On the solution of the implicit Roesser model

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***Abstract*— The main objective of this work is to provide a closed formula for the backward and symmetric solution of the 2-D singular Roesser model given in [6]. Of fundamental importance in our approach is the relative forward and backward fundamental matrix. An algorithm is also given for the determination of the backward fundamental matrix sequense.**

I. Introduction

Consider the 2-D linear discrete time system proposed in [16] as a generalization of the 2-D state-space model [5]

$$Ex(i+1,j+1) = A_0x(i,j) + A_1x(i+1,j) + A_2x(i,j+1) + B_0u(i,j) + B_1u(i+1,j) + B_2u(i,j+1) \quad (1)$$

where i,j are integer-value vertical and horizontal coordinates, respectively, $x(i,j) \in \mathbb{R}^n$ is the local state vector, $u(i,j) \in \mathbb{R}^m$ is the input vector, $A_k \in \mathbb{R}^{n\times n}, B_k \in \mathbb{R}^{n\times m}, k = 0,1,2$ and $E \in \mathbb{R}^{n\times n}$ exists and is not necessarily nonsingular. This model includes similar generalization of other 2-D state space models such as the Fornasini and Marchesini [7] and the Roesser 2-D model [6]. If $E \neq I$ we call these models *implicit 2-D systems*. We shall call (1) the *general singular model* (GSM) or otherwise the *implicit Fornasini-Marchesini model*. If E is non-square or $\det(E) = 0$ we call these models *singular 2-D systems*. A particular case of (1) is the *implicit Roesser model* proposed in [10], [11] as a generalization of the Roesser 2-D model

$$\underbrace{\begin{bmatrix} E_1 & E_2 \\ E_3 & E_4 \end{bmatrix}}_{E} \underbrace{\begin{bmatrix} x^h(i+1,j) \\ x^v(i,j+1) \end{bmatrix}}_{\tilde{x}(i,j)} = \underbrace{\begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x^h(i,j) \\ x^v(i,j) \end{bmatrix}}_{x(i,j)} + \underbrace{\begin{bmatrix} B_1 \\ B_2 \end{bmatrix}}_{B} u(i,j) \quad (2)$$

It is shown in [12] that the implicit Roesser and the implicit FM model are equivalent. Define for example

$$E_I = \begin{bmatrix} E_1 & 0 \\ E_3 & 0 \end{bmatrix}, E_{II} = \begin{bmatrix} 0 & E_2 \\ 0 & E_4 \end{bmatrix}$$

Then (2) may be rewritten as

$$E_I x(i+1,j) + E_{II} x(i,j+1) = Ax(i,j) + Bu(i,j)$$

a special case of (1). Due to the equivalence of the above models we consider in the rest of the paper only the Roesser model, due to its simplest form. According to [1] there are various ways to specify the boundary conditions (BCs) and the region of interest for the FM and Roesser models. First suppose that the 2-D implicit system has BCs specified along the $i-$ and $j-$ axes. For the Roesser model this means we know :

$$\begin{aligned} x(i,0) &= x_{i0}, i = 0,1,...,N \\ x(0,j) &= x_{0j}, j = 0,1,...,M \end{aligned} \quad (3)$$

where x_{i0} and x_{0j} are known vectors. Then, if the region of interest is the rectangle $[0,N]\times[0,M]$ in the (i,j) −plane, we are concerned with finding what could be called a "*forward solution*". If the BCs are specified along the upper and right-hand sides of the rectangle :

$$\begin{aligned} x(i,M) &= x_{iM}, i = 0,1,...,N \\ x(N,j) &= x_{Nj}, j = 0,1,...,M \end{aligned} \quad (4)$$

then the solution on $[0,N]\times[0,M]$ could be called "*backward solution*". A general case which includes both of these situations is where the BCs are of the split form:

$$\begin{aligned} C^u_{i,0}x(i,0) + C^u_{i,M}x(i,M) &= c^u_i, \quad 0 \le i \le N \\ C^h_{0,j}x(0,j) + C^h_{N,j}x(N,j) &= c^h_i, \quad 0 \le j \le M \end{aligned} \quad (5)$$

with $\begin{bmatrix} C^u_{i,0} & C^u_{i,M} \end{bmatrix}$ and $\begin{bmatrix} C^h_{0,j} & C^h_{N,j} \end{bmatrix}$ prescribed matrices of full row rank and c^u_i, c^h_i given vectors. If the BCs are of the split form given above or otherwise involve the semistate along all boundaries of the rectangular region $[0,N]\times[0,M]$ then the solution on $[0,N]\times[0,M]$ could be called "*symmetric solution*".

An example of the implicit Roesser model is given by the 2-D realization of a nonrecursible mask in digital image processing [2]. Implicit Roesser models are also arising from the discretization of continuous-time systems that are described by partial differential equations i.e. the standard discretization of the elliptic equation that results in a five-point discrete mask or the discretization of the diffusion equation that results in a four-point discrete mask [13].

