ON A NEW NOTION OF EQUIVALENCE FOR POLYNOMIAL MATRICES

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Abstract: The aim of this work is to present necessary conditions, in order two polynomial matrices possess the same finite and infinite elementary divisors.

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1. INTRODUCTION

Consider the linear, homogeneous, matrix differential equation :

$$A(\rho)\beta(t) = 0 \ ; \ t \geq 0- \tag{1}$$

where $r := d/dt$ is the differential operator, $A(\rho)$ is an $r \times r$ non-singular (over $R[\rho]$) polynomial matrix i.e. $A(\rho) \in R[\rho]^{r\times r}$ and $\beta(t) : (0-, \infty) \longrightarrow R^r$ i.e. an $r-$dimensional vector-valued function. It is known (Vardulakis, 1991) that (1) exhibits both smooth and impulsive behavior due the finite and infinite zeros of $A(s)$ respectively. The problem of defining an equivalence relation betwen the smooth and impulsive solution sets of two systems of the form (1) has been shown (Pugh et.al., 1992) to be equivalent to the problem of finding an equivalence transformation betwen the polynomial matrices which describes the two systems. Actually full equivalence (Hayton et.al., 1988) is a transformation between polynomial matrices which preserves both the finite and the infinite zeros. A geometric interpretation of full equivalence in terms of maps between the smooth and impulsive solution sets of the systems described by the equivalent matrices can be found in (Pugh et.al., 1992).

Consider now a linear, homogeneous, matrix difference equation :

$$A(\sigma)\beta(k) = 0 \ ; \ k \in [0, N] \tag{2}$$

where σ denotes the forward shift operator i.e. $\sigma\beta(k) = \beta(k+1)$ and $\beta(k) : [0, N] \longrightarrow R^r$. It is known (Antoniou et.al., 1998) that (2) exhibits both forward and backward solutions due the finite and infinite elementary divisors of $A(\sigma)$ (and not the infinite zeros). Like in the continuous-time case we are interested to find out a transformation between square and nonsingular polynomial matrices which preserves both the finite and infinite elementary divisors. Actually in case where $A(\sigma) = \sigma E - A$, then strict equivalence (Gantmacher, 1959) is the transformation we are looking for.

The aim of this work is to present a new transformation between polynomial matrices with the nice property of preserving both the finite and infinite elementary divisors. However the proposed transformation gives only necessary but not sufficient conditions. A matrix pencil equivalent of a polynomial matrix is given as an implication of this new transformation.

2. PRELIMINARY RESULTS

Consider the set $P(m, m)$ of $(r+m) \times (r+m)$ polynomial matrices where $r \geq -m$.

Definition 1. Consider the polynomial matrix

$$A(s) = A_0 + A_1 s + \cdots + A_q s^q \in R[s]^{r\times r} \quad (3)$$

and define as the "dual" of $A(s)$:

$$\tilde{A}(s) = A_0 s^q + A_1 s^{q-1} + \cdots + A_q \in R[s]^{r\times r} \quad (4)$$

Then $A(s)$ is said to have an *infinite elementary divisor* of degree q whenever its dual $\tilde{A}(s)$ has a finite elementary divisor of the form s^q. ■

The finite elementary divisors of $A(s)$ describe the finite zero structure of the matrix polynomial, but the infinite elementary divisors give a complete description of the total structure at infinity (pole and zero structure) and not simply that associated with the zeros (Hayton et.al., 1988),(Vardulakis, 1991).

A necessary and sufficient condition for a polynomial matrix to have no infinite elementary divisors is given in the following Lemma.

Lemma 1. Consider $A(s)$ defined in (3). $A(s)$ has no infinite elementary divisors iff

$$rank_R A_q = r$$

Proof.

($\Rightarrow$) If $rank_R A_q = r$ then $rank\tilde{A}(0) = r$ and therefore $\tilde{A}(s)$ has no finite elementary divisors of the form s^q.

