ON THE DETERMINATION OF THE DIMENSION OF THE SOLUTION SPACE OF DISCRETE TIME AR-REPRESENTATIONS

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Abstract: The main purpose of this work is to determine the dimension of the solution space of a nonregular discrete time AR-representation i.e. $A(\sigma)\xi(k) = 0$, in a closed interval $[0, N]$ where $A(\sigma)$ is a nonregular polynomial matrix and σ is the forward shift operator. It is shown that the dimension of the solution space of such a system is strongly related to the structural invariants of the polynomial matrix $A(\sigma)$ which describes the system i.e. finite and infinite elementary divisors and right minimal indices.

Keywords: Autoregressive models, behaviour, boundary conditions, discrete time systems, indices.

Consider a system of linear homogeneous algebraic and differential equations described in matrix form by :

$$A(\rho)\,\beta(t) = 0 \tag{1}$$

where ρ denotes the differential operator i.e. $\rho\beta(t) = d\beta(t)/dt$, $A(\rho) = A_0 + A_1\rho + \cdots + A_q\rho^q \in R[\rho]^{p\times m}$ with $rank_{R(\rho)}A(\rho) = r \leq \min(p, m)$ and $\beta(t) : [0-, +\infty) \longrightarrow R^m$. Following the terminology of Willems we call the set of equations (1) an AR-Representation of B_C (behaviour) where B_C is defined as :

$$B_C := \left\{ \begin{array}{c} \beta(t) : [0-, +\infty) \longrightarrow R^m \\ (1) \text{ is satisfied } \forall t \in [0-, +\infty) \end{array} \right\}$$

If $A(\rho)$ is *regular* or otherwise square and invertible i.e. $p = m = r$ then B_C contains both smooth and impulsive solutions and its dimension is equal to the total number of finite and infinite zeros of $A(\rho)$ (order accounted for) ((Vardulakis, 1991),(Verghese, 1978)). In system theory however we need sometimes descriptions of dynamical systems where there is no distinction between inputs and outputs i.e. interconnection of systems. In such cases the model (1) with $A(\rho)$ nonregular i.e. $r < p = m$ or $p \neq m$, is very useful (Luenberger, 1977),(Willems, 1986) and (Willems, 1991). In case where $A(\rho)$ is nonregular, then B_C contains an infinite number of smooth and impulsive solutions. In that case we correspond all the solutions of (1) which are due to a specific initial condition to an element named $[\beta(t)]$. According to this way the behaviour space B_C is divided into equivalence classes and a new space is created, named $\hat{B}_C$, whose dimension is equal to the total number of finite zeros, infinite zeros and right minimal indices (order accounted for) ((Karampetakis and Vardulakis, 1993)).

Consider now a system of linear homogeneous difference and algebraic equations described in matrix form by :

$$A(\sigma)\,\xi_k = 0 \tag{2}$$

where σ denotes the forward shift operator i.e. $\sigma\xi_k = \xi_{k+1}$, $A(\sigma) = A_0 + A_1\sigma + \cdots + A_q\sigma^q \in R[\sigma]^{p\times m}$ with $rank_{R(\sigma)}A(\sigma) = r \leq \min(p, m)$ and $\xi_k : [0, N] \to R^m$. We call the set of equations (2) an AR-Representation of B_D (behaviour) where B_D is defined as :

$$B_D := \{\xi_k : [0,N] \to R^m \mid (2) \text{ is satisfied } \forall k \in [0,N]\}$$

If $A(\sigma)$ is regular, then the solution vector space of (2) in a finite time interval has been studied by (Antoniou *et al.*, 1998). More specifically, it has been shown that the behaviour of (2) constitutes of backward and forward solutions and its dimension is equal to rq or equivalently to the total number of finite and infinite elementary divisors of $A(\sigma)$ (orders accounted for). However, certain questions concerning the behaviour over a finite time interval of nonregular discrete time AR-Representations still remains.

In this paper we determine the dimension of the behaviour B_D of (2) in case where $A(\sigma)$ is nonregular. More specifically we show that B_D contains a number of linear independent forward and backward solutions that depends on N and are due to the right null space of $A(\sigma)$. In the sequel we correspond all the forward and backward solutions which are due to a specific boundary value (initial-final condition) to an element $[\xi_k]$. According to this way the behaviour space B_D is divided into equivalence classes and a new space is created, named $\hat{B}_D$, whose dimension is proved to be the total number of the finite elementary divisors (n) and infinite elementary divisors (μ) (order accounted for), plus two times the right minimal indices (ϵ) of $A(\sigma)$ (order accounted for) i.e. $n+\mu+2\epsilon$. The whole theory is illustrated via an example.

