NOTIONS OF EQUIVALENCE FOR DISCRETE TIME AR-REPRESENTATIONS

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Abstract: A transformation of polynomial matrices which preserves both the finite and infinite elementary divisor structure is presented and related to other known transformations.

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1. INTRODUCTION

Consider a linear homogeneous matrix difference equation of the form

$$A(\sigma)\beta(k) = 0,\ k \in [0, N] \quad (1)$$

$$A(\sigma) = A_q\sigma^q + A_{q-1}\sigma^{q-1} + \ldots + A_0 \in \mathbb{R}[\sigma]^{r\times r} \quad (2)$$

where σ denotes the forward shift operator. It is known from (Antoniou *et al.* 1998*a*) that (1) exhibits both forward and backward behavior due to the finite and infinite elementary divisors of $A(\sigma)$ and not due to its finite and infinite zeros. Therefore our main interest is to propose a transformation that will preserve both the finite and infinite elementary divisors of polynomial matrices. Actually (Pugh and Shelton 1978) proposed the extended unimodular equivalence transformation (e.u.e.) which has the nice property of preserving only the finite elementary divisors, while (Vardulakis and Antoniou 2001) and (Karampetakis 2001*a*) have proposed an extension of the e.u.e., i.e. strict equivalence and divisor equivalence respectively, in order to preserve both the finite and infinite elementary divisors.

The main object of this paper is to prove that the equivalence relation defined in (Karampetakis 2001*a*) minus one of the three conditions, provides necessary and sufficient conditions for the invariance of the finite and infinite elementary divisors, and that it is an equivalence relation. Also we will prove that strict equivalence and divisor equivalence coincide.

2. PRELIMINARY RESULTS

We define by $P(m, l)$ the class of $(r + m) \times (r + l)$ polynomial matrices where l and m are fixed integers and r ranges over all integers which are greater than $\max(-m, -l)$.

Definition 1. Let $A(\sigma) \in \mathbb{R}[\sigma]^{p\times m}$ with $rank_{R(s)}A(s) = r \le \min(p, m)$. The values $\lambda_i \in \mathbb{C}$ that satisfy the condition $rank_{\mathbb{C}}A(\lambda_i) < r$ are called finite zeros of $A(s)$. Assume that $A(\sigma)$ has l distinct zeros $\lambda_1, \lambda_2, \ldots, \lambda_l \in \mathbb{C}$, and let

$$S^{\lambda_i}_{A(\sigma)} = \begin{bmatrix} diag\{(\sigma-\lambda_i)^{m_{i1}}, ..., (\sigma-\lambda_i)^{m_{ir}}\} & 0_{r,m-r} \\ 0_{p-r,r} & 0_{p-r,m-r} \end{bmatrix}$$

be the local Smith form $S^{\lambda_i}_{A(\sigma)}$ of $A(\sigma)$ at $\sigma = \lambda_i, i = 1, 2, ..., l$ where $m_{ij} \in \mathbb{Z}^+$ and $0 \le m_{i1} \le m_{i2} \le ... \le m_{ir}$. The terms $(\sigma - \lambda_i)^{m_{ij}}$ are called the *finite elementary divisors* (f.e.d.) of $A(\sigma)$ at $\sigma = \lambda_i$. Define also as $n := \sum_{i=1}^{l}\sum_{j=1}^{r} mi$.

Definition 2. If $A_0 \neq 0$, the *dual* matrix $\tilde{A}(\sigma)$ of $A(\sigma)$ is defined as $\tilde{A}(\sigma) := A_0\sigma^q + A_1\sigma^{q-1} + \ldots + A_q$. Since $rank\tilde{A}(0) = rankA_q$ the dual matrix $\tilde{A}(\sigma)$ of $A(\sigma)$ has zeros at $\sigma = 0$ iff $rankA_q < r$. Let $rankA_q < r$ and let

$$S^0_{\tilde{A}(\sigma)}(\sigma) = \begin{bmatrix} diag\{\sigma^{\mu_1}, ..., \sigma^{\mu_r}\} & 0_{r,m-r} \\ 0_{p-r,r} & 0_{p-r,m-r} \end{bmatrix} \quad (3)$$

be the local Smith form of $\tilde{A}(\sigma)$ at $\sigma = 0$ where $\mu_j \in \mathbb{Z}^+$ and $0 \leq \mu_1 \leq \mu_2 \leq \ldots \leq \mu_r$. The *infinite elementary divisors* (i.e.d.) of $A(\sigma)$ are defined as the finite elementary divisors σ^{μ_j} of its dual $\tilde{A}(\sigma)$ at $\sigma = 0$. Define also as $\mu = \sum_{i=1}^{r} \mu_i$.