A complete analysis of solutions and properties in the forward, backward and symmetric case for the 1-D singular systems $Ex(i+1) = Ax(i) + Bu(i)$ was given in [3] in terms of the matrices E, A, B and the forward and backward fundamental matrix of $(zE - A)^{-1}$. [4] and [2] have proposed, a forward solution to the 2-D implicit Roesser model and GSM respectively, in terms of the forward fundamental matrix of the system. Following similar methods to those of [3], we produce a closed formula for the backward and symmetric solution of the implicit Roesser model (2) in terms of the *forward fundamental matrix* $T_{p,q}$ and *backward fundamental matrix* $V_{p,q}$ of $(z_1E_I + z_2E_{II} - A)^{-1}$. A generalized Leverrier technique for computing the forward fundamental matrix sequence is available [15], [14] so that we may assume that this matrix sequence is given. An algorithm for the computation of the backward fundamental matrix is given in section 2, either by using the forward fundamental matrix of the inverse of the dual polynomial matrix $\tilde{G}(z_1,z_2) = z_2E_I + z_1E_{II} - Az_1z_2$ or directly in terms of the coefficient matrices of the adjoint matrix of $G^{-1}(z_1,z_2)$ and the coefficients of the determinant of $G(z_1,z_2)$ [2].

II. Preliminary results

Assume that the polynomial matrix

$$G(z_1,z_2) = z_1E_I + z_2E_{II} - A \tag{6}$$

and the Laurent expansion at infinity of $G(z_1,z_2)^{-1}$ exists, is unique ([15], [2]), and is given by :

$$G(z_1,z_2)^{-1} = \sum_{i=-n_1}^{\infty}\sum_{j=-n_2}^{\infty} T_{i,j} z_1^{-i} z_2^{-j} \tag{7}$$

$$(n_1 \le n, n_2 \le n) \quad \text{and} \quad |z_1| > \sigma_1 > 0, |z_2| > \sigma_2 > 0$$

where the matrix sequence $\{T_{i,j}\}$ is known as the *forward fundamental matrix*. Note that a necessary and sufficient condition for the uniqueness of the fundamental matrix sequence $\{T_{i,j}\}$ is that condition $\deg_z |G(z,z)| = \deg_{z_1} |G(z_1,z_2)| + \deg_{z_2} |G(z_1,z_2)|$ is satisfied ([15], [2]), where $\deg_{z_i} |G(z_1,z_2)|$ is the degree of $\det G(z_1,z_2)$ in z_i, with $i = 1,2$ and $\deg_z |G(z,z)|$ is the degree of $\det G(z,z)$. In [2] the inverse of the polynomial matrix $G(z_1,z_2) = z_1E_I + z_2E_{II} - A$ has been obtained by

$$(z_1E_I + z_2E_{II} - A)^{-1} = R(z_1,z_2)/d(z_1,z_2) \tag{8}$$

where

$$R(z_1,z_2) = \sum_{i=f_1^d}^{f_1^u}\sum_{j=f_2^d}^{f_2^u} R_{i,j} z_1^i z_2^j \tag{9}$$

$$d(z_1,z_2) = \sum_{i=d_1^d}^{d_1^u}\sum_{j=d_2^d}^{d_2^u} d_{i,j} z_1^i z_2^j \tag{10}$$

where f_i^d and d_i^d (resp. f_i^u and d_i^u) are the lower (resp. upper) bounds of the degree of $R(z_1,z_2)$ and $d(z_1,z_2)$ in terms of $z_i, i = 1,2$. Assuming now that the Laurent expansion at zero of $G(z_1,z_2)^{-1}$ is given by

$$G(z_1,z_2)^{-1} = \sum_{i=\ell_1}^{-\infty}\sum_{j=\ell_2}^{-\infty} V_{i,j} z_1^{-i} z_2^{-j} \tag{11}$$

$$|z_1| < \sigma_1, |z_2| < \sigma_2$$

where the matrix sequence $\{V_{i,j}\}$ is known as the *backward fundamental matrix*, then we have from (8) that

$$d(z_1,z_2)(z_1E_I + z_2E_{II} - A)^{-1} = R(z_1,z_2) \tag{12}$$

and thus $f_i^d = -\ell_i + d_i^d, i = 1,2$. Substituting $R(z_1,z_2), d(z_1,z_2)$ and $G(z_1,z_2)^{-1}$ by (9), (10) and (11) respectively in (12), and equating the coefficient matrices of the corresponding powers of $z_1^i z_2^j$, on both sides of the resulting equation, yields

$$R_{i,j} = \sum_{l=d_1^d}^{d_1^u}\sum_{m=d_2^d}^{d_2^u} d_{l,m} V_{l-i,m-j} \tag{13}$$

