta}(s) = N(s)\tilde{\beta}_1(s) - S_{q-1}X_N\tilde{\beta}_1(0-) \overset{(27)}{\Leftrightarrow} \\ \tilde{\beta}(s) = N(s)\tilde{\beta}_1(s) - S_{q-1}QX_{A_1}\tilde{\beta}_1(0-) \end{gathered} \quad (32)$$

If $\beta(t) \in B_2$ then $\exists X_{A_2}\tilde{\beta}_2(0-)$ such that

$$A_2(\rho)\beta(t) = 0 \Leftrightarrow A_2(s)\tilde{\beta}(s) = S_{q-1}X_{A_2}\tilde{\beta}_2(0-) \quad (33)$$

However (19) induces to a map between the equivalence class solution spaces B_1/Z_1 and B_2/Z_2 and thus from Theorem 5 there exists a polynomial matrix $M(s)$ such that

$$\begin{gathered} M(s)A_1(s) = A_2(s)N(s) \overset{(\times\tilde{\beta}_1(s))}{\Leftrightarrow} \\ M(s)A_1(s)\tilde{\beta}_1(s) = A_2(s)N(s)\tilde{\beta}_1(s) \underset{(32)}{\overset{(31)}{\Leftrightarrow}} \\ M(s)S_{q-1}X_{A_1}\tilde{\beta}_1(0-) = \\ = A_2(s)\tilde{\beta}(s) + A_2(s)S_{q-1}QX_{A_1}\tilde{\beta}_1(0-) \overset{(33)}{\Leftrightarrow} \\ S_{q-1}X_{A_2}\tilde{\beta}_2(0-) = M(s)S_{q-1}X_{A_1}\tilde{\beta}_1(0-)- \\ -A_2(s)S_{q-1}QX_{A_1}\tilde{\beta}_1(0-) \end{gathered} \quad (34)$$

Equating the coefficient powers of s in (34) we obtain that

$$X_{A_2}\tilde{\beta}_2(0-) = X_{A_2}\tilde{N}\tilde{\beta}_1(0-) - X_M\tilde{A}_1\tilde{\beta}_1(0-) \quad (35)$$

where

$$\begin{gathered} \tilde{N} := \begin{pmatrix} N_0 & N_1 & \cdots & N_{q-1} \\ 0 & N_0 & \cdots & N_{q-2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & N_0 \end{pmatrix} \\ \tilde{A}_1 := \begin{pmatrix} A_{0,1} & A_{1,1} & \cdots & A_{q-1,1} \\ 0 & A_{0,1} & \cdots & A_{q-2,1} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_{0,1} \end{pmatrix} \end{gathered} \quad (36)$$

Relation (35) implies that there exists $X_{A_2}\tilde{\beta}_2(0-)$ such that (33) is satisfied iff

$$\begin{gathered} Im\,[X_M] \subseteq Im\,[X_{A_2}] \Leftrightarrow Im \begin{bmatrix} X_M & X_{A_2} \end{bmatrix} = Im\,[X_{A_2}] \Leftrightarrow \\ rank_{\mathbf{R}} \begin{bmatrix} X_M & X_{A_2} \end{bmatrix} = rank_{\mathbf{R}}\,[X_{A_2}] \Leftrightarrow \\ \delta_M \begin{pmatrix} M(s) & A_2(s) \end{pmatrix} = \delta_M (A_2(s)) \end{gathered} \quad (37)$$

which verifies the Theorem. ∎

Note that the proof of Theorem 6 remains the same if in place of B_1, B_2 we have B_1/Z_1 and B_2/Z_2 respectively.

However, until now we haven't mention anything concerning the surjectiveness and injectiveness of the map (19). The following Theorem gives certain conditions for the map (19) to be injective.

Theorem 8 *A necessary and sufficient condition for the map (19) to be injective is that the compound matrix*

$$\begin{pmatrix} A_1(s) \\ N(s) \end{pmatrix} \quad (38)$$

a) *has full column rank,*
b) *possess no finite nor infinite zeros.*

Proof. Following similar lines with the proof of Theorem 2 in [11] we can prove the necessity and sufficiency of (b). However, while in [11] the matrix $A_1(s)$ is square and nonsingular and thus the compound matrix (38) has always full column rank, in the nonregular case presented in this work we have to assure that (38) has full column rank such that (1) a left polynomial (biproper) inverse of (38) exists which gives rise in relation with condition (b) to the injectiveness of (19) according to [11] (necessity) and (2) the homogeneous AR-representation

$$\begin{pmatrix} A_1(\rho) \\ N(\rho) \end{pmatrix} \beta(t) = 0 \tag{39}$$

has no solution because of the absence of the right kernel of the compound matrix (38) which in connection with the absence of the finite and infinite zeros of (38) gives rise according to [11] to the sufficiency of (a), (b). ∎

The surjectiveness of the map (19) is based on the same conditions defined in Theorem 7 but for the dual compound matrix $\begin{pmatrix} M(s) & A_2(s) \end{pmatrix}$ as we prove in the following

Theorem 9 *A necessary and sufficient condition for the map (19) to be surjective is that the compound matrix*

$$\begin{pmatrix} M(s) & A_2(s) \end{pmatrix} \tag{40}$$

a) *has full row rank,*
b) *possess no finite nor infinite zeros.*

Proof. Consider the AutoRegressive Moving Average representation

$$A_2(\rho)\bar{\beta}_2(t) = M(\rho)\bar{u}(t) \tag{41}$$

where

$$\bar{\beta}_2(t) = N(\rho)\beta_1(t) \ ; \ \bar{u}(t) = A_1(\rho)\beta_1(t) \tag{42}$$

We can see that the conditions for the surjectiveness of the map (19) i.e. $\beta_2(t) = N(\rho)\beta_1(t)$, coincides with the controllability property, in the way defined by Verghese [13] of the ARMA-representation (41) i.e.