1. DETERMINATION OF THE DIMENSION OF THE BEHAVIOUR SPACE OF A DISCRETE TIME AR-REPRESENTATIONS

In this section we are trying to determine the dimension of the behaviour space B_D of (2).

Equation (2) may be rewritten as :

$$\underbrace{\begin{bmatrix} A_0 & A_1 & \cdots & A_q & 0 & \cdots & 0 \\ 0 & A_0 & A_1 & \cdots & A_q & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & A_0 & A_1 & \cdots & A_q \end{bmatrix}}_{S_N} \underbrace{\begin{bmatrix} \xi(0) \\ \xi(1) \\ \vdots \\ \xi(N) \end{bmatrix}}_{\xi_N} = 0 \Longleftrightarrow \tag{3}$$

$$S_N \xi_N = 0 \quad S_N \in R^{(N-q+1)p\times(N+1)m}, \xi_N \in R^{(N+1)m}$$

or under applying the Z transform of $\xi(k)$ given by (Freeman, 1965),

$$\xi(z) \overset{def}{=} Z[\xi(k)] = \sum_{k=0}^{N} \xi(k) z^{-k}$$

to (2) may be rewritten as :

$$\begin{gathered} Z[A(\sigma)\xi(k)] = Z[0] \Leftrightarrow \\ A(z)\xi(z) = \begin{bmatrix} z^q I_p & \cdots & zI_p & I_p \end{bmatrix} \times \\ \times \begin{bmatrix} A_q & 0 & \cdots & 0 \\ A_{q-1} & A_q & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_1 & A_2 & \cdots & A_q \end{bmatrix} \begin{bmatrix} \xi(0) \\ \xi(1) \\ \vdots \\ \xi(q-1) \end{bmatrix} + \\ + \begin{bmatrix} z^{-N} I_p & \cdots & z^{-N+q-2} I_p & z^{-N+q-1} I_p \end{bmatrix} \times \\ \times \begin{bmatrix} A_0 & 0 & \cdots & 0 \\ A_1 & A_0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_{q-1} & A_{q-2} & \cdots & A_0 \end{bmatrix} \begin{bmatrix} \xi(N) \\ \xi(N-1) \\ \vdots \\ \xi(N-q+1) \end{bmatrix} \\ =: (z^q I_p) \Re_q \tilde{\xi}(0) + \left(z^{-N} I_p\right) \Re_0 \tilde{\xi}(N) \end{gathered} \tag{4}$$

Let also a right minimal polynomial basis of $A(\sigma)$ (Beelen, 1987) that consisted of the vectors

$$\begin{gathered} \tilde{u}_i(\sigma) := \tilde{u}_{i,0} + \tilde{u}_{i,1}\sigma + \cdots + \tilde{u}_{i,\varepsilon_i}\sigma^{\varepsilon_i} \\ i = r+1, r+2, ..., m \end{gathered}$$

Then the rational vectors of the form

$$\begin{gathered} \hat{u}_i(\sigma) := \tilde{u}_{i,0}\frac{1}{\sigma^{\varepsilon_i}} + \tilde{u}_{i,1}\frac{1}{\sigma^{\varepsilon_i - 1}} + \cdots + \tilde{u}_{i,\varepsilon_i} \\ i = r+1, r+2, ..., m \end{gathered} \tag{5}$$

still constitute a right minimal proper basis of $A(\sigma)$. Then one can prove the following

Proposition 1. The AR-representation (2) with boundary conditions

$$\begin{bmatrix} \xi(0) \\ \xi(1) \\ \vdots \\ \xi(q-1) \end{bmatrix} \in Ker \begin{bmatrix} A_q & 0 & \cdots & 0 \\ A_{q-1} & A_q & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_1 & A_2 & \cdots & A_q \end{bmatrix} \tag{6}$$

and

$$\begin{bmatrix} \xi(N) \\ \xi(N-1) \\ \vdots \\ \xi(N-q+1) \end{bmatrix} \in Ker \begin{bmatrix} A_0 & 0 & \cdots & 0 \\ A_1 & A_0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_{q-1} & A_{q-2} & \cdots & A_0 \end{bmatrix} \tag{7}$$

has the solution

$$\xi(k) := \sum_{i=0}^{k} \hat{u}_{r+1}(i) z_1(k-i) + \sum_{i=0}^{k} \hat{u}_{r+2}(i) z_2(k-i) + \tag{8}$$

$$+ \cdots + \sum_{i=0}^{k} \hat{u}_m(i) z_{m-r}(k-i)$$

where $z_i(k), i = 1, 2, ..., m - r$ are arbitrary discrete time functions and $\hat{u}_i(k) = Z^{-1}[\hat{u}_i(z)], i = r + 1, ..., m$.