$$(\ f_1^d \le i \le f_1^u \text{ and } f_2^d \le j \le f_2^u)$$

$$0 = \sum_{l=d_1^d}^{d_1^u}\sum_{m=d_2^d}^{d_2^u} d_{l,m} V_{l-i,m-j} \tag{14}$$

where $\{i < \ell_1 - (f_1^u - f_1^d) \wedge j \le \ell_2\} \vee \{i \le \ell_1 \wedge j < \ell_2 - (f_2^u - f_2^d)\}$ which allows the computation of V_{ij} in the stated region in terms of its values for smaller i,j. Thus (14) constitutes another form of the Cayley Hamilton theorem for the 2-D matrix pencils. In the case where $d_{d_1^d,d_2^d} = 0$, then the Laurent expansion at zero of $G(z_1,z_2)^{-1}$ may not be unique as we can see in the following Theorem.

Theorem 1: Suppose that $d^d = d_1^d + d_2^d$, where d^d is the less degree of $\det G(z,z) = \det(zE_I + zE_{II} - A)$, or equivalently that $d_{d_1^d,d_2^d} \ne 0$. Then the Laurent expansion at zero of $G(z_1,z_2)^{-1}$ is unique.

Proof: Let

$$\tilde{G}(z_1,z_2) \equiv z_1z_2 G\left(\frac{1}{z_1},\frac{1}{z_2}\right) = z_1E_{II} + z_2E_I - Az_1z_2 \tag{15}$$

Since

$$\tilde{d}(z_1,z_2) = \det\left[\tilde{G}(z_1,z_2)\right] = \det\left[z_1z_2G\left(\frac{1}{z_1},\frac{1}{z_2}\right)\right] =$$

$$= z_1^n z_2^n \left(\sum_{i=d_1^d}^{d_1^u}\sum_{j=d_2^d}^{d_2^u} d_{i,j} z_1^{-i} z_2^{-j}\right) = \sum_{i=d_1^d}^{d_1^u}\sum_{j=d_2^d}^{d_2^u} d_{i,j} z_1^{n-i} z_2^{n-j}$$

the above condition is equivalent to the condition $\tilde{d} = \tilde{d}_1 + \tilde{d}_2$ where $\tilde{d}_i = \deg_{z_i} |z_1E_{II} + z_2E_I - Az_1z_2|, i = 1,2$ and $\tilde{d} = \deg_z |zE_{II} + zE_I - Az_1z_2|$. The proof of this Theorem follows from the unique construction of an explicit formula for the computation of the Laurent expansion at zero of $G(z_1,z_2)^{-1}$. Equation (13) may be rewritten as

$$\underbrace{\begin{bmatrix} P_{d_2^d}I_n & 0 & & & & \\ P_{d_2^d+1}I_n & P_{d_2^d}I_n & & & 0 & \\ \vdots & \vdots & \ddots & & & \\ P_{d_2^u}I_n & P_{d_2^u-1}I_n & \ddots & \ddots & & \\ 0 & P_{d_2^u}I_n & & \ddots & \ddots & \\ 0 & & & & \ddots & \ddots \\ & & & P_{d_2^u}I_n & \cdots & P_{d_2^d}I_n \end{bmatrix}}_{P} \times$$

$$\times \underbrace{\begin{bmatrix} V_{\ell_2} \\ V_{\ell_2-1} \\ \vdots \\ V_{\ell_2-(f_2^u-f_2^d)} \end{bmatrix}}_{V} = \underbrace{\begin{bmatrix} R_{f_2^d} \\ R_{f_2^d+1} \\ \vdots \\ R_{f_2^u} \end{bmatrix}}_{R}$$

where

$$P_i = \begin{bmatrix} d_{d_1^d,i}I_n & 0 & & \\ d_{d_1^d+1,i}I_n & d_{d_1^d,i}I_n & & 0 \\ \vdots & \vdots & & \\ d_{d_1^u,i}I_n & d_{d_1^u-1,i}I_n & \ddots & \\ 0 & d_{d_1^u,i}I_n & \ddots & \ddots \\ 0 & & \ddots & \ddots \\ & & d_{d_1^u,i}I_n & \cdots & d_{d_1^d,i}I_n \end{bmatrix}$$

$$i = d_2^d, d_2^d+1, ..., d_2^u$$

$$V_i = \begin{bmatrix} V_{\ell_1,i} \\ V_{\ell_1-1,i} \\ \vdots \\ V_{\ell_1-(f_1^u-f_1^d),i} \end{bmatrix}, R_j = \begin{bmatrix} R_{f_1^d,j} \\ R_{f_1^d+1,j} \\ \vdots \\ R_{f_1^u,j} \end{bmatrix}$$

for $i = \ell_2, \ell_2-1, ..., \ell_2-(f_2^u-f_2^d)$ and $j = f_2^d, f_2^d+1, ..., f_2^u$. Due to the special Toeplitz form of $P_{d_2^d}$, we find that the unique (i.e. $\det P_{d_2^d} \neq 0$) inverse of $P_{d_2^d}$ is