($\Rightarrow$) If $A(s)$ has no infinite elementary divisors then $\tilde{A}(s)$ has no finite elementary divisors of the form s^q and therefore $\tilde{A}(s)$ does not lose rank at $s = 0$. Therefore $rank_R\tilde{A}(0) = r \Longleftrightarrow rank_R A_q = r$. ■

It is easily seen that the proof is independent of the dimension of $A(s)$ and therefore can be applied to any polynomial matrix. An important transformation between polynomial matrices is given in the sequel.

Definition 2. (Pugh and Shelton, 1978)

Two matrices $A_1(s), A_2(s) \in P(m,m)$ are said to be *extended unimodular equivalent* (*e.u.e.*) if there exist polynomial matrices $M(s), N(s)$ of appropriate dimensions, such that

$$M(s)A_1(s) = A_2(s)N(s) \quad (5)$$

$$\text{or } \begin{bmatrix} M(s) & A_2(s) \end{bmatrix} \begin{bmatrix} A_1(s) \\ -N(s) \end{bmatrix} = 0$$

where $M(s)$ and $A_2(s)$ (respectively $A_1(s)$ and $N(s)$) are relatively left (respectively right) prime i.e.

$$\begin{bmatrix} M(s) & A_2(s) \end{bmatrix} \; ; \; \begin{bmatrix} A_1(s) \\ -N(s) \end{bmatrix} \text{ have full rank } \forall s \in C \quad (6)$$

■

Some important properties of this transfornation are contained in the following statement.

Lemma 2. (Pugh and Shelton, 1978)

(i) E.u.e. is an equivalence relation on $P(m,m)$.

(ii) $A_1(s), A_2(s) \in P(m,m)$ are e.u.e. if and only if they have the same finite elementary divisors. ■

However e.u.e. does not preserves the infinite elementary divisors as we can see in the following example :

Example 1. Consider the following e.u.e. transformation

$$\underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{M(s)} \underbrace{\begin{bmatrix} 1 & s^2 \\ 0 & s+1 \end{bmatrix}}_{A_1(s)} = \underbrace{\begin{bmatrix} 1 & s^3 \\ 0 & s+1 \end{bmatrix}}_{A_2(s)} \underbrace{\begin{bmatrix} 1 & s^2 - s^3 \\ 0 & 1 \end{bmatrix}}_{N(s)}$$

Although $A_1(s), A_2(s)$ have the same finite elementary divisors i.e.

$$S^C_{A_1(s)}(s) = \begin{bmatrix} 1 & 0 \\ 0 & s+1 \end{bmatrix} = S^C_{A_1(s)}(s)$$

they have different infinite elementary divisors i.e.

$$S^0_{\tilde{A}_1(s)}(s) = S^0_{\begin{bmatrix} s^2 & 1 \\ 0 & s+s^2 \end{bmatrix}} = \begin{bmatrix} 1 & 0 \\ 0 & s^3 \end{bmatrix}$$

$$S^0_{\tilde{A}_2(s)}(s) = S^0_{\begin{bmatrix} s^3 & 1 \\ 0 & s^2+s^3 \end{bmatrix}} = \begin{bmatrix} 1 & 0 \\ 0 & s^5 \end{bmatrix}$$

■

The above example indicates that further restrictions must be placed on the compound matrices (6) in order to ensure that the associated transformation will leave invariant finite and infinite elementary divisors.

3. A NEW MATRIX TRANSFORMATION

A new transformation between polynomial matrices is given in the following definition.