We can easily observe from Definition 2 that $A(\sigma) \in \mathbb{R}[\sigma]^{p\times m}$ has no i.e.d. iff $rank A_q = r$ where A_q is the leading degree coefficient matrix of $A(s)$. Let $A_1(s), A_2(s) \in P(m,l)$ and suppose $\exists$ rational matrices $M(s), N(s)$ s.t.

$$\begin{bmatrix} M(s) & A_2(s) \end{bmatrix} \begin{bmatrix} A_1(s) \\ -N(s) \end{bmatrix} = 0 \qquad (4)$$

Definition 3. (Pugh and Shelton 1978) If $M(s), N(s)$ are polynomial matrices and the compound matrices

$$\begin{bmatrix} M(s) & A_2(s) \end{bmatrix} \quad ; \quad \begin{bmatrix} A_1(s) \\ -N(s) \end{bmatrix} \qquad (5)$$

have full rank $\forall s \in C$ then $A_i(s)$ are said to be **extended unimodular equivalent** (e.u.e).

E.u.e. allows matrices of different dimensions to be related and preserves the f.e.d.. However it does not preserve the i.e.d..

Definition 4. (Gantmacher 1959) If $M(s), N(s)$ are constant, square and nonsingular then the matrices $A_i(s)$ $i = 1, 2$ are said to be **strict equivalent** (s.e.).

S.e. has the nice property of preserving both the f.e.d. and i.e.d. of polynomial matrices, note however that it relates matrices of the same dimensions.

Definition 5. (Karampetakis *et al.* 1994) If the compound matrices in (5) have full rank $\forall s \in C$ then $A_i(s)$ are said to be **{0}-equivalent**.

{0}-equivalence preserves only the f.e.d. of the form $s^i, i > 0$.

It is known (Praagman 1991),(Antoniou *et al.* 1998*b*) that the total number of f.e.d. and i.e.d. (order accounted for) of the polynomial matrix $A(s)$ defined in (2) is equal to rq where r is the dimension of the matrix and q is the highest degree of all the entries of $A(s)$ i.e. $n + \mu = rq$. Therefore, in order two polynomial matrices $A_1(s), A_2(s)$ have the same f.e.d. and i.e.d., they must have the same total number of f.e.d. and i.e.d. or otherwise the same product rq i.e. $r_1 q_1 = r_2 q_2$. For this reason we define the following set of polynomial matrices

$$R_c[s] := \{A(s) \text{ defined in (2) with } c = rq,\ r \geq 2\} \qquad (6)$$

Definition 6. (Karampetakis 2001*a*) If the compound matrices in (5) satisfy the following three conditions

(i) they have full rank and no f.e.d.,

(ii) they have no i.e.d.,

(iii) the following degree conditions are satisfied

$$d \begin{bmatrix} M(s) & A_2(s) \end{bmatrix} \leq d\left[A_2(s)\right] \ ; \ d \begin{bmatrix} A_1(s) \\ -N(s) \end{bmatrix} \leq d\left[A_1(s)\right]$$

where $d[P]$ denotes the degree of $P(s)$ seen as a polynomial with nonzero matrix coefficients, then $A_1(s), A_2(s) \in R_c[s]$ are said to be **divisor equivalent** (d.e.).

D.e. preserves both the f.e.d. and i.e.d. (Karampetakis 2001*a*). However it is not known if d.e. a) is an equivalence relation and b) provide us with necessary conditions in order two polynomial matrices possess the same f.e.d. and i.e.d.. We don't know also the exact meaning of the degree conditions appearing in the definition of d.e..

The following lemma is required in the subsequent proofs.

Lemma 7. If $A_1(s), A_2(s) \in \mathbb{R}_c[s]$ are d.e. and $A_2(s), A_3(s) \in \mathbb{R}_c[s]$ are s.e., then $A_1(s), A_3(s) \in \mathbb{R}_c[s]$ are d.e..

Proof. It is easily seen using the same reasoning with Lemma 5.5.2 of (Walker 1988). $\square$

Let $A(s)$ as defined in (2). Then the pencil

$$sE - A := \begin{bmatrix} sI_r & -I_r & 0 & \cdots & 0 & 0 \\ 0 & sI_r & -I_r & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & sI_r & -I_r \\ A_0 & A_1 & A_2 & \cdots & A_{q-2} & sA_q + A_{q-1} \end{bmatrix} \qquad (7)$$

is shown in (Karampetakis 2001*a*) that is d.e. to $A(s)$ and vice versa.