$$D = P_{d_2^d}^{-1} = \begin{bmatrix} r_0 I_n & & & 0 \\ r_1 I_n & r_0 I_n & & \\ \vdots & \vdots & \ddots & \vdots \\ r_{f_1^u-f_1^d} I_n & r_{f_1^u-f_1^d-1} I_n & \cdots & r_0 I_n \end{bmatrix}$$

where $r_0 = \frac{1}{d_{d_1^d,d_2^d}}$ and

$$r_j = (-1)^j \left(\frac{1}{d_{d_1^d,d_2^d}}\right)^j \sum_{i=0}^{j-1} \left[d_{d_1^d+j-i,d_2^d} \times r_i\right]$$

$$j = 1, 2, ..., f_1^u - f_1^d$$

so that we may write for the elements of V_{ℓ_2} the expressions

$$V_{i,\ell_2} = \sum_{j=0}^{\ell_1-i} r_j R_{d_i^d+i+j,d_2^d} \tag{16}$$

for $i = \ell_1-(f_1^u-f_1^d), .., \ell_1-1, \ell_1$. Due to the special Toeplitz form of P, we find also that the unique P^{-1} is

$$P^{-1} = \begin{bmatrix} D_0 I_n & & & 0 \\ D_1 I_n & D_0 I_n & & \\ \vdots & \vdots & \ddots & \vdots \\ D_{(f_2^u-f_2^d)} I_n & D_{f_1^u-f_1^d-1} I_n & \cdots & D_0 I_n \end{bmatrix}$$

where $D_0 = P_{d_2^d}^{-1}$ and

$$D_i = -\left(\sum_{j=0}^{i-1} D_j P_{d_2^d+j-i}\right) P_{d_2^d}^{-1}, i = 1, 2, ..., (f_2^u - f_2^d)$$

Thus

$$V_i = \sum_{j=0}^{\ell_2-i} D_j R_{d_2^d+i+j}, i = \ell_2-(f_2^u-f_2^d), .., \ell_2-1, \ell_2$$

For the calculation of $V_{i,j}$ for less values of i, j we get from (14) that

$$V_{i,j} = \frac{1}{d_{d_1^d,d_2^d}} \sum_{l=d_1^d}^{d_1^u} \sum_{m=d_2^d}^{d_2^u} d_{l,m} V_{l+i-d_1^d,m+j-d_2^d} \tag{17}$$

for $(l,m) \neq (d_1^d, d_2^d)$ and $\{i < \ell_1 - (f_1^u - f_1^d) \wedge j \leq \ell_2\} \vee \{i \leq \ell_1 \wedge j < \ell_2 - (f_2^u - f_2^d)\}$. From (16) and (17) we obtain a unique form of the Laurent expansion of $G(z_1,z_2)^{-1}$ and thus the Theorem has been proved. ■

The Laurent expansion about zero of $G(z_1,z_2)^{-1}$ given in (11) is related with the Laurent expansion at infinity given in (7) of the inverse of the dual matrix $\tilde{G}(z_1,z_2)$ as we can see in the following Lemma.

Lemma 2: Let the Laurent expansion at infinity of $\tilde{G}(z_1,z_2)^{-1}$ be

$$\tilde{G}(z_1,z_2)^{-1} = \sum_{p=-f_1}^{\infty} \sum_{q=-f_2}^{\infty} \tilde{T}_{p,q} z_1^{-p} z_2^{-q} \tag{18}$$

and (11) be the Laurent expansion at zero of $G(z_1,z_2)^{-1}$. Then

$$f_i + 1 = \ell_i \text{ and } V_{-i,-j} = \tilde{T}_{i+1,j+1}$$
$$i = \ell_1, \ell_1-1, ... \text{ and } j = \ell_2, \ell_2-1, ... \tag{19}$$

Proof: We have that

$$G(z_1,z_2) = z_1 z_2 \tilde{G}\left(\frac{1}{z_1}, \frac{1}{z_2}\right) \overset{(18)}{\underset{(11)}{=}} \tag{20}$$

$$= \sum_{p=-f_1}^{\infty} \sum_{q=-f_2}^{\infty} \tilde{T}_{p,q} z_1^{p-1} z_2^{q-1} \equiv \sum_{p=\ell_1}^{-\infty} \sum_{q=\ell_2}^{-\infty} V_{p,q} z_1^{-p} z_2^{-q}$$

Equating the coefficients of the powers of $z_i, i = 1, 2$ we obtain the proof of Lemma. ■