Definition 3. Two matrices $A_1(s), A_2(s) \in P(m,m)$ are said to be *divisor equivalent* (*d.e.*) if there exist polynomial matrices $M(s), N(s)$ of appropriate dimensions, such that

$$M(s)A_1(s) = A_2(s)N(s) \quad (7)$$
$$\text{or } \begin{bmatrix} M(s) & A_2(s) \end{bmatrix} \begin{bmatrix} A_1(s) \\ -N(s) \end{bmatrix} = 0$$

where

(i)

$$\begin{bmatrix} M(s) & A_2(s) \end{bmatrix} \; ; \; \begin{bmatrix} A_1(s) \\ -N(s) \end{bmatrix} \quad (8)$$

have no finite nor infinite elementary divisors,

(ii)

$$d\begin{bmatrix} M(s) & A_2(s) \end{bmatrix} = d[A_2(s)] \; ; d\begin{bmatrix} A_1(s) \\ -N(s) \end{bmatrix} = d[A_1(s)] \quad (9)$$

where d denotes the degree of the highest coefficient matrix of the certain polynomial matrix i.e. $d[A_0 + A_1 s + \cdots + A_q s^q] = q$. ■

Note that according to Lemma 1 the condition that the compound matrices defined in (8) have no infinite elementary divisors, is equivalent to "the highest coefficient matrix in each compound matrix" has full row (column) rank".

An important property of the above transformation is given by the following

Theorem 3. If $A_1(s), A_2(s) \in P(m,m)$ are divisor equivalent then they have the same finite and infinite elementary divisors.

Proof.

According to the definition of "divisor equivalence", $A_1(s)$ and $A_2(s)$ are also e.u.e. and thus have the same finite elementary divisors.

(7) may be rewritten, by setting $s = \frac{1}{w}$, as

$$\begin{bmatrix} M(\frac{1}{w}) & A_2(\frac{1}{w}) \end{bmatrix} \begin{bmatrix} A_1(\frac{1}{w}) \\ -N(\frac{1}{w}) \end{bmatrix} = 0$$

or equivalently by premultiplying and postmultiplying the above relation by

$$w^d \begin{bmatrix} M(s) & A_2(s) \end{bmatrix} \quad \text{and} \quad w^d \begin{bmatrix} A_1(s) \\ -N(s) \end{bmatrix}$$

respectively we get

$$\begin{bmatrix} \widetilde{M(w) \; A_2(w)} \end{bmatrix} \begin{bmatrix} \widetilde{A_1(w)} \\ -N(w) \end{bmatrix} = 0 \quad (10)$$

where $\tilde{}$ denotes the dual matrix. Now since $d\begin{bmatrix} M(s) & A_2(s) \end{bmatrix} = d[A_2(s)]$ and $d\begin{bmatrix} A_1(s) \\ -N(s) \end{bmatrix} = d[A_1(s)]$ equation (10) may be rewritten as :

$$\begin{bmatrix} M'(w) & \tilde{A}_2(w) \end{bmatrix} \begin{bmatrix} \tilde{A}_1(w) \\ -N'(w) \end{bmatrix} = 0 \quad (11)$$

The compound matrices in (10) have two kind of zeros : (i) finite zeros at $w = 0$ and (ii) finite zeros at $w \neq 0$. First of all $\begin{bmatrix} M(s) & A_2(s) \end{bmatrix}$ has no infinite elementary divisors and therefore its dual $\begin{bmatrix} M'(w) & \tilde{A}_2(w) \end{bmatrix}$ has no finite zeros at $w = 0$. Secondly there is a connection between the nonzero eigenvalues of $\begin{bmatrix} M(s) & A_2(s) \end{bmatrix}$ and its dual $\begin{bmatrix} M'(w) & \tilde{A}_2(w) \end{bmatrix}$ (Vardulakis, 1991) i.e. if $s_0 \neq 0$ is an eigenvalues of $\begin{bmatrix} M(s) & A_2(s) \end{bmatrix}$ then $w = s_0^{-1}$ is an eigenvalue of $\begin{bmatrix} M'(w) & \tilde{A}_2(w) \end{bmatrix}$. However $\begin{bmatrix} M(s) & A_2(s) \end{bmatrix}$ has no finite nonzero eigenvalues and therefore $\begin{bmatrix} M'(w) & \tilde{A}_2(w) \end{bmatrix}$ has also no finite nonzero eigenvalues. Therefore $\begin{bmatrix} M'(w) & \tilde{A}_2(w) \end{bmatrix}$ has no zeros at all. Similar results applied for the second compound matrix $\begin{bmatrix} \tilde{A}_1(w) \\ -N'(w) \end{bmatrix}$. Thus (11) is an e.u.e. transformation which preserves, according to Lemma 1, the finite elementary divisors of $\tilde{A}_1(w), \tilde{A}_2(w)$ or otherwise the infinite elementary divisors of $A_1(s), A_2(s)$. ■