3. STRUCTURAL PROPERTIES OF D.E.

In what follows we will prove i) that the degree conditions imposed by d.e. are not needed and ii) that d.e. provide us with necessary and sufficient conditions in order for two polynomial matrices to possess the same f.e.d. and i.e.d.. The following technical relationships are required in the subsequent proofs.

Lemma 8. Let $A_1(s)$ and $A_2(s) \in \mathbb{R}_c[s]$ with dimensions $m \times m$ and $(m+r) \times (m+r)$ respectively where $r \neq 0$. Then the first two conditions of d.e. implies the degree conditions of d.e. i.e. $\deg M(s) \leq \deg A_2(s)$ and $\deg N(s) \leq \deg A_1(s)$

Proof. Assume that $\exists M(s)$ with $\deg M(s) > \deg A_2(s)$. In order for $\left[\, M(s)\ A_2(s) \,\right]$ to have no i.e.d its highest coefficient matrix must be of full row rank i.e. $rank \left[\, M_{hc}\ 0 \,\right] = m + r$. This is impossible since $M_{hc} \in \mathbb{R}^{(m+r)\times m}$. So $\deg M(s) <= \deg A_2(s)$.

Let $\deg N(s) > \deg A_1(s)$. Then $\exists d <> 0$ s.t. $\deg A_1(s) + d = \deg N(s)$. Condition (i) of d.e. implies that the matrices $A_1(s)$ and $A_2(s)$ are e.u.e. so they have the same f.e.d. (and of course the same number of f.e.d. i.e. $S_R\left(A_1(s)\right) = S_R\left(A_2(s)\right)$ where $S_R(A(s))$ denotes the total number of f.e.d. of $A(s)$ (order accounted for)).Taking the duals of the compound matrices we get

$$\left[\, M'(w)\ \widetilde{A}_2(w) \,\right] \begin{bmatrix} w^d \widetilde{A}_1(w) \\ -\widetilde{N}(w) \end{bmatrix} = 0 \qquad (8)$$

(8) is a $\{0\}$-equivalence relation and thus $\widetilde{A}_2(w)$ and $w^d \widetilde{A}_1(w)$ have the same f.e.d. at 0. Denoting by $S_l(A(w))$ the total number of f.e.d. (order accounted for) at l of $A(w)$ we have

$$\begin{aligned} S_0(\widetilde{A}_2(w)) = S_0(w^d \widetilde{A}_1(w)) > S_0(\widetilde{A}_1(w)) \Longrightarrow \\ S_0(\widetilde{A}_2(w)) > S_0(\widetilde{A}_1(w)) \Longrightarrow \\ \Longrightarrow S_\infty(A_2(s)) > S_\infty(A_1(s)) \end{aligned} \qquad (9)$$

According to our assumption $A_1(s), A_2(s) \in \mathbb{R}_c[s]$. Thus $c = S_\infty(A_2(s)) + S_\mathbb{R}(A_2(s)) = S_\infty(A_1(s)) + S_\mathbb{R}(A_1(s))$ or equivalently since $S_\mathbb{R}(A_2(s)) = S_\mathbb{R}(A_1(s))$ we have that $S_\infty(A_2(s)) = S_\infty(A_1(s))$ which contradicts with (9). ☐

In the above discussion we proved that it is impossible to find transforming matrices of d.e. $M(s) \in \mathbb{R}^{(m+r)\times m}[s], N(s) \in \mathbb{R}^{(m+r)\times m}[s]$ with $\deg M(s) > \deg A_2(s)$ or $\deg N(s) > \deg A_1(s)$. The latter is not the case when $r = 0$.

Lemma 9. Let $A_1(s)$ and $A_2(s) \in \mathbb{R}_c[s]$ having the same dimensions $m \times m$ and therefore the same degree d. If $A_1(s), A_2(s)$ satisfies (4) and the first two conditions of d.e. then $\deg M(s) = \deg N(s)$.