We conclude from the above Lemma that the Laurent expansion at zero of $G(z_1,z_2)^{-1}$ exists and is unique iff the Laurent expansion at infinity of $\tilde{G}(z_1,z_2)^{-1}$ exists and is unique or otherwise when $\tilde{d}^d = \tilde{d}_1 + \tilde{d}_2$, where $\tilde{d}_i = \deg_{z_i} |z_1 E_{II} + z_2 E_I - A z_1 z_2|, i = 1, 2$ and $\tilde{d}^d = \deg_z |z E_{II} + z E_I - A z_1 z_2|$. A direct result of Lemma 2 is that the Leverrier algorithm presented in [15], [2] may also be used for the computation of both the forward and backward fundamental matrix sequence. Therefore, Lemma 2 give us an alternative method from the algorithm presented in Theorem 1 for the computation of the backward fundamental matrix sequence. An interesting result that connects the solutions of (2) and the ones of the dual 2-D implicit Roesser model

$$E_I \tilde{x}(i,j+1) + E_{II} \tilde{x}(i+1,j) = \tag{21}$$
$$= A\tilde{x}(i+1,j+1) + B\tilde{u}(i+1,j+1)$$

in the closed interval $[0,N] \times [0,M]$ is given by :

Lemma 3: (a) If $\tilde{x}(i,j)$ is a solution of (21) for the non-zero input $\tilde{u}(i,j)$, then the sequence $x(i,j) = \tilde{x}(N-i, M-j)$ is a solution of the dual equation (2) for the nonzero input $u(i,j) = \tilde{u}(N-i, M-j)$.

(b) If $x(i,j)$ is a solution of (2) for the non-zero input $u(i,j)$, then the sequence $\tilde{x}(i,j) = x(N-i, M-j)$ is a solution of the dual equation (21) for the nonzero input $\tilde{u}(i,j) = u(N-i, M-j)$.

Proof: (a) Let $\tilde{x}(i,j)$ be a solution of (21) for the nonzero input $\tilde{u}(i,j)$. This implies that (21) is satisfied. Now consider equation (2). If we set $x(i,j) = \tilde{x}(N-i, M-j)$, $u(i,j) = \tilde{u}(N-i, M-j)$ we have

$$\begin{aligned} &E_I x(i+1,j) + E_{II} x(i,j+1) = \\ &= E_I \tilde{x}(N-(i+1), M-j) + E_{II} \tilde{x}(N-i, M-(j+1)) = \\ &\overset{(21)}{=} A\tilde{x}(N-i, M-j) + B\tilde{u}(N-i, M-j) \\ &\underset{u(i,j)=\tilde{u}(N-i,M-j)}{\overset{(x(i,j)=\tilde{x}(N-i,M-j))}{=}} Ax(i,j) + Bu(i,j) \end{aligned}$$

(b) Similarly we can prove the (b) part of the Theorem. ∎

A direct result of Lemma 3 is that the backward solution of the singular Roesser model (2) comes directly from the forward solution of the dual singular Roesser model (21).

III. Solutions of the singular Roesser model

In the next three subsections we give the forward, backward and symmetric solution of the singular Roesser model (2) in terms of the matrix coefficients E_I, E_{II}, A, B and the forward/backward fundamental matrix sequence $\{T_{i,j}\} / \{V_{i,j}\}$ of $G(z_1,z_2)^{-1}$.

A. The forward solution of the implicit Roesser model

Consider the singular Roesser model (2) and the Laurent matrix expansion at infinity of $G(z_1,z_2)^{-1}$ given in (7). Then the unique forward solution to (2) with admissible (3) is given according to [2] by :

$$x(i,j) = \sum_{p=0}^{i+n_1} \sum_{q=0}^{j+n_2} T_{i-p,j-q} Bu(p,q)$$
$$\sum_{q=0}^{i+n_2} T_{i+1,j-q} E_I x(0,q) + \sum_{p=0}^{i+n_1} T_{i-p,j+1} E_{II} x(p,0) \quad (22)$$

for $(-n_1, -n_2) \le (i,j)$. It is important to note that (2) does not always have a solution. A necessary and sufficient condition for (2) to have a solution is that the initial conditions (3) satisfy the relation (22) for ($i = 0 \& j = 0, 1, 2, ..., M$) and ($j = 0 \& i = 0, 1, ..., N$).

B. The backward solution of the implicit Roesser model

Let $\tilde{X}(z_1,z_2), \tilde{U}(z_1,z_2)$ be respectively the $2-D$ Z-transform of the functions $\tilde{x}(i,j)$ and $\tilde{u}(i,j)$. Then by applying the $2-D$ Z-transform [8] in the dual Roesser model (21) of (2) we obtain

$$\tilde{X}(z_1,z_2) = \underbrace{\left(\sum_{p=-f_1}^{\infty} \sum_{q=-f_2}^{\infty} \tilde{T}_{p,q} z_1^{-p} z_2^{-q} \right)}_{(E_I z_2 + E_{II} z_1 - A z_1 z_2)^{-1}} \times$$
$$\begin{aligned} \times\{ &Bz_1z_2\tilde{U}(z_1,z_2) - Bz_1z_2\tilde{U}(z_1,0) - Bz_1z_2\tilde{U}(0,z_2) + \\ &+ Bz_1z_2\tilde{u}(0,0) + Az_1z_2\tilde{x}(0,0) + E_I z_2 \tilde{X}(z_1,0) \\ &- Az_1z_2\tilde{X}(z_1,0) + E_{II} z_1 \tilde{X}(0,z_2) - Az_1z_2\tilde{X}(0,z_2) \} \end{aligned} \quad (23)$$

