# A CLASSIFICATION OF THE SOLUTIONS OF NON-REGULAR, DISCRETE-TIME DESCRIPTOR SYSTEMS

S.Antoniou[†], N.Karampetakis[†] and A.Vardulakis[†]
[†]Aristotle University of Thessaloniki
Dept. of Mathematics - Faculty of Sciences
54006 - Thessaloniki - GREECE
e-mail: antoniou@ccf.auth.gr, karampetakis@ccf.auth.gr, avardula@ccf.auth.gr

**Abstract**

A classification of the solutions of linear, time invariant non-regular, discrete descriptor systems is given in terms of the structural invariants of the associated matrix pencil $\sigma E - A$. The lack of conditionability (in the general case) implies a partitioning of the behavior and thus a classification of the solutions according to their boundary values. A generalization of the boundary mapping equation is also given.

**Keywords:** Descriptor Systems, Non-regular Systems, Discrete time

## 1 Introduction

In this paper we investigate the solution space of "non-regular", discrete-time, homogeneous descriptor systems described by

$$Ex_{k+1} = Ax_k$$

where $E, A$ are in general constant, non-square, real matrices. The term "non-regular" is used throughout this paper to distinguish this general case from the regular one, i.e. the case where $E, A$ are both square with $\det(\sigma E - A) \neq 0$ for almost every $\sigma$.

Non-regular descriptor systems are the natural framework for many physical, social and economical systems. We shall only mention some of them indicatively (for more details on the applications see [6]). Non-square systems occur in interconnected systems where no distinction between inputs and outputs is made. In economics the Leontief model is in general a non-square implicit system, while the square case corresponds to a system where the number of goods produced equals to the number of factories, which is rather an artificial case. Non-regular descriptor equations play also a very important role in the study of the discrete Riccati equation, where the associated Extended Hamiltonian Pencil (EHP) (see [11], [10]) is involved.

The regular case has been extensively studied by many authors (see for example [1], [2], [3], [4], [5], [7] etc.) and several approaches have been proposed. In all these studies it shown that the initial conditions of the descriptor equation cannot be arbitrarily chosen, since this may result the system not to be well-posed. Furthermore it is shown that if we restrict the time interval from $Z^+$ to a finite interval and place appropriate boundary conditions then the solution can be uniquely characterized,. The non-causal nature of descriptor systems can be expressed either through its forward - backward decomposition [3], where the original system is decomposed into the casual and the anticausal part or via the boundary mapping equation [2], which plays the role of a generalized transition matrix for singular equations.

From a behavioral point of view non-square descriptor systems have been studied both in continuous and discrete time (see e.g. [14], [12], [13]), but no attention has been focussed on the role of the structure at infinity of the corresponding matrix pencil. As a result in the discrete time case the behavioral approach does not take into account the non-causal nature of singular systems. This is because causality of the behavior is an *a priori* assumption and consequently, $Z^+$ as time domain, is the natural framework, in all these studies. Furthermore a fundamental distinction is made between the system and its mathematical representation. According to the behavioral approach, the system (see e.g. [12], [13]) is defined as the set of all possible trajectories produced as outcomes from some particular physical, economical or social phenomenon. On the other hand the mathematical representation of the system may take several forms depending on the way we choose to model it.

In this paper we follow an approach similar to that in [1], [2], [3], [4], [5], [7], rather than the behavioral one. Particularly, we examine the solutions of a given non-regular descriptor equation without making any assumptions of causality of the corresponding behavior.

The notion of conditionabilty [1] is naturally extended to the non-square case and plays a fundamental role in the classification of the solutions.

## 2 Conditionabilty and Behavior

Consider the non-regular homogeneous descriptor equation

$$Ex_{k+1} = Ax_k, \quad k = 0, 1, 2, ..., N-1 \tag{1}$$

where $E, A \in R^{p\times m}$ are constant real matrices and $x_k \in R^m$, $k = 0, 1, 2, ..., N$ is the descriptor vector. The above equation can be written in a more compact form as

$$\begin{bmatrix} -A & E & 0 & \cdots & 0 \\ 0 & -A & E & & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & -A & E \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_N \end{bmatrix} = 0 \tag{2}$$