Proof. First we will prove that if one of the chosen transforming matrices has degree more than d then $\deg M(s) = \deg N(s)$. Let

$$d_M = \deg M,\ d_N = \deg N$$

$$D_L = \deg \left[\, M(s)\ A_2(s) \,\right] \ ;\ D_R = \deg \begin{bmatrix} A_1(s) \\ -N(s) \end{bmatrix}$$

$$d = \deg A_1(s) = \deg A_2(s)$$

Then

$$\left[\, \widetilde{M(s)\ A_2(s)} \,\right] = \left[\, w^{d_M + D_L} \widetilde{M}(w)\ w^{d+D_l} \widetilde{A}_2(w) \,\right]$$

$$\begin{bmatrix} \widetilde{A_1(s)} \\ -N(s) \end{bmatrix} = \begin{bmatrix} w^{-d+D_r} \widetilde{A}_1(w) \\ w^{-d_N + D_r} \widetilde{N}(w) \end{bmatrix}$$

¿From (4) taking the duals gives

$$\left[\, w^{-d_M + D_L} \widetilde{M}(w)\ w^{-d+D_l} \widetilde{A}_2(w) \,\right] \begin{bmatrix} w^{-d+D_r} \widetilde{A}_1(w) \\ w^{-d_N + D_r} \widetilde{N}(w) \end{bmatrix} = 0 \qquad (10)$$

Then

$$\left\{ \begin{array}{c} S_0(w^{-d_M + D_L} \widetilde{M}(w)) = S_0(w^{-d_N + D_r} \widetilde{N}(w)) \\ S_0(w^{-d+D_l} \widetilde{A}_2(w)) = S_0(w^{-d+D_r} \widetilde{A}_1(w)) \end{array} \right\} \Rightarrow$$

$$S_\infty M(s) + (-d_M + D_L)m = S_\infty N(s) + (-d_N + D_R)m \qquad (11)$$

$$S_\infty A_2(s) + (-d + D_L)m = S_\infty A_1(s) + (-d + D_R)m \qquad (12)$$

Also (4) is an e.u.e. relation and

$$S_\mathbb{R} M(s) = S_\mathbb{R} N(s)\ \ ;\ \ S_\mathbb{R} A_2(s) = S_\mathbb{R} A_1(s) \qquad (13)$$

But since $A_1(s)$ and $A_2(s) \in \mathbb{R}_c^{mxm}[s]$ we have

$$S_\infty A_1(s) + S_\mathbb{R} A_1(s) = S_\infty A_2(s) + S_\mathbb{R} A_2(s) \overset{(12)}{\Longrightarrow} \qquad (14)$$

$$\begin{aligned} S_\mathbb{R} A_1(s) + S_\infty A_2(s) + (-d + D_L)m - \\ -(-d + D_R)m = S_\infty A_2(s) + S_\mathbb{R} A_2(s) \overset{(13)}{\Longrightarrow} \\ D_L = D_R \end{aligned} \qquad (15)$$

Using (11) and (15) the following equation holds

$$S_\infty M(s) - S_\infty N(s) = (d_M - d_N)m \qquad (16)$$

Equation (15) tells us that if $d_M > d$ or $d_N > d$ then $d_M = D_L = D_R = d_N$.

In the second part of the proof we will show that if one of the chosen transforming matrices has degree less than d then $\deg M(s) = \deg N(s)$.Suppose that $N(s)$ has degree d_N such that $d_N(s) < d_M(s) \leq d$. Then

$$M^{hc} A_1^{hc} = 0 \qquad (17)$$

where A^{hc} denotes the highest coefficient matrix of the polynomial matrix $A(s)$. Note that from the first part of the proof if one of the matrices has degree less than d then the other one cannot have degree more than d. Since $d_N(s) < d$ we have that A_1^{hc} has full rank and therefore $\dim(Ker(M^{hc})) = m$. Thus $rank(M^{hc}) = 0$ and therefore $\deg M(s) < d_M$ which contradicts with our second assumption. ☐

Lemma 10. If $A_1(s)(s - s_0)^k \in \mathbb{R}^{m\times m}[s]$ has the same f.e.d. and i.e.d. as $A_2(s)(s - s_0)^k \in \mathbb{R}^{m\times m}[s]$, where $s_0 \neq 0$ is not a zero of either $A_1(s)$ or $A_2(s)$, then $A_1(s)$ and $A_2(s)$ have the same f.e.d. and i.e.d..

Proof. The proof is trivial having in mind that the i.e.d. of $A_1(s)(s-s_0)^k \in \mathbb{R}^{m\times m}[s]$, where $s_0 \neq 0$ is not a zero of $A_1(s)$, are exactly the i.e.d. of $A_1(s)$. The f.e.d. of $A_1(s)(s-s_0)^k$ are the f.e.d. of $A_1(s)$ plus m divisors of the form $(s-s_0)^k$. □

Now we are able to restate the definition of d.e. with only two conditions.