"The ARMA-representation (41) is controllable iff any distribution solution of the homogeneous system (41) and thus of B_2 can be reached from an input $\bar{u}(t)$ and thus from the trajectory $\beta_1(t) \in B_1$, with zero initial conditions." Thus the map (19) is surjective iff the compound matrix (40) has full row rank and no finite nor infinite zeros which verifies the theorem. ∎

We can easily see from the following diagram that in case where (19) is injective (surjective) then (20) is also injective (surjective) because of the transitivity property of the diagram and the fact that the canonical projection $I_{Z_i} : B_i \to B_i/Z_i \ i = 1,2$ is always injective (surjective).

Trying to summarize all the above theorems we give the following

Proposition 10 *Let*

$$\Sigma_i : A_i(\rho)\beta_i(t) = 0 \quad i = 1,2 \tag{43}$$

Then Σ_1, Σ_2 are equivalent in the sense of Definition 1 iff the polynomial matrices $A_1(\rho)$, $A_2(\rho)$ are fully equivalent [4] i.e. there exist polynomial matrices $M(s)$, $N(s)$ such that

$$M(s)A_1(s) = A_2(s)N(s) \tag{44}$$

where the compound matrices

$$\begin{pmatrix} M(s) & A_2(s) \end{pmatrix} \ ; \ \begin{pmatrix} A_1(s) \\ N(s) \end{pmatrix} \tag{45}$$

a) *have full normal rank,*
b) *possess no finite nor infinite zeros,*
c)

$$\text{(1) } \delta_M \begin{pmatrix} M(s) & A_2(s) \end{pmatrix} = \delta_M (A_2(s))$$
$$\text{(2) } \delta_M \begin{pmatrix} A_1(s) \\ N(s) \end{pmatrix} = \delta_M (A_1(s))$$

Proof. (⇒) Suppose that Σ_1, Σ_2 are *equivalent.* Then there exists an isomorphism

$$N(\rho) : \beta_1(t) \in B_1 \to \beta_2(t) = N(\rho)\beta_1(t) \in B_2 \tag{46}$$

which induces to a unique isomorphism

$$\begin{aligned} \bar{N}(\rho) : \beta_1(t) + Z_1 \in B_1/Z_1 \to \\ \to \beta_2(t) = N(\rho)\beta_1(t) + Z_2 \in B_2/Z_2 \end{aligned} \tag{47}$$

(46) is a map and thus according to Theorem 2 condition (c1) is satisfied. The map (46) induces to the map (47) between the equivalence class solution space and thus according to Theorem 5 there exist a polynomial matrix $M(s)$ such that

$$M(s)A_1(s) = A_2(s)N(s) \tag{48}$$

The codomain of B_1 belongs to B_2 because of the definition of the map (46) and thus according to Theorem 6 condition (c2) is satisfied. The map (46) is injective and thus according to Theorem 7 the compound matrix

$$\begin{pmatrix} A_1(s) \\ N(s) \end{pmatrix} \tag{49}$$

has full column rank and possess no finite nor infinite zeros. (46) is also surjective and thus from Theorem 8 the compound matrix

$$\begin{pmatrix} M(s) & A_2(s) \end{pmatrix} \tag{50}$$

has full row rank and possess no finite nor infinite zeros. Summarizing the above properties we conclude that $A_1(s)$ and $A_2(s)$ satisfy the properties (a), (b) and (c) and thus are fully equivalent.

($\Leftarrow$) Suppose that $A_1(s)$, $A_2(s)$ are related by the full equivalence relation (44). Then

$$\begin{array}{c} M(\rho)A_1(\rho) = A_2(\rho)N(\rho) \overset{(\times\beta_1(t))}{\Leftrightarrow} \\ M(\rho)A_1(\rho)\beta_1(t) = A_2(\rho)N(\rho)\beta_1(t) \overset{(43)}{\Leftrightarrow} \\ 0 = M(\rho)\times 0 = A_2(\rho)\left[N(\rho)\beta_1(t)\right] \end{array} \tag{51}$$

Thus we define the relation

$$N(\rho) : \beta_1(t) \in B_1 \rightarrow \beta(t) = N(\rho)\beta_1(t) \in B_2 \tag{52}$$

From the McMillan degree condition (c2) and Theorem 2 we have that (52) is a map. From (44) and Theorem 5 we conclude that the map (52) induces to a unique isomorphism

$$\begin{array}{c} \bar{N}(\rho) : \beta_1(t) + Z_1 \in B_1/Z_1 \rightarrow \\ \rightarrow \beta_2(t) = N(\rho)\beta_1(t) + Z_2 \in B_2/Z_2 \end{array} \tag{53}$$

From (51) we conclude that the codomain of B_1 via (52) belongs to B_2. From (a), (b) and Theorems 6 and 7 we conclude that the map (52) is both injective and surjective. Therefore relations (52) and (53) are both isomorphisms. ∎

4 Conclusions.

A neat characterization of the transformation of full equivalence has been given in terms of the existence of a bijective map between the finite and infinite solution sets of the AR-representations constructed by the full equivalent polynomial matrices. This characterization has enabled the true nature and role of the conditions of full equivalence in case where nonregular polynomial matrices are involved. Thus for example the McMillan degree conditions (c1), (c2) in Proposition 9, which previously appeared somewhat arbitrarily attached to the transformation, are seen to be vital in a quite fundamental way. It is concluded therefore that the transformation of full equivalence, with its various characterizations, is the basic transformational tool for the simultaneous study of the finite and infinite frequency behaviour of AR-representations.

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