or equivalently

$$S_N \bar{x}_N = 0 \tag{3}$$

where $S_N \in R^{Np\times(N+1)m}$ is the matrix in the left hand side of (3) and $\bar{x}_N = [x_0^T, x_1^T, ..., x_N^T]^T \in R^{(N+1)m}$. In [1] the matrix $S_N$ is defined as the *solvability matrix*. In the non-homogeneous case [1] where inputs are involved the matrix $S_N$ must have full row rank, in order for the system to be solvable. However in the homogeneous case we don't have to check for full row rank of $S_N$ since there are no input terms and equation (3) is always solvable (in the worst case the system will have the trivial solution $x_k = 0$). We introduce the set of all solutions of (1)

$$B = \{x_k : Ex_{k+1} = Ax_k, \ k = 0, 1, 2, ..., N-1\}$$

The notion of conditionabilty can be extended to the non-regular case. Luenburger in [1] defines a solvable system to be *conditionable* if any choice of (admissible) boundary values $x_0, x_N$, characterizes uniquely the solution for all the intermediate steps $x_1, x_2, ..., x_{N-1}$. In the regular - time invariant case the system is always conditionable.

In our case (1) the system is not in general conditionable, which means that the boundary values $x_0, x_N$ are not sufficient to determine the solution $x_k \in B$ uniquely. To see this consider two solutions of (2) $\bar{x}_N$, $\bar{y}_N$ having the same boundary values $x_0, x_N$ and probably different intermediate values, i.e. $\bar{x}_N = [x_0^T, x_1^T, ..., x_{N-1}^T, x_N^T]^T$ and $\bar{y}_N = [x_0^T, y_1^T, ..., y_{N-1}^T, x_N^T]^T$. Then it is easy to see that the difference $\bar{x}_N - \bar{y}_N = [0, x_1^T - y_1^T, ..., x_{N-1}^T - y_{N-1}^T, 0]^T$ will also satisfy (2). Equation (2) now reduces to

$$\begin{bmatrix} E & 0 & \cdots & 0 \\ -A & E & 0 & \vdots \\ 0 & \ddots & \ddots & 0 \\ \vdots & \ddots & -A & E \\ 0 & \cdots & 0 & -A \end{bmatrix} \begin{bmatrix} x_1 - y_1 \\ x_2 - y_2 \\ \vdots \\ x_{N-1} - y_{N-1} \end{bmatrix} = 0$$

Obviously $x_k = y_k$, $k = 1, 2, ..., N-1$ if and only if the matrix in the left hand side of the above equation has full column rank. This matrix is defined in [1] for the regular case as the *conditionabilty matrix*. Similarly we define the conditionabilty matrix of the non-regular system to be

$$C_N = \begin{bmatrix} E & 0 & \cdots & 0 \\ -A & E & 0 & \vdots \\ 0 & \ddots & \ddots & 0 \\ \vdots & \ddots & -A & E \\ 0 & \cdots & 0 & -A \end{bmatrix} \in R^{Np\times(N-1)m} \tag{4}$$

It is well known (see for example [9]) that regularity of the matrix pencil $\sigma E - A$ implies full column rank of $C_N$ and hence conditionabilty of the corresponding system. When the matrix pencil $\sigma E - A$ is not regular then it is possible to have an unconditionable system. This would imply that there are non-trivial solutions even when the boundary values $x_0, x_N$ are both zero. These solutions will have the form $\bar{x}_N = [0, z_1^T, ..., z_{N-1}^T, 0]^T$, with $[z_1^T, ..., z_{N-1}^T]^T \in \ker C_N$.

On the other hand when we have a homogeneous autoregressive representation such as (1), it is natural to expect that the solutions are triggered by non-zero boundary values $x_0, x_N$. This leads to the following definition

**Definition 1** *Two solutions of (1) $(x_k, y_k) \in B \times B$ are said to be "boundary" equivalent iff*

$$x_k - y_k \in [0] \tag{5}$$

*where* $[0] = \{z_k : z_0 = z_N = 0, [z_1^T, ..., z_{N-1}^T]^T \in \ker C_N\}$.

It is a trivial task to verify that (5) defines an equivalence relation between the solutions of (1). This equivalence relation defines a partitioning of the behavior of (1). It is natural to consider the solution space of (1) as the set of equivalence classes of solutions, i.e.

$$\tilde{B} = B/[0] \tag{6}$$

The vector space $\tilde{B}$ consists of equivalence classes mod [0] of the form $[x_k] = \{y_k : y_k = x_k + z_k, \ y_k \in B, z_k \in$

$[0]\} = x_k + [0]$. The solution $x_k$ will be called a representative of the class $[x_k]$. Obviously when the system is conditionable, $[0]$ reduces to $\{0\}$ and $\tilde{B} = B$.