Definition 11. Two matrices $A_1(s), A_2(s) \in \mathbb{R}_c[s]$ are said to be *divisor equivalent* (*d.e.*) if there exist polynomial matrices $M(s), N(s)$ of appropriate dimensions, such that equation (4) is satisfied where the compound matrices in (5) have full rank and no f.e.d. nor i.e.d..

Some nice properties of d.e. are given in the following Theorem.

Theorem 12. $A_1(s), A_2(s) \in \mathbb{R}_c[s]$ are d.e. iff they have the same f.e.d. and i.e.d..

Proof. Sufficiency. Let $A_1(s) \in R[s]^{m\times m}$ and $A_2(s) \in \mathbb{R}[s]^{(m+r)\times(m+r)}$ with $r \neq 0$. Then according to Lemma 8 $\deg M(s) \leq \deg A_2(s)$ and $\deg N(s) \leq \deg A_1(s)$. Therefore $A_1(s), A_2(s)$ possess the same f.e.d. and i.e.d. (Theorem 3, (Karampetakis 2001*a*)).

Let $A_1(s), A_2(s) \in R[s]^{m\times m}$ with the same degree d. Then according to Lemma 9 $d_t = \deg M(s) = \deg N(s)$. In case where $d \leq d_t$ then the proof is the same with the one presented in Theorem 3 of (Karampetakis 2001*a*). Consider now the case where $d_t > d$. Let $s_0 \neq 0$ not a zero of either $A_1(s)$ or $A_2(s)$. Then

$$\begin{bmatrix} M(s) & A_2(s)(s-s_0)^{d_t-d} \end{bmatrix} \begin{bmatrix} A_1(s)(s-s_0)^{d_t-d} \\ -N(s) \end{bmatrix} = 0 \tag{18}$$

is an e.u.e. relation which means that $A_2(s)(s-s_0)^{d_t-d}$ and $A_1(s)(s-s_0)^{d_t-d}$ have the same f.e.d.. Therefore according to Lemma 10, $A_1(s)$ and $A_2(s)$ have the same f.e.d.. Taking the duals of (18) we have

$$\begin{bmatrix} \widetilde{M}(s) & A_2(s)\widetilde{(s-s_0)}^{d_t-d} \end{bmatrix} \begin{bmatrix} A_1(s)\widetilde{(s-s_0)}^{d_t-d} \\ \widetilde{-N(s)} \end{bmatrix} = 0 \tag{19}$$

(19) is a $\{0\}$-equivalence relation, so

$$A_2(s)\widetilde{(s-s_0)}^{d_t-d} \text{ and } A_1(s)\widetilde{(s-s_0)}^{d_t-d}$$

have the same f.e.d. at 0 and thus $A_2(s)(s-s_0)^{d_t-d}$ and $A_1(s)(s-s_0)^{d_t-d}$ have the same i.e.d. i.e. $A_2(s)$ and $A_1(s)$ have the same i.e.d..

Necessity. Assume that $A_1(s), A_2(s) \in \mathbb{R}_c[s]$ have identical f.e.d. and i.e.d.. Then according to (Karampetakis 2001*a*) $A_1(s)$ and $A_2(s)$ are d.e. to matrix pencils $sE_1 - A_1$ and $sE_2 - A_2$ of the form (7) and vice versa. The pencils $sE_1 - A_1$ and $sE_2 - A_2$ are also strict equivalent to their respective Weierstrass forms let $W(sE_1 - A_1)$ and $W(sE_2 - A_2)$. Since $sE_1 - A_1, sE_2 - A_2 \in \mathbb{R}_c[s]$ and share the same f.e.d. and i.e.d. they have the same Weierstrass form, let $sE_w - A_w \equiv W(sE_1 - A_1) \equiv W(sE_2 - A_2)$. Repeated use of the transitivity property proved in Lemma 7 shows that $A_1(s)$ is divisor equivalent to $sE_2 - A_2$. This argument is summarized in Figure 1.

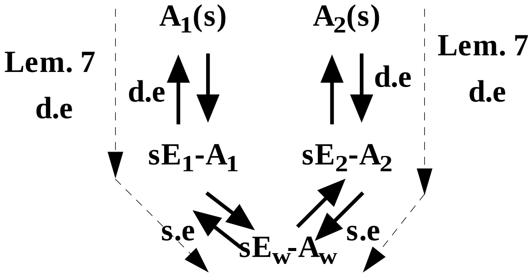

Figure 1.