Example 2. Consider the transformation

$$\underbrace{\begin{bmatrix} s+1 & 0 \\ s^2 & 0 \\ 0 & -1 \end{bmatrix}}_{M(s)} \underbrace{\begin{bmatrix} s^2 & 1 \\ 0 & s^3 \end{bmatrix}}_{A_1(s)} = \underbrace{\begin{bmatrix} s^2 & 1 & 0 \\ 0 & s & 1 \\ 0 & 0 & s^2 \end{bmatrix}}_{A_2(s)} \underbrace{\begin{bmatrix} 1 & 0 \\ s^3 & s+1 \\ 0 & -s \end{bmatrix}}_{N(s)}$$

It is easily seen that :

$$S^C_{\begin{bmatrix} M(s) & A_2(s) \end{bmatrix}}(s) = \begin{bmatrix} I_3 & 0 \end{bmatrix}$$
$$S^0_{\widetilde{\begin{bmatrix} M(s) & A_2(s) \end{bmatrix}}}(s) = \begin{bmatrix} I_3 & 0 \end{bmatrix}$$

while

$$S^C_{\begin{bmatrix} A_1(s) \\ -N(s) \end{bmatrix}} = \begin{bmatrix} I_2 \\ 0 \end{bmatrix} \quad ; \quad S^0_{\widetilde{\begin{bmatrix} A_1(s) \\ -N(s) \end{bmatrix}}} = \begin{bmatrix} I_2 \\ 0 \end{bmatrix}$$

The degree condition is also satisfied and thus $A_1(s)$ and $A_2(s)$ are divisor equivalent. Therefore they have the same finite and infinite elementary divisors i.e.

$$S^C_{A_1(s)} = \begin{bmatrix} 1 & 0 \\ 0 & s^5 \end{bmatrix} \; ; S^C_{A_2(s)} = \begin{bmatrix} I_2 & 0 \\ 0 & s^5 \end{bmatrix}$$

$$S^0_{\tilde{A}_1(s)} = \begin{bmatrix} 1 & 0 \\ 0 & s \end{bmatrix} \; ; S^0_{\tilde{A}_2(s)} = \begin{bmatrix} I_2 & 0 \\ 0 & s \end{bmatrix}$$

■

In the special case of matrix pencils of the same dimension, strict equivalence and divisor equivalence define the same equivalence class as we can easily show in the following theorem.

Corollary 4. Let $sE_1 - A_1, sE_2 - A_2 \in R[s]^{m\times m}$ with $\det[sE_i - A_i] \neq 0$. Then $sE_1 - A_1, sE_2 - A_2$ are strict equivalent iff they are divisor equivalent.

Proof. If $sE_1 - A_1, sE_2 - A_2$ are strict equivalent then there exists constant, square and nonsingular matrices $M, N \in E^{m\times m}$ such that :

$$M[sE_1 - A_1] = [sE_2 - A_2]N$$

Then select s_0 such that $\{\det[s_0 E_i - A_i] \neq 0, i = 1, 2\}$ and construct the transformation :

$$[M(s - s_0)][sE_1 - A_1] = [sE_2 - A_2][N(s - s_0)]$$

It is easily seen that the above transformation is a divisor equivalence transformation.

Suppose that $sE_1 - A_1, sE_2 - A_2$ are divisor equivalent. Then according to Theorem 3.1, $sE_1 - A_1$ and $sE_2 - A_2$ possess the same finite and infinite elementary divisors. Therefore (Gantmacher, 1959) the pencils $sE_1 - A_1$ and $sE_2 - A_2$ are strict equivalent. ■

An interesting question is if we are able to reduce a polynomial matrix to a divisor equivalent matrix pencil ? The answer to this question is given in the next section.