In order to investigate the structure of $\tilde{B}$ we use the well known Kronecker form of the matrix pencil $\sigma E - A$ (see for example [9]). A geometric characterization of $\tilde{B}$, in terms of proper and non-proper deflating subspaces, is also possible but it will be avoided here for simplicity reasons.

It is known that for every matrix pencil $\sigma E - A$, with $E, A \in R^{p\times m}$, there exist two invertible matrices $U \in R^{p\times p}, V \in R^{m\times m}$ such that

$$U(\sigma E - A)V = \begin{bmatrix} \sigma I_n - J_C & 0 & 0 & 0 \\ 0 & \sigma J_\infty - I_\mu & 0 & 0 \\ 0 & 0 & L_\epsilon(\sigma) & 0 \\ 0 & 0 & 0 & L_\eta(\sigma) \end{bmatrix} \tag{7}$$

where the matrix in the right-hand side of the above equation is the Kronecker form of the original pencil. The first block of the diagonal matrix corresponds to the finite (generalized) eigenvalues of $\sigma E - A$ and $J_C$ is considered to be in (real) Jordan form. Similarly the second block corresponds to the infinite eigenvalues of the original pencil and $J_\infty$ is a (nilpotent) matrix in Jordan form with all its diagonal elements equal to zero. The third (fourth) block $L_\epsilon(\sigma)$ $(L_\eta(\sigma))$ is a block diagonal matrix consisting of smaller non-square blocks $L_{\epsilon_i}(\sigma)$, $i = 1, 2, \ldots r$ $(L_{\eta_i}(\sigma), i = 1, 2, \ldots, l)$ of the form $L_{\epsilon_i}(\sigma) = \sigma M_{\epsilon_i} - N_{\epsilon_i}$ $(L_{\eta_i}(\sigma) = \sigma M_{\eta_i}^T - N_{\eta_i}^T)$ with $M_v = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 1 & 0 \end{bmatrix} \in R^{v\times(v+1)}$ and

$N_v = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 \end{bmatrix} \in R^{v\times(v+1)}$, where $v = \epsilon_i$ or $v = \eta_i$. The blocks $L_{\epsilon_i}(\sigma)$ $(L_{\eta_i}(\sigma))$ are the right (left) Kronecker blocks and the indices $\epsilon_i$ $(\eta_i)$ are the right (left) Kronecker indices of $\sigma E - A$. Furthermore let $\epsilon = \sum_{i=1}^r \epsilon_i$ $(\eta = \sum_{i=1}^l \eta_i)$ and $p = n + \mu + \epsilon + \eta + l$, $m = n + \mu + \epsilon + \eta + r$.

The following lemma will be very useful in the sequel

**Lemma 1** *Consider the generalized resultant matrix (see [8]) of $\sigma E - A$*

$$S_k = \begin{bmatrix} -A & E & 0 & \cdots & 0 \\ 0 & -A & E & & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & -A & E \end{bmatrix} \in R^{kp\times(k+1)m} \tag{8}$$

*then $rankS_k = kp - \sum_{\{i: k>\eta_i\}} (k - \eta_i)$ where $\eta_i$, $i = 1, 2, \ldots, l$ are the left Kronecker indices of $\sigma E - A$.*

**Proof:** The proof is straightforward if we apply theorem 1 in [8][1] for the pencil $\sigma E - A$. ∎

Now we can state the following

**Theorem 2** *For any long enough time interval, namely for $N > N_{\min}$ the following hold*

$$\dim B = Nr + m - \eta \tag{9}$$

$$\dim[0] = (N-1)r - \epsilon \tag{10}$$

*where* $N_{\min} = \max_{\substack{i=1,\ldots,r \\ j=1,\ldots,l}} \{\epsilon_i + 1, \eta_i\}$.