Therefore $\exists A(s),\ B(s)$ s.t. the following transformation :

$$A(s)\,(sE_2 - A_2) = A_1(s)B(s) \tag{20}$$

is a d.e. transformation. The polynomial matrices $sE_2 - A_2$ and $A_2(s)$ are also connected through the d.e. transformation (Karampetakis 2001*a*),

$$\begin{pmatrix} 0 \\ kI \end{pmatrix} A_2(s) = (sE_2 - A_2)\,(kL(s)) \tag{21}$$

where $k = (s - s_0), s_0 \neq 0$ not a zero of either $A_2(s)$ or $(sE_2 - A_2)$ and $L(s) := \begin{pmatrix} I & sI & \cdots & s^{q_2-1}I \end{pmatrix}^T$ and $q_2 = d\,[A_2(s)]$. Premultiplying (21) by $A(s)$ and then using (20) we get

$$\left\{ A(s) \begin{pmatrix} 0 \\ kI \end{pmatrix} \right\} A_2(s) = A_1(s)\,\{B(s)\,(kL(s))\} \tag{22}$$

In order to keep the notation as simple as possible in (22) we divide both sides with k and thus we get

$$\begin{bmatrix} A(s) \begin{pmatrix} 0 \\ I \end{pmatrix} & A_1(s) \end{bmatrix} \begin{bmatrix} A_2(s) \\ -B(s)L(s) \end{bmatrix} = 0 \tag{23}$$

In order to prove the absence of f.e.d. and i.e.d. of the compound matrices in (23), it is enough to prove that they possess the same f.e.d and i.e.d. respectively with

$$\begin{bmatrix} A(s) & A_1(s) \end{bmatrix} \text{ and } \begin{bmatrix} sE_2 - A_2 \\ -B(s) \end{bmatrix}$$

which have neither f.e.d. nor i.e.d., according to the d.e. transformation (20).

(i) Our first goal is to prove that the matrices

$$\begin{bmatrix} sE_2 - A_2 \\ -B(s) \end{bmatrix} \text{ and } \begin{bmatrix} A_2(s) \\ -B(s)L(s) \end{bmatrix} \tag{24}$$

possess the same f.e.d. and i.e.d..

(i-a) - **Same f.e.d..** Consider the transformation

$$\begin{bmatrix} \begin{pmatrix} 0 \\ I \end{pmatrix} & 0 & sE_2 - A_2 \\ 0 & I & -B(s) \end{bmatrix} \begin{bmatrix} A_2(s) \\ -B(s)L(s) \\ -L(s) \end{bmatrix} = 0$$

It is easily seen that both compound matrices of the above transformation include the unit matrix and therefore does not possess f.e.d.. Therefore the matrices (24) are e.u.e. and thus the matrices in (24) have the same f.e.d..
(i-b) - **Same i.e.d.** Consider the transformation

$$\begin{bmatrix} \begin{pmatrix} 0 \\ kI \end{pmatrix} & 0 & sE_2 - A_2 \\ 0 & kI & -B(s) \end{bmatrix} \begin{bmatrix} A_2(s) \\ -B(s)L(s) \\ -L(s)k \end{bmatrix} = 0 \tag{25}$$

The highest coefficient matrices of the above compound matrices are

$$\begin{bmatrix} 0 & 0 & I & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & I & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & I & 0 \\ I & 0 & 0 & 0 & \cdots & 0 & A_{q_2} \\ 0 & I & & & & -B_1 & \end{bmatrix}, \begin{bmatrix} A_{2,q_2} \\ -B_1 L \\ -L \end{bmatrix}$$

respectively, where $L = \begin{pmatrix} 0 & 0 & \cdots & I \end{pmatrix}^T, B(s) = B_0 + B_1 s$ and $A_2(s) = A_{2,0} + A_{2,1}s + \cdots + A_{2,q_2}s^{q_2}$ with $A_{2,q_2} \neq 0$. It is easily seen that the above highest degree coefficient matrices have full row and column rank respectively and therefore the compound matrices in (25) have no i.e.d. and thus the matrices in (24) possess the same i.e.d..
(ii) Our second goal is to prove that the matrices

$$\begin{bmatrix} A(s) \begin{pmatrix} 0 \\ I \end{pmatrix} A_1(s) \end{bmatrix} \text{ and } \begin{bmatrix} A(s) & A_1(s) \end{bmatrix} \tag{26}$$

possess the same f.e.d. and i.e.d..
(ii-a) **Same f.e.d.** Consider the transformation

$$I \begin{bmatrix} A(s) \begin{pmatrix} 0 \\ I \end{pmatrix} A_1(s) \end{bmatrix} = \begin{bmatrix} A(s) & A_1(s) \end{bmatrix} \begin{bmatrix} \begin{pmatrix} 0 \\ I \end{pmatrix} & 0 \\ 0 & I \end{bmatrix}$$