Using the inverse 2-D transformation [8] for (23) and taking into account that $\tilde{T}_{p,q} = 0$ for $p < -f_1$ or $q < -f_2$, we obtain

$$\begin{aligned} \tilde{x}(i,j) = &\sum_{p=0}^{i+f_1+1} \sum_{q=0}^{j+f_2+1} \tilde{T}_{i-p+1,j-q+1} B\tilde{u}(p,q) + \\ &+ \sum_{p=0}^{i+f_1} \tilde{T}_{i-p,j+1} E_I \tilde{x}(p,0) + \sum_{q=0}^{j+f_2} \tilde{T}_{i+1,j-q} E_{II} \tilde{x}(0,q) - \\ &- \sum_{p=1}^{i+f_1+1} \tilde{T}_{i-p+1,j+1} \begin{bmatrix} A & B \end{bmatrix} \begin{bmatrix} \tilde{x}(p,0) \\ \tilde{u}(p,0) \end{bmatrix} - \\ &- \sum_{q=1}^{j+f_2+1} \tilde{T}_{i+1,j-q+1} \begin{bmatrix} A & B \end{bmatrix} \begin{bmatrix} \tilde{x}(0,q) \\ \tilde{u}(0,q) \end{bmatrix} \\ &+ \tilde{T}_{i+1,j+1} \begin{bmatrix} A & B \end{bmatrix} \begin{bmatrix} \tilde{x}(0,0) \\ \tilde{u}(0,0) \end{bmatrix} \end{aligned} \quad (24)$$

Now by using the part (a) of Lemma 3 and the solution of the dual Roesser model (24) we can easily prove the following Theorem.

Theorem 4: If $\det[G(z_1,z_2)] \neq 0$, and the condition of Theorem 1 is satisfied, then the unique backward solution to (1) with admissible boundary conditions (4) is given by

$$x(i,j) = \sum_{p=0}^{N-i+\ell_1} \sum_{q=0}^{M-j+\ell_2} \left\{ \begin{matrix} V_{p-i-N,q-j-M} \times \\ Bu(N-p, M-q) \end{matrix} \right\} + \quad (25)$$
$$\begin{aligned} &+ \sum_{p=0}^{N-i+\ell_1-1} V_{1+p+i-N,j-M} E_I x(N-p, M) + \\ &+ \sum_{q=0}^{M-j+\ell_2-1} V_{i-N,1+q+j-M} E_{II} x(N, M-q) - \\ &- \sum_{p=1}^{N-i+\ell_1} V_{p+i-N,j-M} \begin{bmatrix} A & B \end{bmatrix} \begin{bmatrix} x(N-p,M) \\ u(N-p,M) \end{bmatrix} - \\ &- \sum_{q=1}^{M-j+\ell_2} V_{i-N,q+j-M} \begin{bmatrix} A & B \end{bmatrix} \begin{bmatrix} x(N,M-q) \\ u(N,M-q) \end{bmatrix} + \\ &+ V_{i-N,j-M} \begin{bmatrix} A & B \end{bmatrix} \begin{bmatrix} x(N,M) \\ u(N,M) \end{bmatrix} \end{aligned}$$

where $V_{i,j}$ is the backward fundamental matrix sequence of $G(z_1,z_2)^{-1}$ given in (11).

Proof: Let $\tilde{x}(i,j)$ be the solution of (21) for the nonzero input $\tilde{u}(i,j)$ presented in (24). Then by using (19) and the fact that $x(i,j) = \tilde{x}(N-i, M-j)$ is a solution of the dual equation (1) for the nonzero input $u(i,j) = \tilde{u}(N-i, M-j)$ from Lemma 3 we get the result. ∎

A necessary and sufficient condition for (2) to have a solution is that the final conditions (4) satisfy (25) for ($i = N \& j = 0,1,...,M$) and ($i = 0,1,....,N \& j = M$).