**Proof:** In view of (2) it is obvious that $B$ is isomorphic to $\ker S_N$. Now according to lemma 1 $rankS_N = Np - \sum_{\{j:N>\eta_j\}} (N - \eta_j)$. Now for $N > N_{\min}$ we have $rankS_N = Np - \sum_{j=1,\ldots,l} (N - \eta_j) = Np - Nl + \eta$. Thus $\dim\ker S_N = (N+1)m - rankS_N = Nr + m - \eta$. This proves that $\dim B = Nr + m - \eta$. Similarly it is easily seen that $[0]$ is isomorphic to $\ker C_N$. Now applying the previous lemma to

$$C_N^T = \begin{bmatrix} E^T & -A^T & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & E^T & A^T \end{bmatrix} \in R^{(N-1)m\times Np} \tag{11}$$

we take $rankC_N = rankC_N^T = (N-1)m - \sum_{\{i:N>\epsilon_i\}} (N - 1 - \epsilon_i)$ where the right indices of $\sigma E^T - A^T$ are simply the left indices of $\sigma E - A$. Thus for $N > N_{\min}$ we take $rankC_N = Nm - m - (N-1)r + \epsilon$ and $\dim\ker C_N = (N-1)m - rankC_N = Nr - r - \epsilon$. This proves that $\dim[0] = Nr - r - \epsilon$. ∎

In what follows it will be assumed that $N > N_{\min}$ so we can apply the results of the above theorem. We can apply Luenberger's method [2] in order to determine a generalized boundary mapping equation for (1). The boundary mapping equation in regular descriptor systems, is a generalization of the transition matrix in state space systems. Namely the boundary mapping equation gives the relation between the boundary values $x_0$ and $x_N$. Furthermore it summarizes the restrictions posed by the system at both end points of the time interval $k = 0, 1, 2, \ldots, N$.

[1] Actualy equation (2.4) in [8] has a typographical error. From the proof of theorem 1 it is obvious that the correct formula is $rankS_k = (r+q)k - \sum_{\{i:v_i<k\}} (k - v_i)$.

We apply row compression on $S_N$ in (2) to keep only the independent rows of $S_N$. This can be done by premultiplying (2) by an appropriate (invertible) matrix $F_1 \in R^{pN\times pN}$ as follows

$$F_1 S_N \bar{x}_N = 0 \tag{12}$$

$$\begin{bmatrix} X_0 & C'_N & X_N \\ 0 & 0 & 0 \end{bmatrix} \bar{x}_N = 0 \tag{13}$$

where
$X_0, X_N \in R^{(Np-Nl+\eta)\times m}$, $C'_N \in R^{(Np-Nl+\eta\times m(N-1)}$ (recall that $Np - Nl + \eta = rankS_N$ and thus the matrix $[X_0, C'_N, X_N]$ has full row rank) and the zeros are zero matrices of appropriate dimensions. We can drop the zero rows of the matrix in (13), since they play no role in our system. Thus (13) reduces to

$$\begin{bmatrix} X_0 & C'_N & X_N \end{bmatrix} \bar{x}_N = 0 \tag{14}$$

Now the matrix $C'_N$ plays the role of the conditionability matrix and obviously $rankC'_N = rankC_N$ . Thus if we apply row compression in $C'_N$ (by premultiplying $C'_N$ by an appropriate invertible matrix $F_1$) we have

$$F_1 C'_N = \begin{bmatrix} W \\ 0 \end{bmatrix}$$

where $W \in R^{(Nm-m-(N-1)r+\epsilon)\times(N-1)m}$ is a full row rank matrix. Applying this in (14) we take

$$F_1 \begin{bmatrix} X_0 & C'_N & X_N \end{bmatrix} \bar{x}_N = 0$$

$$\begin{bmatrix} \times & & & & \times \\ \vdots & & W & & \vdots \\ \times & & & & \times \\ Z_0 & 0 & \cdots & 0 & Z_N \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_N \end{bmatrix} = 0 \tag{15}$$

where the number of rows of both $Z_0, Z_N$ will be $(Np - Nl + \eta) - (Nm - m - (N-1)r + \epsilon) = n + \mu + 2\eta$. Furthermore the matrix $[Z_0, Z_N]$ will have full row rank since the matrix on the left-hand side of (15) has also full row rank. The last block row of (15) gives rise to the following

**Theorem 3** *For every (possibly) non-regular descriptor system of the form (1) there exists a generalized boundary mapping equation of the form*

$$[Z_0, Z_N] \begin{bmatrix} x_0 \\ x_N \end{bmatrix} = 0 \tag{16}$$

*with* $[Z_0, Z_N] \in R^{(n+\mu+2\eta)\times 2m}$ *and* $rank[Z_0, Z_N] = n + \mu + 2\eta$, *which summarizes the restrictions posed on* $x_0, x_N$ *by the system.*

The above theorem gives a very important result. Obviously $[x_0^T, x_N^T]$ can be chosen from $\ker[Z_0, Z_N]$ whose dimension is

$$\dim\ker[Z_0, Z_N] = 2m - (n+\mu+2\eta) = n+\mu+2(\epsilon+r) \tag{17}$$

Thus there are exactly $n+\mu+2(\epsilon+r)$ degrees of freedom in the choice of $x_0, x_N$. It is natural to expect that this will be also the dimension of $\tilde{B}$ since its elements are equivalence classes of solutions with the same boundary conditions. This will be much more clear in the following section.