Proof. Taking into account relations (6) and (7), relation (4) may be rewritten as

$$A(z)\,\xi(z) = 0$$

and so the solution $\xi(z)$ belongs to the right kernel of $A(z)$ and therefore, can be written as a linear combination of the right minimal proper basis of $A(z)$ defined in (5) i.e.

$$\xi(z) = \sum_{i=r+1}^{m} \hat{u}_i(z)\, z_{i-r}(z) \tag{9}$$

where $z_i(k) = \sum_{k=0}^{N} z_{i,k} \times z^{-k}$ with $z_{i,k}$ arbitrary for $i = 1, 2, ..., m - r$. Taking inverse Z-transforms in (9) we obtain the solution (8) which verifies the Proposition.

A necessary and sufficient condition for the uniqueness of solution is given in the following Theorem.

Theorem 2. In case where (2) has a solution then this solution is unique iff the following conditions are satisfied

$$p \geq m \text{ and } rank_{R(\sigma)} A(\sigma) = m$$

Proof. Suppose that (2) has not a unique solution but has two solutions $\xi_1(k), \xi_2(k)$ $(\xi_1(k) \neq \xi_2(k))$ under the same initial-final conditions or equivalently that

$$A(z)\,\xi_1(z) = (z^q I_p)\,\Re_q \tilde{\xi}(0) + \left(z^{-N} I_p\right) \Re_0 \tilde{\xi}(N) \tag{10}$$

and

$$A(z)\,\xi_2(z) = (z^q I_p)\,\Re_q \tilde{\xi}(0) + \left(z^{-N} I_p\right) \Re_0 \tilde{\xi}(N) \tag{11}$$

Equating the left hand sides of (10) and (11) because the right hand sides coincide, we obtain that

$$A(z)\,\xi_1(z) = A(z)\,\xi_2(z) \Leftrightarrow A(z)\,(\xi_1(z) - \xi_2(z)) = 0$$

and so the difference between $\xi_1(z)$ and $\xi_2(z)$ belongs to the right kernel of $A(z)$. Thus, these two solutions will coincide iff the right kernel of $A(z)$ is the null space or equivalently iff the required conditions are satisfied.

Corollary 3. It follows from the above Theorem that in case where the right kernel of $A(z)$ is not the null space and $\xi_0(k)$ be a solution of (2) then

$$\xi(k) = \xi_0(k) + \\ + \sum_{i=0}^{k} \hat{u}_{r+1}(i)\, z_1(k-i) + \cdots + \sum_{i=0}^{k} \hat{u}_m(i)\, z_{m-r}(k-i)$$

is also a solution of (2).

Let now

$$X = \left\{ \begin{array}{c} \xi(k) : \tilde{\xi}(0) \in Ker[\Re_q] \\ \text{and } \tilde{\xi}(N) \in Ker[\Re_0] \\ \xi(k) = \sum_{i=0}^{k} \hat{u}_{r+1}(i)\, z_1(k-i) + \\ + \cdots + \sum_{i=0}^{k} \hat{u}_m(i)\, z_{m-r}(k-i) \end{array} \right\}$$

be the solution space of (2) which comes under initial-final conditions of the form (6) and (7). An interesting question arisen from above concerns the dimension of the spaces B_D and X. In order to find out these dimensions we first propose the following Lemma.

Lemma 4. (Bitmead *et al.*, 1978) Consider the matrix S_k defined by :

$$S_k = \begin{bmatrix} A_0 & A_1 & \cdots & A_q & 0 & \cdots & 0 \\ 0 & A_0 & A_1 & \cdots & A_q & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & A_0 & A_1 & \cdots & A_q \end{bmatrix} \in R^{p(k-q+1)\times(k+1)m} \tag{12}$$

Then

$$rank_R S_k = p(k-q+1) - \sum_{\{i:k>\eta_i\}} (k - q + 1 - \eta_i) \tag{13}$$

where $\eta_i, i = 1, 2, ..., l$ are the left Kronecker indices of $A(s)$.