C. The symmetric solution of the implicit Roesser model

Consider the Laurent expansion at infinity of $G^{-1}(z_1, z_2)$ given in (7). Then the following relations

$$-T_{p-1,q-1}A + T_{p,q-1}E_I + T_{p-1,q}E_{II} = \delta_{p-1,q-1}I_n$$

are following from comparison of coefficient matrices at like powers of z_1 and z_2 of the equality

$$\underbrace{\left(\sum_{p=-n_1}^{\infty}\sum_{q=-n_2}^{\infty} T_{p,q}z_1^{-p}z_2^{-q}\right)}_{G(z_1,z_2)^{-1}} \times \underbrace{(E_I z_1 + E_{II}z_2 - A)}_{G(z_1,z_2)} = I_n$$

Define now the matrices

$$\begin{array}{r}\mathcal{A}_0 = blockdiag\left(\begin{pmatrix} E_I & -A \end{pmatrix}, ..., \begin{pmatrix} E_I & -A \end{pmatrix}\right)\\ \mathcal{A}_1 = blockdiag\left(\begin{pmatrix} 0 & E_{II} \end{pmatrix}, ..., \begin{pmatrix} 0 & E_{II} \end{pmatrix}\right)\\ \mathcal{B} = blockdiag\left(\begin{pmatrix} 0 & B \end{pmatrix}, ..., \begin{pmatrix} 0 & B \end{pmatrix}\right)\end{array}$$

where $\mathcal{A}_0 \in R^{nN\times n(N+1)}, \mathcal{A}_1 \in R^{nN\times n(N+1)}, \mathcal{B} \in R^{nN\times m(N+1)}$ and the vectors

$$y_i = \begin{pmatrix} x_{N,i} \\ x_{N-1,i} \\ \vdots \\ x_{1,i} \\ x_{0,i}\end{pmatrix}, u_i = \begin{pmatrix} u_{N,i} \\ u_{N-1,i} \\ \vdots \\ u_{1,i} \\ u_{0,i}\end{pmatrix}$$

where $y_i \in R^{(N+1)n}$ and $u_i \in R^{(N+1)m}$ for $i = 0,1,..,M$. Then (2) may be rewritten in the form

$$\underbrace{\begin{pmatrix} \mathcal{A}_1 & \mathcal{A}_0 & \cdots & 0 & 0 & 0 \\ 0 & \mathcal{A}_1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \mathcal{A}_1 & \mathcal{A}_0 & 0 \\ 0 & 0 & \cdots & 0 & \mathcal{A}_1 & \mathcal{A}_0 \end{pmatrix}}_{\tilde{A}_N}\underbrace{\begin{pmatrix} y_M \\ y_{M-1} \\ \vdots \\ y_1 \\ y_0 \end{pmatrix}}_{y_{0,M}} =$$

$$= \underbrace{\begin{pmatrix} 0 & \mathcal{B} & \cdots & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & \mathcal{B} & 0 \\ 0 & 0 & \cdots & 0 & 0 & \mathcal{B} \end{pmatrix}}_{\tilde{B}_N}\underbrace{\begin{pmatrix} u_M \\ u_{M-1} \\ \vdots \\ u_1 \\ u_0 \end{pmatrix}}_{v_{0,M}} \quad (26)$$

Let also

$$\mathcal{H}_i = \begin{pmatrix} T_{1,i} & T_{2,i} & \cdots & T_{N-1,i} & T_{N,i} \\ T_{0,i} & T_{1,i} & \cdots & T_{N-2,i} & T_{N-1,i} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ T_{-N+1,i} & T_{-N+2,i} & \cdots & T_{0,i} & T_{1,i} \\ T_{-N,i} & T_{-N+1,i} & \cdots & T_{-1,i} & T_{0,i} \end{pmatrix}$$

$$\mathcal{S}_i = \begin{pmatrix} F_{1,i} & 0 & \cdots & 0 & Q_{N,i} \\ F_{0,i} & \delta_{i-1}I & \cdots & 0 & Q_{N-1,i} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ F_{-N+2,i} & 0 & \cdots & \delta_{i-1}I & Q_{2,i} \\ F_{-N+1,i} & 0 & \cdots & 0 & Q_{1,i} \end{pmatrix}$$

$$F_{k,i} = T_{k,i-1}E_I$$

$$Q_{k,i} = T_{k,i}E_{II} - T_{k,i-1}A$$

Then we can check that

$$\mathcal{H}_i\mathcal{A}_1 + \mathcal{H}_{i-1}\mathcal{A}_0 = \mathcal{S}_i$$

Premultiplying (26) by the matrix

$$\tilde{A}_N^L = \begin{pmatrix} \mathcal{H}_1 & \mathcal{H}_2 & \mathcal{H}_3 & \cdots & \mathcal{H}_M \\ \mathcal{H}_0 & \mathcal{H}_1 & \mathcal{H}_2 & \cdots & \mathcal{H}_{M-1} \\ \mathcal{H}_{-1} & \mathcal{H}_0 & \mathcal{H}_1 & \cdots & \mathcal{H}_{M-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathcal{H}_{-M+1} & \mathcal{H}_{-M+2} & \mathcal{H}_{-M+3} & \cdots & \mathcal{H}_0 \end{pmatrix}$$

we obtain that

$$\tilde{A}_N^L\tilde{A}_N y_{0,M} = \tilde{A}_N^L\tilde{B}_N v_{0,M} \Leftrightarrow \quad (27)$$