## 3 Classification of the Solutions

The matrix pencil $\sigma E - A$ in the original equation (1) can be considered without loss of generality to be in its Kronecker form, since equation (1) can be transformed to its "canonical" form by premultiplying by $U$ and taking a coordinate transformation of the descriptor vector according to $x_k = V\hat{x}_k$.

In this section for simplicity of notation we shall assume that $\sigma E - A$ is already in its canonical form and no distinction will be made between $\hat{x}_k$ and $x_k$. Furthermore we assume that $N > N_{\min}$. In this case it is obvious that the system can be decomposed to several subsystems corresponding to the finite, infinite, right and left Kronecker blocks. We note that we shall use the notation of the previous section for the vector spaces corresponding to each subsystem. In order to avoid confusion of notation we shall distinguish them using the indices $C, \infty, \epsilon_i, \eta_j$. At this point it would be useful to partition correspondingly the decriptor vector as $x_k = \left[(x_k^C)^T, (x_k^\infty)^T, (x_k^{\epsilon_1})^T, \ldots (x_k^{\epsilon_r})^T, (x_k^{\eta_1})^T, \ldots, (x_k^{\eta_l})^T\right]^T$

Now (1) can be written equivalently as

$$\begin{aligned} x_{k+1}^C &= J_C x_k^C & (18) \\ J_\infty x_{k+1}^\infty &= x_k^\infty & (19) \\ M_{\epsilon_i} x_{k+1}^{\epsilon_i} &= N_{\epsilon_i} x_k^{\epsilon_i} & (20) \\ M_{\eta_j}^T x_{k+1}^{\eta_j} &= N_{\eta_j}^T x_k^{\eta_j} & (21) \end{aligned}$$

for $i = 1, 2, ..., r$, $j = 1, 2, ..., l$. The first two equations (18),(19) are of no particular interest since, these two parts correspond to a maximal forward F/B decomposition of a regular system in [3]. The solutions can be easily obtained in terms of arbitrary initial and the final values of $x_0^C, x_N^\infty$ from the formulas $x_k^C = J_C^k x_0^C$ and $x_k^\infty = J_\infty^{N-k} x_N^\infty$. Accordingly we define the behavior of these two subsystems $B_C = \{x_k^C : x_k^C = J_C^k x_0^C\}$ and $B_\infty = \{x_k^\infty : x_k^\infty = J_\infty^{N-k} x_N^\infty\}$. Furthermore $\dim B_C = n$ and $\dim B_\infty = \mu$ since the columns of the corresponding matrices $J_C^k, J_\infty^{N-k}$ are linearly independent sequences. Obviously these two subsystems are conditionable (they are regular descriptor systems) and thus $[0]_C = \{0\}$, $\tilde{B}_C = B_C$ and $[0]_{B_\infty} = \{0\}$, $\tilde{B}_\infty = B_\infty$.

We examine now the third group of equations (20). For equation (20) define similarly the vector spaces $B_{\epsilon_i}$ and $[0]_{\epsilon_i}$. Then according to theorem 2 and if we take into account that $\sigma M_{\epsilon_i} + N_{\epsilon_i}$ has only the right index $\epsilon_i$, we have

$$\dim B_{\epsilon_i} = N + \epsilon_i + 1 \tag{22}$$

$$\dim[0]_{\epsilon_i} = N - \epsilon_i - 1 \tag{23}$$

It is obvious that $[0]_{\epsilon_i} \subseteq B_{\epsilon_i}$. In order to determine the set of equivalence classes $\tilde{B}_{\epsilon_i} = B_{\epsilon_i}/[0]_{\epsilon_i}$ of (20) we need to find a representative for every non-zero equivalence class. This is a relatively easy task if we consider the following

**Lemma 4** *If $J_{\epsilon_i} \in R^{(\epsilon_i+1)\times(\epsilon_i+1)}$ is Jordan block with all its diagonal entries equal to zero then*

$$M_{\epsilon_i} J_{\epsilon_i} = N_{\epsilon_i} \quad \text{and} \quad N_{\epsilon_i} J_{\epsilon_i}^T = M_{\epsilon_i} \tag{24}$$