Using the above Lemma we can now easily prove the following :

Theorem 5.

$$\dim B_D = (N+1)(m-r) + qr - \eta \tag{14}$$

$$\dim X = (N+1)(m-r) - \varepsilon \tag{15}$$

where $N_{\min} = \max_{i=1,..,r \;\; j=1,..,l} \{\varepsilon_i + 1, \eta_j\}$, $\varepsilon_i, i = 1, 2, ..., l$ are the right Kronecker indices of $A(s)$

and $\varepsilon = \sum_{i=1}^{r} \varepsilon_i$ ($\eta = \sum_{i=1}^{l} \eta_i$) is the total number of the right (left) minimal indices (order accounted for).

Proof. B_D is isomorphic to $\ker S_N$. According to the previous lemma we have that :

$$rank S_N = p\,(N-q+1) - \sum_{\{j:k>\eta_j\}} (N-q+1-\eta_j)$$

For $N > N_{\min}$ we have that

$$\begin{aligned} rank S_N &= p\,(N-q+1) - \sum_{j=r+1,..,p} (N-q+1-\eta_j) = \\ &= p\,(N-q+1) - (N-q+1)\,(p-r) + \eta \end{aligned}$$

Thus

$$\begin{gathered} \dim Ker S_N = (N+1)\,m - rank_R S_N = \\ = (N+1)\,m - p\,(N-q+1) + \\ + (N-q+1)\,(p-r) - \eta = \\ = (N+1)\,(m-p+p-r) + pq - q\,(p-r) - \eta = \\ = (N+1)\,(m-r) + pq - pq + qr - \eta = \\ = (N+1)\,(m-r) + qr - \eta \end{gathered}$$

This proves that $\dim B_D = (N+1)\,(m-r) + qr - \eta$. In the same way we can prove that X is isomorphic to $\ker C_N$ (by adding equations (6) and (7) in equation (3)) where

$$C_N^T = \begin{bmatrix} A_0^T & A_1^T & \cdots & A_q^T & 0 & \cdots & 0 \\ 0 & A_0^T & A_1^T & \cdots & A_q^T & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & A_0^T & A_1^T & \cdots & A_q^T \end{bmatrix}$$

with $C_N^T \in R^{(N+1)m \times p(N+q+1)}$. Applying the previous lemma to C_N we have that

$$\begin{gathered} rank_R C_N = rank_R C_N^T = \\ = (N+1)m - \sum_{\{i:k>\varepsilon_i\}} (N+1-\varepsilon_j) \end{gathered}$$

where the left indices of $A(s)^T$ are the right indices of $A(s)$. Thus for $N > N_{\min}$ we have that

$$rank_R C_N = (N+1)\,m - (N+1)\,(m-r) + \varepsilon$$

where ε is the total number of the right minimal indices of $A(z)$ (order accounted for) and

$$\begin{gathered} \dim Ker C_N = (N+1)\,m - rank_R C_N = \\ = (N+1)\,m - (N+1)\,m + (N+1)\,(m-r) - \varepsilon \\ = (N+1)\,(m-r) - \varepsilon \end{gathered}$$

This proves that $\dim X = (N+1)\,(m-r) - \varepsilon$.

It is easily checked that

$$\underbrace{\begin{bmatrix} A_0 & A_1 & \cdots & A_q & 0 & \cdots & 0 \\ 0 & A_0 & A_1 & \cdots & A_q & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & A_0 & A_1 & \cdots & A_q \end{bmatrix}}_{S_N} \times$$

$$\times \begin{bmatrix} u_{i,0} & 0 & \cdots & 0 \\ u_{i,1} & u_{i,0} & \cdots & 0 \\ \vdots & \vdots & \ddots & u_{i,0} \\ u_{i,\varepsilon_i} & u_{i,\varepsilon_i-1} & \cdots & \vdots \\ 0 & u_{i,\varepsilon_i} & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & u_{i,\varepsilon_i} \end{bmatrix} = 0$$