We observe that both compound matrices of the above transformation include the unit matrix and therefore does not possess f.e.d.. Therefore the matrices (26) are e.u.e. and thus have the same f.e.d..
(ii-b) **Same i.e.d.** Consider the transformation

$$\begin{bmatrix} k^{q_1}I & A(s) & A_1(s) \end{bmatrix} \begin{bmatrix} A(s) \begin{pmatrix} 0 \\ I \end{pmatrix} & A_1(s) \\ -\begin{pmatrix} 0 \\ k^{q_1}I \end{pmatrix} & 0 \\ 0 & -k^{q_1}I \end{bmatrix} = 0$$

where $A(s) = A_0 + A_1 s + \cdots + A_{q_1}s^{q_1}$ and $A_1(s) = A_{1,0} + A_{1,1}s + \cdots + A_{1,q_1}s^{q_1}$ with $A_{1,q_1} \neq 0$. The highest coefficient matrices (h.c.m.) of the above compound matrices are :

$$\begin{bmatrix} I & A_{q_1} & A_{1,q_1} \end{bmatrix} \text{ and } \begin{bmatrix} A_{q_1} \begin{pmatrix} 0 \\ I \end{pmatrix} & A_{1,q_1} \\ -\begin{pmatrix} 0 \\ I \end{pmatrix} & 0 \\ 0 & -I \end{bmatrix}$$

It is easily seen that the above h.c.m. have full row and column rank respectively and therefore the compound matrices in (25) have no i.e.d.. So the matrices (26) possess the same i.e.d..
N.B. It is possible to start with the matrix $A_2(s)$ and follow identical arguments to yield a transformation of d.e. between $A_2(s)$ and $A_1(s)$ which is symmetric to (23). □

Remark 13. Note that the proof of the sufficiency of above theorem is independent of the class of polynomial matrices. Therefore in case where we define divisor equivalence on the set $P(m, l)$ of polynomial matrices, instead of $R_c(s)$, the property of the invariance of the finite and infinite elementary divisors still remains. However we don't know if in that case the converse of the above theorem, still remains true, due to the extra invariants of the polynomial matrices i.e. left and right minimal indices.

Remark 14. In case where (4) is a d.e. relation then it is easily seen that the relation $A_2(s)N(s) = M(s)A_1(s)$ is also a d.e. relation. Therefore according to Theorem 12 and Remark 13 the nonregular polynomial matrices $M(s), N(s)$ have the same f.e.d. and i.e.d.. ¿From (4) we get the relation $G(s) := A_2(s)^{-1}M(s) = N(s)A_1(s)^{-1}$. Thus in case where we are able to construct a left and right matrix fraction description of a rational matrix $G(s)$, s.t. the compound matrices defined in (5) have full rank and no f.e.d. or i.e.d., then the numerators $M(s), N(s)$ (resp. denominators $A_1(s), A_2(s)$) will have the same f.e.d. and i.e.d.

Theorem 15. D.e. is an equivalence relation on $\mathbb{R}_c[s]$.

Proof. (i) *Reflexivity.* Let $A(s) \in \mathbb{R}_c[s]$ and consider the following relation

$$\begin{bmatrix} (s - s_0)^{d[P]} I & A(s) \end{bmatrix} \begin{bmatrix} A(s) \\ -(s - s_0)^{d[P]} I \end{bmatrix} = 0$$

where s_0 is not a zero of $A(s)$. It is easily proved that the above transformation is a d.e. transformation.
(ii) *Symmetry.* Let $A_1(s), A_2(s) \in \mathbb{R}_c[s]$ be related by a d.e. transformation of the form

$$M(s)A_1(s) = A_2(s)N(s) \tag{27}$$

Then $A_1(s)$ and $A_2(s)$ have identical f.e.d. and i.e.d.. Hence there exists a relation of d.e. of the form (23) i.e a relation symmetric to (27).
(iii) *Transitivity.* Suppose that $A_1(s), A_2(s) \in \mathbb{R}_c[s]$ are d.e. and that $A_2(s), A_3(s) \in \mathbb{R}_c[s]$ are d.e.. Then $A_1(s)$ and $A_3(s)$ have identical f.e.d. and i.e.d.. Hence $A_1(s)$ and $A_3(s)$ are d.e. according to Theorem 12. □