**Proof:** The proof follows simply by straightforward computation if we take into account the special form of $M_{\epsilon_i}, N_{\epsilon_i}, J_{\epsilon_i}$. ∎

In view of (24) we can verify that $M_{\epsilon_i} J_{\epsilon_i}^{k+1} = N_{\epsilon_i} J_{\epsilon_i}^k$ and $N_{\epsilon_i}(J_{\epsilon_i}^T)^{k+1} = M_{\epsilon_i}(J_{\epsilon_i}^T)^k$for every $k = 0,1,2,...,N-1$, which means that the columns of both $J^k$, $(J^T)^{N-k}$ satisfy (20). Furthermore these columns are linearly independent solutions. This leads to the following

**Definition 2** $B_{\epsilon_i}^f = \{x_k^{\epsilon_i} : x_k^{\epsilon_i} = J_{\epsilon_i}^k x_0^{\epsilon_i}\}$ *and* $B_{\epsilon_i}^b = \{x_k^{\epsilon_i} : x_k^{\epsilon_i} = (J_{\epsilon_i}^T)^{N-k} x_N^{\epsilon_i}\}$.

The indices $f, b$ in the above defined subspaces stand for *"forward"* and *"backward"* respectively. The subspace $B_{\epsilon_i}^f$ contains solutions arising from the forward propagation of the initial value $x_0^{\epsilon_i}$, while $B_{\epsilon_i}^b$ contains solution moving in the opposite direction.

**Lemma 5** $\dim B_{\epsilon_i}^f = \epsilon_i + 1$ *and* $\dim B_{\epsilon_i}^b = \epsilon_i + 1$.

**Proof:** The proof is straightforward if we take into account that the columns of both $J^k$, $(J^T)^{N-k}$, for $k = 0,1,2,...,N$, are linearly independent solutions of (20). The dimension of both $B_{\epsilon_i}^f, B_{\epsilon_i}^b$ is equal to the dimension of $J^k$ or $(J^T)^{N-k}$, which is $\epsilon_i + 1$. ∎

**Theorem 6**

$$B_{\epsilon_i} = B_{\epsilon_i}^f \oplus B_{\epsilon_i}^b \oplus [0]_{\epsilon_i} \tag{25}$$

**Proof:** Consider first a solution $x_k^{\epsilon_i}$ such that $x_k^{\epsilon_i} \in B_{\epsilon_i}^f$ and $x_k^{\epsilon_i} \in B_{\epsilon_i}^b$ then

$$x_k^{\epsilon_i} = J_{\epsilon_i}^k x_0^{\epsilon_i} = (J_{\epsilon_i}^T)^{N-k} x_N^{\epsilon_i} \tag{26}$$

If we take into account that $J_{\epsilon_i}^k = (J_{\epsilon_i}^T)^k = 0$ for every $k \geq \epsilon_i + 1$ and the assumption that $N > N_{\min} \geq \epsilon_i$ from (26) we take $x_N^{\epsilon_i} = J_{\epsilon_i}^N x_0^{\epsilon_i} = 0$ and $x_0^{\epsilon_i} = (J_{\epsilon_i}^T)^N x_N^{\epsilon_i} = 0$. Thus $x_k^{\epsilon_i} = 0$, for every $k = 0,1,2,...,N$. This proves that $B_{\epsilon_i}^f \cap B_{\epsilon_i}^b = \{0\}$and we can define

$$B_{\epsilon_i}^{f/b} = B_{\epsilon_i}^f \oplus B_{\epsilon_i}^b = \{x_k^{\epsilon_i} : x_k^{\epsilon_i} = J_{\epsilon_i}^k x_0^{\epsilon_i} + (J_{\epsilon_i}^T)^{N-k} x_N^{\epsilon_i}\} \tag{27}$$

Now it is easy to see that $B_{\epsilon_i}^{f/b} \cap [0]_{\epsilon_i} = \{0\}$. Indeed if $x_k^{\epsilon_i} \in B_{\epsilon_i}^{f/b}$ and $x_k^{\epsilon_i} \in [0]_{\epsilon_i}$ then $x_0^{\epsilon_i} = x_N^{\epsilon_i} = 0$ and from (27) $x_k^{\epsilon_i} = 0$, for $k = 0,1,2,...,N$. Now we can define the direct sum $B_{\epsilon_i}^{f/b} \oplus [0]_{\epsilon_i} \equiv B_{\epsilon_i}^f \oplus B_{\epsilon_i}^b \oplus [0]_{\epsilon_i}$. If we take into account the dimensions of these subspaces we have $\dim(B_{\epsilon_i}^f \oplus B_{\epsilon_i}^b \oplus [0]_{\epsilon_i}) = N + \epsilon_i + 1$ which coincides with $\dim B_{\epsilon_i} = N + \epsilon_i + 1$. Thus $B_{\epsilon_i} = B_{\epsilon_i}^f \oplus B_{\epsilon_i}^b \oplus [0]_{\epsilon_i}$. ∎