and thus the columns of the matrix

$$U_i := \begin{bmatrix} u_{i,0} & 0 & \cdots & 0 \\ u_{i,1} & u_{i,0} & \cdots & 0 \\ \vdots & \vdots & \ddots & u_{i,0} \\ u_{i,\varepsilon_i} & u_{i,\varepsilon_i-1} & \cdots & \vdots \\ 0 & u_{i,\varepsilon_i} & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & u_{i,\varepsilon_i} \end{bmatrix} \in R^{(N+1)m \times (N-\varepsilon_i+1)} \tag{16}$$

$$i = r+1, .., m$$

are linearly independent solutions of (2) with its initial and final conditions satisfying (6) and (7) i.e the linear independence can be proved using the properties of the columns vectors of the right minimal polynomial bases. The number of those linearly independent solutions which are due to the right minimal indices is equal to

$$\sum_{i=r+1}^{m} (N-\varepsilon_i+1) = (N+1)\,(m-r) - \varepsilon \tag{17}$$

and thus produce the vector space X. In order now to find out the number of linearly independent solutions which belong to B_D but do not belong to X we need the following lemma

Lemma 6. (Praagman, 1991) Let $A(\sigma) \in R[\sigma]^{p \times m}$ be a polynomial matrix of rank r and degree q.

Then the sum of its structure indices is equal to qr i.e.$n+\mu+\varepsilon+\eta = qr$ where n is the total number of finite elementary divisors (order accounted for), μ is the total number of infinite elementary divisors (order accounted for), ε is the total number of right minimal indices (order accounted for) and η is the total number of left minimal indices (order accounted for).

Using the above lemma we can now prove the following

Corollary 7.

$$\dim [B_D - X] = n + \mu + 2\varepsilon$$

Proof. We conclude from the above theorem that the number of linearly independent solutions which belong to B_D but do not belong to X is equal to :

$$\begin{aligned}\dim [B_D - X] &= \dim B_D - \dim X = \\ &= ((N+1)(m-r) + qr - \eta) - \\ &- ((N+1)(m-r) - \varepsilon) = \\ &= qr - \eta + \varepsilon = \\ &= (n + \mu + \varepsilon + \eta) - \eta + \varepsilon = \\ &= n + \mu + 2\varepsilon\end{aligned}$$

Define now the following relation R between the solutions of B

$$\begin{gathered} R(\xi_1(k), \xi_2(k)) := \\ = \left\{ \begin{array}{c} (\xi_1(k), \xi_2(k)) : \xi_1(k) - \xi_2(k) \in X \\ \text{where } \xi_1(k), \xi_2(k) \in B_D \end{array} \right\} \end{gathered} \tag{18}$$

Then we can easily prove the following

Theorem 8. The relation (18) is an equivalence relation.

We call an equivalence class of the element $\xi(k) \in B_D$, and we denote this with $[\xi(k)]$, the set of all the elements of B_D which are equivalent to $\xi(k)$ or equivalently

$$\begin{aligned} [\xi(k)]_R &:= \{\xi_1(k) \in B_D : (\xi_1(k), \xi(k)) \in R\} = \\ &= \xi(k) \oplus X = \\ &= \left\{ \begin{array}{c} \xi(k) + x(k) \\ \text{where } \xi(k) \in B_D \text{ and } x(k) \in X \end{array} \right\} = \qquad (19) \\ &= \left\{ \begin{array}{c} \xi(k) + \sum_{i=0}^{k} \hat{u}_{r+1}(i) z_1(k-i) + \\ + \cdots + \sum_{i=0}^{k} \hat{u}_m(i) z_{m-r}(k-i) \end{array} \right\} \end{aligned}$$

We can see that any *equivalence class* of an element $\xi(k)$ gives all the solutions of (2) under some specific initial-final conditions. In the case where $A(\sigma)$ has no right kernel then every equivalence class is composed either by a unique element or by no element contrary to the nonregular case where to each equivalence class corresponds more than one elements of B_D. We conclude therefore that B_D is divided into equivalence classes which are defined by (19). Define now the following *sum* between equivalence classes of the form (19) :

$$[\xi_1(k)]_R + [\xi_2(k)]_R := [\xi_1(k) + \xi_2(k)]_R$$

and the *product*

$$\lambda [\xi(k)]_R = [\lambda \xi(k)]_R \quad \lambda \in R$$

It is now easy to show the following

Theorem 9. The space which is spanned by the equivalence classes defined in (19) is a vector space $\hat{B}_D := B_D/R$ and has dimension

$$f := \dim \hat{B}_D = n + \mu + 2\varepsilon$$

f is called the generalized order of (2).