4. MATRIX PENCIL DIVISOR EQUIVALENTS OF A POLYNOMIAL MATRIX.

Consider the polynomial matrix $A(s)$ defined in (3) and define the following pencil

$$sE - A := \begin{bmatrix} sI_r & -I_r & 0 & \cdots & 0 & 0 \\ 0 & sI_r & -I_r & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & sI_r & -I_r \\ A_0 & A_1 & A_2 & \cdots & A_{q-2} & sA_q + A_{q-1} \end{bmatrix} \tag{12}$$

The above pencil has originally been defined in (Gohberg et.al., 1982) while it was shown in (Praagman, 1991) and (Antoniou and Vardulakis, 2001) that $A(s)$ and $sE - A$ share the same finite and infinite elementary divisor structure. In what follows we shall show that $A(s)$ and $sE - A$ are divisor equivalent and thus have the same finite and infinite elementary divisor structure.

Theorem 5. Consider $A(s)$ defined in (3) and $sE - A$ defined in (12). Then $A(s)$ and $sE - A$ are divisor equivalent.

Proof. Select a pencil $sI_r - J$ [1] such that :

$$S^C_{\left[\, sI_r - J \;\; A(s) \,\right]}(s) = \left[\, I_r \;\; 0 \,\right]$$

i.e. have no common zeros with $A(s)$. Then consider the transformation

$$\underbrace{\begin{bmatrix} 0_{(q-1)r,r} \\ sI_r - J \end{bmatrix}}_{M(s)} A(s) = \tag{13}$$

$$= \underbrace{\begin{bmatrix} sI_r & -I_r & 0 & \cdots & 0 & 0 \\ 0 & sI_r & -I_r & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & sI_r & -I_r \\ A_0 & A_1 & A_2 & \cdots & A_{q-2} & sA_q + A_{q-1} \end{bmatrix}}_{sE-A} \underbrace{\begin{bmatrix} (sI_r - J) \\ (sI_r - J)s \\ \vdots \\ (sI_r - J)s^{q-1} \end{bmatrix}}_{N(s)}$$

(i) Consider the compound matrix

$$\left[\, sE - A \;\; M(s) \,\right] =$$

$$= \begin{bmatrix} sI_r & -I_r & 0 & \cdots & 0 & 0 & 0 \\ 0 & sI_r & -I_r & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & sI_r & -I_r & 0 \\ A_0 & A_1 & A_2 & \cdots & A_{q-2} & sA_q + A_{q-1} & (sI_r - J) \end{bmatrix}$$

(i-1) We can easily find two greatest order minors of $\left[\, sE - A \;\; M(s) \,\right]$ i.e.

$$L_1(s) := \begin{bmatrix} sI_r & -I_r & 0 & \cdots & 0 & 0 \\ 0 & sI_r & -I_r & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & sI_r & -I_r \\ A_0 & A_1 & A_2 & \cdots & A_{q-2} & sA_q + A_{q-1} \end{bmatrix}$$

$$L_2(s) := \begin{bmatrix} -I_r & 0 & \cdots & 0 & 0 & 0 \\ sI_r & -I_r & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & sI_r & -I_r & 0 \\ A_1 & A_2 & \cdots & A_{q-2} & sA_q + A_{q-1} & (sI_r - J) \end{bmatrix}$$

with

[1] Select $s_0 : det[A(s_0)] \neq \mathbf{0}$ and define $sI_r - J := (s - s_0) * I$.

$$det\,[L_1(s)] = det\,[A(s)] \text{ and } det\,[L_2(s)] = det\,[sI_r - J]$$

However J is selected such that the greatest common divisor of $det\,[L_1(s)]$ and $det\,[L_2(s)]$ to be 1. Therefore the compound matrix $\left[\, sE - A \;\; M\,(s)\,\right]$ has no finite elementary divisors.