We introduce now the vector space of equivalence classes mod $[0]_{\epsilon_i}$ corresponding to (20)

$$\tilde{B}_{\epsilon_i} = B_{\epsilon_i}/[0]_{\epsilon_i} \tag{28}$$

**Theorem 7**

$$\tilde{B}_{\epsilon_i} = (B_{\epsilon_i}^f \oplus B_{\epsilon_i}^b)/[0]_{\epsilon_i} = B_{\epsilon_i}^f/[0]_{\epsilon_i} \oplus B_{\epsilon_i}^b/[0]_{\epsilon_i} \tag{29}$$

$$\dim \tilde{B}_{\epsilon_i} = 2(\epsilon_i + 1) \tag{30}$$

**Proof:** Obviously $B_{\epsilon_i}^f \oplus B_{\epsilon_i}^b \subseteq B_{\epsilon_i}$ which implies $(B_{\epsilon_i}^f \oplus B_{\epsilon_i}^b)/[0]_{\epsilon_i} \subseteq B_{\epsilon_i}/[0]_{\epsilon_i} = \tilde{B}_{\epsilon_i}$. Now let $[x_k^{\epsilon_i}] \in \tilde{B}_{\epsilon_i}$. From (25) we can uniquely write $x_k^{\epsilon_i} = y_k^{\epsilon_i} + \hat{y}_k^{\epsilon_i} + z_k^{\epsilon_i}$ with $y_k^{\epsilon_i} \in B_{\epsilon_i}^f, \hat{y}_k^{\epsilon_i} \in B_{\epsilon_i}^b$ and $z_k^{\epsilon_i} \in [0]_{\epsilon_i}$. Obviously $[x_k^{\epsilon_i}] = y_k^{\epsilon_i} + \hat{y}_k^{\epsilon_i} + [0]_{\epsilon_i}$ which implies that $[x_k^{\epsilon_i}] \in (B_{\epsilon_i}^f \oplus B_{\epsilon_i}^b)/[0]_{\epsilon_i}$. Thus $\tilde{B}_{\epsilon_i} \subseteq (B_{\epsilon_i}^f \oplus B_{\epsilon_i}^b)/[0]_{\epsilon_i}$. This proves that $\tilde{B}_{\epsilon_i} = (B_{\epsilon_i}^f \oplus B_{\epsilon_i}^b)/[0]_{\epsilon_i}$. For the second part of (29) we have $B_{\epsilon_i}^f \subseteq B_{\epsilon_i}$ and $B_{\epsilon_i}^b \subseteq B_{\epsilon_i}$ and obviously $B_{\epsilon_i}^f/[0]_{\epsilon_i} + B_{\epsilon_i}^b/[0]_{\epsilon_i} \subseteq B_{\epsilon_i}/[0]_{\epsilon_i}$. Inversely if $[x_k^{\epsilon_i}] \in \tilde{B}_{\epsilon_i}$ then according to the above unique decomposition of $x_k^{\epsilon_i}$ we have $[x_k^{\epsilon_i}] = y_k^{\epsilon_i} + \hat{y}_k^{\epsilon_i} + [0]_{\epsilon_i}$ or $[x_k^{\epsilon_i}] = (y_k^{\epsilon_i} + [0]_{\epsilon_i}) + (\hat{y}_k^{\epsilon_i} + [0]_{\epsilon_i}) \in B_{\epsilon_i}^f/[0]_{\epsilon_i} + B_{\epsilon_i}^b/[0]_{\epsilon_i}$. Thus $B_{\epsilon_i}/[0]_{\epsilon_i} \subseteq B_{\epsilon_i}^f/[0]_{\epsilon_i} + B_{\epsilon_i}^b/[0]_{\epsilon_i}$. This proves that $B_{\epsilon_i}/[0]_{\epsilon_i} = B_{\epsilon_i}^f/[0]_{\epsilon_i} + B_{\epsilon_i}^b/[0]_{\epsilon_i}$. Furthermore it is easy to see that $B_{\epsilon_i}^f/[0]_{\epsilon_i} \cap B_{\epsilon_i}^b/[0]_{\epsilon_i} = [0]_{\epsilon_i}$. This allows us to write $\tilde{B}_{\epsilon_i} = B_{\epsilon_i}^f/[0]_{\epsilon_i} \oplus B_{\epsilon_i}^b/[0]_{\epsilon_i}$. In order to prove (30) it is enough to see that the mapping $(B_{\epsilon_i}^f \oplus B_{\epsilon_i}^b) \ni x_k^{\epsilon_i} \mapsto [x_k^{\epsilon_i}] \in \tilde{B}_{\epsilon_i}$ is clearly an isomorphism and thus $\dim \tilde{B}_{\epsilon_i} = \dim B_{\epsilon_i}^f + \dim B_{\epsilon_i}^b = 2(\epsilon_i + 1)$. ∎