Proof. The proof that $\hat{B}_D$ is a vector space is left to the reader. As concerns the dimension of the vector space $\hat{B}_D$, it is enough to select $n+\mu+2\varepsilon$ linearly independent vectors of $B_D - X$ let $\xi_i(k)$ (7) and then show that the vectors $[\xi_i(k)]_R$ span the vector space $\hat{B}_D$ (see (Karampetakis, 2001)).

Remark 10. Note that if $A(\sigma)$ is regular then $\varepsilon = 0$ and f coincides with $n + \mu$, as has already been proved in (Antoniou *et al.*, 1998).

2. ILLUSTRATIVE EXAMPLE.

Consider the AR-representation

$$\underbrace{\begin{bmatrix} \sigma & \sigma^4 & \sigma^2+\sigma \\ 1 & \sigma^3 & \sigma+1 \\ 0 & \sigma+1 & 0 \end{bmatrix}}_{A(\sigma)} \underbrace{\begin{bmatrix} \xi_1(k) \\ \xi_2(k) \\ \xi_3(k) \end{bmatrix}}_{\xi(k)} = 0_{3,1}$$

Then

$$\underbrace{\begin{bmatrix} \sigma & \sigma^4 & \sigma^2+\sigma \\ 1 & \sigma^3 & \sigma+1 \\ 0 & \sigma+1 & 0 \end{bmatrix}}_{A(\sigma)} \underbrace{\begin{bmatrix} -1-\sigma \\ 0 \\ 1 \end{bmatrix}}_{\bar{u}_3(\sigma)} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Therefore

$$X = \left\{ \begin{array}{c} \xi(k) : \xi(\kappa) = \\ = \sum_{i=0}^{k} \begin{bmatrix} -\delta(i-1) - \delta(i) \\ 0 \\ \delta(i-1) \end{bmatrix} z(k-i) = \\ = \begin{bmatrix} -z(k-1) - z(k) \\ 0 \\ z(k-1) \end{bmatrix} \end{array} \right\}$$

and

$$\hat{B}_D = \left\langle \left[\begin{bmatrix} (-1)^k \\ (-1)^k \\ 0 \end{bmatrix} \right]_R \right\rangle \oplus \left\langle \left[\begin{bmatrix} -\delta(k) \\ 0 \\ \delta(k) \end{bmatrix} \right]_R \right\rangle \oplus$$
$$\oplus \left\langle \left[\begin{bmatrix} -\delta(N-k) \\ 0 \\ \delta(N-k) \end{bmatrix} \right]_R, \left[\begin{bmatrix} -\delta(N-k+1) \\ 0 \\ \delta(N-k+1) \end{bmatrix} \right]_R, \right.$$
$$\left[\begin{bmatrix} -\delta(N-k+2) \\ -2\delta(N-k) \\ \delta(N-k+2) \end{bmatrix} \right]_R, \left[\begin{bmatrix} -\delta(N-k+3) \\ -2\delta(N-k+1) \\ \delta(N-k+3) \end{bmatrix} \right]_R,$$
$$\left. \left[\begin{bmatrix} -\delta(N-k+4) \\ -2\delta(N-k+2) \\ \delta(N-k+4) \end{bmatrix} \right]_R \right\rangle \oplus$$
$$\oplus \left\langle \left[\begin{bmatrix} -\delta(N-k) \\ 0 \\ 0 \end{bmatrix} \right]_R \right\rangle$$

where

$$[\xi(k)]_R := \xi(k) \oplus X$$

An algorithm for the construction of the vector space $\hat{B}_D$ and thus of the whole vector space B_D is presented in (Karampetakis, 2001).

3. CONCLUSIONS

It is known that a nonregular discrete time AR-representation may have more than one solutions under the same boundary conditions due to the right null structure of the polynomial matrix that describes the system, in contrast to the regular AR-representations where the solutions are unique. In this paper we have shown that we can correspond all the solutions which are due to the specific boundary values, to a unique element. Under this correspodence the solution space of the AR-representation is divided into equivalence classes. The dimension of the new equivalent class space has been determined and is shown to be equal to the sum of the finite elementary divisors (order accounted for), infinite elementary divisors (order accounted for) plus two times the sum of the right minimal indices (order accounted for) of the polynomial matrix that describes the AR-representation. An algorithm for the construction of the solution space of discrete time AR-representations over a finite time interval is given in (Karampetakis, 2001).

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