(i-2) It is easily seen that the highest degree coefficient matrix of $\left[\, sE - A \;\; M\,(s)\,\right]$ i.e.

$$\left[\, E \;\; \tilde{M}(0)\,\right] = \begin{bmatrix} I_r & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & I_r & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & I_r & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & A_q & I_r \end{bmatrix}$$

has full row rank and therefore according to Lemma 1 the compound matrix $\left[\, sE - A \;\; M\,(s)\,\right]$ has no infinite elementary divisors.

(i-3) Both $sE - A$ and $M(s)$ are of degree 1 and therefore $d\left[\, sE - A \;\; M\,(s)\,\right] = 1 = d\,[sE - A]$.

(ii) Consider the second compound matrix

$$\begin{bmatrix} A(s) \\ -N(s) \end{bmatrix} = \begin{bmatrix} A(s) \\ (sI_r - J) \\ (sI_r - J)\, s \\ \vdots \\ (sI_r - J)\, s^{q-1} \end{bmatrix}$$

(ii-1) We can easily find two greatest order minors i.e.

$$Q_1(s) = A(s) \text{ and } Q_2(s) = sI_r - J$$

with

$$\begin{aligned} det\,[Q_1(s)] &= det\,[A(s)] \\ det\,[Q_2(s)] &= det\,[sI_r - J] \end{aligned}$$

However J is selected such that the greatest common divisor of $det\,[Q_1(s)]$ and $det\,[Q_2(s)]$ to be 1. Therefore the compound matrix $\begin{bmatrix} A(s) \\ -N(s) \end{bmatrix}$ has no finite elementary divisors.

(ii-2) The highest degree coefficient matrix of $\begin{bmatrix} A(s) \\ -N(s) \end{bmatrix}$ i.e.

$$\left[\, A_q^T \;\; 0 \; \cdots \; 0 \;\; I_r \,\right]^T$$

has full column rank and thus according to Lemma 1, the compound matrix $\begin{bmatrix} A(s) \\ -N(s) \end{bmatrix}$ has no infinite elementary divisors.

(ii-3) Note also that

$$d \begin{bmatrix} A(s) \\ -N(s) \end{bmatrix} = q = d\,[A(s)]$$

From (i) and (ii) we conclude that $A(s)$ and $sE - A$ are divisor equivalent. ■

Corollary 6. $A(s)$ and $sE - A$ are divisor equivalent and therefore according to Theorem 3 have the same finite and infinite elementary divisors. ■

Corollary 7. The symmetric transformation between $A(s)$ and $sE - A$ is :

$$\underbrace{\left[\begin{matrix} -\,(sI_r - J)\, s^{q-2} A_0 & -\,(sI_r - J)\, s^{q-3}\,(A_0 + A_1 s) & \cdots \\ -\,(sI_r - J)\,\left(A_0 + \cdots A_{q-2} s^{q-2}\right) & (sI_r - J)\, s^{q-1} & \end{matrix}\right]}_{M(s)} \times$$

$$\times \underbrace{\begin{bmatrix} sI_r & -I_r & 0 & \cdots & 0 & 0 \\ 0 & sI_r & -I_r & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & sI_r & -I_r \\ A_0 & A_1 & A_2 & \cdots & A_{q-2} & sA_q + A_{q-1} \end{bmatrix}}_{sE-A} =$$

$$= A(s) \underbrace{\left[\, 0_{r,(q-1)r} \;\; (sI_r - J)\,\right]}_{N(s)}$$

where J is defined in a similar way with the previous Theorem. ■

A direct implication of the above Theorem is given in the following Example.