Applying these results to (20) for all $i = 1,2,...,r$ we define

$$\tilde{B}_\epsilon = B_\epsilon/[0]_\epsilon \tag{31}$$

where $B_\epsilon = \bigoplus_{i=1}^r B_{\epsilon_i}$ and $[0]_\epsilon = \bigoplus_{i=1}^r [0]_{\epsilon_i}$ (these direct sums are well defined because of the block diagonal structure of the Kronecker form). The previous theorem can be extended if we define $B^f_\epsilon = \bigoplus_{i=1}^{r} B^f_{\epsilon_i}$ and $B^b_\epsilon = \bigoplus_{i=1}^{r} B^b_{\epsilon_i}$. We have

$$\tilde{B}_\epsilon = (B^f_\epsilon \oplus B^b_\epsilon)/[0]_\epsilon \tag{32}$$

$$\dim \tilde{B}_\epsilon = \dim(B^f_\epsilon \oplus B^b_\epsilon) = 2\sum_{i=1}^{r}(\epsilon_i + 1) = 2(\epsilon + r) \tag{33}$$

We examine now the fourth group of equations (21). Applying theorem 2 and the fact that the pencil $\sigma M^T_{\eta_j} - N^T_{\eta_j}$ has only the left index $\eta_j$ we take $\dim B_{\eta_j} = 0 + \eta_j - \eta_j = 0$ and $\dim[0]_{\eta_j} = 0$. Hence $B_{\eta_j} = \{0\}$, which means that this part of the system has only the trivial solution $x^{\eta_i}_k = 0$, $k = 0, 1, 2, ..., N$. Extending this result for all $j = 1, 2, 3, ..., l$ we have $B_\eta = \bigoplus_{j=1}^{l} B_{\eta_j} = \{0\}$.

Now that we have completed the investigation of the solution space we recall the definition of $\tilde{B}$ in (6) as the set equivalence classes mod $[0] = [0]_C \oplus [0]_\infty \oplus [0]_\epsilon \oplus [0]_\eta = [0]_\epsilon$. It is a trivial task to see that the following holds

**Theorem 8** $\tilde{B} = (B_C \oplus B_\infty \oplus B^f_\epsilon \oplus B^b_\epsilon)/[0]$ *and* $\dim \tilde{B} = \dim(B_C \oplus B_\infty \oplus B^f_\epsilon \oplus B^b_\epsilon) = n + \mu + 2(\epsilon + r)$.

It is not surprising that $\dim \tilde{B}$ coincides with the number of degrees of freedom in the choice of boundary values $x_0, x_N$ in (16).

## 4 Conclusions

The lack of conditionability, due to the presence of right Kronecker indices in the pencil $\sigma E - A$, plays a crucial role in the classification of the solutions of the non-regular $Ex_{k+1} = Ax_k$. The set of solutions arising, from zero boundary conditions $x_0 = x_N = 0$ is defined as the zero equivalence class [0] and the set of all solutions $B$ is partitioned to equivalence classes having as elements solutions with the same boundary values $x_0, x_N$. It has been shown that the source of unconditionability and hence the existence of non-trivial zero equivalence class [0], are the right Kronecker indices.

The finite and infinite eigenvalue structure of the pencil gives rise to forward and backward solutions respectively, while the right indices give both forward and backward solutions. On the other hand the left indices give only the trivial zero solution, restricting this way the choice of boundary conditions of the complete equation.

We should finaly notice that all the above results can be easily expressed even when the matrix $\sigma E - A$ is not in its canonical form, using the transformation matrices $U, V$.

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