4. ON THE CONNECTION OF D.E. AND STRICT EQUIVALENCE

Definition 16. (Vardulakis and Antoniou 2001) $A_1(s)$ and $A_2(s) \in \mathbb{R}_c[s]$ are called strictly equivalent iff their equivalent matrix pencils $sE_1 - A_1 \in \mathbb{R}^{c\times c}$ and $sE_2 - A_2 \in \mathbb{R}^{c\times c}$ proposed in (7), are strictly equivalent (Definition 4).

The coincidence of divisor and strict equivalence is proved in the following Theorem.

Theorem 17. Strict equivalence (Definition 16) belongs to the same equivalence class with d.e..

Proof.

($\Longleftarrow$)Suppose that $A_1(s)$,$A_2(s)$ are d.e.. Then

$$sE_1 - A_1 \overset{d.e}{\sim} A_1(s) \overset{d.e}{\sim} A_2(s) \overset{d.e}{\sim} sE_2 - A_2$$

Therefore from the transitivity property of d.e. $sE_1 - A_1 \overset{d.e}{\sim} sE_2 - A_2$ and since d.e. coincides with strict equivalence in pencils (Karampetakis 2001*a*), the two pencils are strict equivalent.

($\Longrightarrow$)Suppose that $A_1(s)$ and $A_2(s)$ are strict equivalent. Then

$$A_1(s) \overset{d.e}{\sim} sE_1 - A_1 \overset{d.e}{\sim} sE_2 - A_2 \overset{d.e}{\sim} A_2(s)$$

Then from the transitivity property of d.e. $A_1(s) \overset{d.e}{\sim} A_2(s)$. □

A geometrical meaning of d.e. is given in the sequel.

Definition 18. (Vardulakis and Antoniou 2001)

Two AR-representations

$$A_i\,(\sigma)\,\xi_k^i = 0, k = 0, 1, 2, ..., N$$

where σ is the shift operator, $A_i(\sigma) \in R[\sigma]^{r_i\times r_i}$, $i = 1, 2$ will be *called fundamentally equivalent (f.e.) over the finite time interval* $k = 0, 1, 2, ..., N$ iff there exists a bijective polynomial map between their respective behaviors $\mathcal{B}_{A_1(\sigma)}, \mathcal{B}_{A_2(\sigma)}$.

Theorem 19. D.e. implies f.e..

Proof. ¿From (4) we have

$$M(\sigma)A_1(\sigma) = A_2(\sigma)N(\sigma) \tag{28}$$

By multiplying (28) on the right by ξ_k^1 we get

$$M(\sigma)A_1(\sigma)\xi_k^1 = A_2(\sigma)N(\sigma)\xi_k^1 \Longrightarrow 0 = A_2(\sigma)N(\sigma)\xi_k^1 \Longrightarrow$$
$$\exists \xi_k^2 \in \mathcal{B}_{A_2(\sigma)} \text{ s.t. } \xi_k^2 = N(\sigma)\xi_k^1 \tag{29}$$

According to the conditions of d.e., $\left[\, A_1(\sigma)^T \;\; -N(\sigma)^T \,\right]^T$ has full rank and no f.e.d. or i.e.d.. This implies (Karampetakis 2001*b*) that $\xi_k^1 = 0$. Therefore the map defined by the polynomial matrix $N(s)$: $\mathcal{B}_{A_1(s)} \rightarrow \mathcal{B}_{A_2(s)} \mid \xi_k^1 \mapsto \xi_k^2$ is injective. From the symmetric d.e. transformation $\hat{M}(s)A_2(s) = A_1(s)\hat{N}(s)$ in a similar way we get that the map defined by the polynomial matrix $\hat{N}(s) : \mathcal{B}_{A_2(s)} \rightarrow \mathcal{B}_{A_1(s)} \mid \xi_k^2 \mapsto \xi_k^1$ is injective. Therefore both maps are bijections between $\mathcal{B}_{A_1(\sigma)}, \mathcal{B}_{A_2(\sigma)}$. □

5. CONCLUSIONS

The problem addressed in this paper is the restate of the divisor equivalence relation presented in (Karampetakis 2001*a*). We have shown that this redefined transformation with one condition less, provide us with necessary and sufficient conditions for the invariance of the f.e.d. and i.e.d..

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