Example 3. Consider the polynomial matrix $A(s)$ of example 1

$$A(s) := \begin{bmatrix} 1 & s^2 \\ 0 & s+1 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{A_0} + \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}}_{A_1} s + \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}_{A_2} s^2$$

Define the pencil

$$sE - A := \begin{bmatrix} sI_2 & -I_2 \\ A_0 & sA_2 + A_1 \end{bmatrix} = \begin{bmatrix} s & 0 & -1 & 0 \\ 0 & s & 0 & -1 \\ 1 & 0 & 0 & s \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

Let $s = 2$ i.e. $det\,[A(2)] = 3 \neq 0$, and define the matrix

$$sI_2 - J := \begin{bmatrix} s-2 & 0 \\ 0 & s-2 \end{bmatrix}$$

Then $A(s)$ and $sE - A$ are divisor equivalent and connected through the following divisor equivalent transformation :

$$\underbrace{\begin{bmatrix} 0_{2,2} \\ sI_2 - J \end{bmatrix}}_{M(s)} A(s) = (sE - A) \underbrace{\begin{bmatrix} sI_2 - J \\ (sI_2 - J)\, s \end{bmatrix}}_{N(s)}$$

or equivalently

$$\underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ s-2 & 0 \\ 0 & s-2 \end{bmatrix}}_{M(s)} \underbrace{\begin{bmatrix} 1 & s^2 \\ 0 & s+1 \end{bmatrix}}_{A_1(s)} =$$

$$= \underbrace{\begin{bmatrix} s & 0 & -1 & 0 \\ 0 & s & 0 & -1 \\ 1 & 0 & 0 & s \\ 0 & 1 & 0 & 1 \end{bmatrix}}_{sE-A} \underbrace{\begin{bmatrix} s-2 & 0 \\ 0 & s-2 \\ (s-2)\, s & 0 \\ 0 & (s-2)\, s \end{bmatrix}}_{N(s)}$$

while the symmetry transformation is given by

$$\underbrace{\begin{bmatrix} -(sI_2 - J)\, A_0 & (sI_2 - J)\, s \end{bmatrix}}_{M(s)} (sE - A) =$$

$$= A(s) \underbrace{\begin{bmatrix} 0_{2,2} & (sI_2 - J) \end{bmatrix}}_{N(s)}$$

or equivalently

$$\underbrace{\begin{bmatrix} -(s-2) & 0 & (s-2)\, s & 0 \\ 0 & -(s-2) & 0 & (s-2)\, s \end{bmatrix}}_{M(s)} \underbrace{\begin{bmatrix} s & 0 & -1 & 0 \\ 0 & s & 0 & -1 \\ 1 & 0 & 0 & s \\ 0 & 1 & 0 & 1 \end{bmatrix}}_{sE-A} =$$

$$= \underbrace{\begin{bmatrix} 1 & s^2 \\ 0 & s+1 \end{bmatrix}}_{A_1(s)} \underbrace{\begin{bmatrix} 0 & 0 & s-2 & 0 \\ 0 & 0 & 0 & s-2 \end{bmatrix}}_{N(s)}$$

Note also that

$$S^C_{A(s)}(s) = \begin{bmatrix} 1 & 0 \\ 0 & s+1 \end{bmatrix} \text{ and } S^C_{sE-A}(s) = \begin{bmatrix} I_3 & 0 \\ 0 & s+1 \end{bmatrix}$$

and

$$S^0_{\tilde{A}(s)}(s) = \begin{bmatrix} 1 & 0 \\ 0 & s^3 \end{bmatrix} \text{ and } S^0_{E-sA}(s) = \begin{bmatrix} I_3 & 0 \\ 0 & s^3 \end{bmatrix}$$

■

5. CONCLUSIONS

A new transformation between square and nonsingular polynomial matrices has been defined. This new transformation has the property to preserve both the finite and infinite elementary divisors of polynomial matrices in contrast to the full equivalent transformation which has the property to preserve both the finite and infinite zero structure. This new transformation has many applications in the study of equivalence between discrete time representations. However certain questions arising from the above paper as concerns : a) if the presented conditions of divisor equivalence are also sufficient and b) if divisor equivalence is an equivalence relation ? These questions will be considered in a further research. Some interesting Maple procedures for the determination of the matrices $M(s), N(s)$ which satisfies the relation $M(s)A_1(s) = A_2(s)N(s)$ under the specific degree conditions of divisor equivalence has been produced by my colleague S. Vologiannidis (svol@math.auth.gr) and is available under request.

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