$$\begin{pmatrix} \mathcal{H}_1\mathcal{A}_1 & \mathcal{S}_2 & \cdots & \mathcal{S}_M & \mathcal{H}_M\mathcal{A}_0 \\ \mathcal{H}_0\mathcal{A}_1 & \mathcal{S}_1 & \cdots & \mathcal{S}_{M-1} & \mathcal{H}_{M-1}\mathcal{A}_0 \\ \mathcal{H}_{-1}\mathcal{A}_1 & \mathcal{S}_0 & \cdots & \mathcal{S}_{M-2} & \mathcal{H}_{M-2}\mathcal{A}_0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathcal{H}_{-M+1}\mathcal{A}_1 & \mathcal{S}_{-M+2} & \cdots & \mathcal{S}_0 & \mathcal{H}_0\mathcal{A}_0 \end{pmatrix} y_{0,M}$$

$$= \begin{pmatrix} 0 & \mathcal{H}_1\mathcal{B} & \cdots & \mathcal{H}_M\mathcal{B} \\ 0 & \mathcal{H}_0\mathcal{B} & \cdots & \mathcal{H}_{M-1}\mathcal{B} \\ 0 & \mathcal{H}_{-1}\mathcal{B} & \cdots & \mathcal{H}_{M-2}\mathcal{B} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \mathcal{H}_{-M+1}\mathcal{B} & \cdots & \mathcal{H}_0\mathcal{B} \end{pmatrix} v_{0,M}$$

From the first and last block equation we get boundary conditions that must be satisfied in order (2) have a solution

$$\begin{array}{r}\mathcal{H}_1\mathcal{A}_1 y_M + \mathcal{S}_2 y_{M-1} + \cdots + \mathcal{S}_M y_1 + \mathcal{H}_M\mathcal{A}_0 y_0 = \\ = (\mathcal{H}_1\mathcal{B})u_{M-1} + \cdots + (\mathcal{H}_{M-1}\mathcal{B})u_1 + (\mathcal{H}_M\mathcal{B})u_0 \quad (28)\end{array}$$

and

$$\begin{array}{r}\mathcal{H}_{-M+1}\mathcal{A}_1 y_M + \mathcal{S}_{-M+2}y_{M-1} + \cdots + \mathcal{S}_0 y_1 + \mathcal{H}_0\mathcal{A}_0 y_0 = \\ = (\mathcal{H}_{-M+1}\mathcal{B})u_{M-1} + \cdots + (\mathcal{H}_0\mathcal{B})u_1 + (\mathcal{H}_0\mathcal{B})u_0 \quad (29)\end{array}$$

Note that the matrices $S_i, i = 0,2,3,...,M$ in (28) and (29) have all their block columns, except of the first and the last one, filled with zero entries and therefore the above equations gives rise only to boundary conditions of the form (5). Now consider the rest equations coming from (27)

$$\begin{array}{r}(\mathcal{H}_{-q}\mathcal{A}_1)y_M + \mathcal{S}_{-q+1}y_{M-1} + \cdots + \quad (30)\\ +\mathcal{S}_{-q+M-1}y_1 + (\mathcal{H}_{-q+M-1}\mathcal{A}_0)y_0 = \\ = (\mathcal{H}_{-q}\mathcal{B})u_{M-1} + \cdots + (\mathcal{H}_{-q+M-1}\mathcal{B})u_0\end{array}$$

where $q = 0,1,...,M-2$. Now by taking the $i-$th row of the above equations i.e. for $q = 0,1,...,M-2$ and $i =$

$0, 1, ..., N-2$ and substituting $N-1+i$ with p, and $M-1-q$ with q, we can easily get the following Theorem.

Theorem 5: If $\det\left[G\left(z_1, z_2\right)\right] \neq 0$, $\deg_z\left|G\left(z, z\right)\right| = \deg_{z_1}\left|G\left(z_1, z_2\right)\right| + \deg_{z_2}\left|G\left(z_1, z_2\right)\right|$ is satisfied [2], then the unique symmetric solution to (2) with admissible boundary conditions (3) is given by

$$\begin{gathered}
x_{p,q} = -T_{2(N-1)-p,1+q-M}E_{II}x_{0,M} - \\
-\sum_{k=0}^{N-2} T_{N-1-p+k,1+q-M}E_{II}x_{N-1-k,M} + \\
+\sum_{j=q+2-M}^{q+1} \{-T_{N-1-p,j-1}E_I x_{N,q+2-j} + \\
+\{T_{2(N-1)-p,j}E_{II} - T_{2(N-1)-p,j-1}A\}x_{0,q+2-j}\} - \\
-T_{N-1-p,q}E_I x_{N,0} + \\
+\sum_{k=0}^{N-2}\{T_{N-1-p+k,q}A - T_{N-p+k,q}E_I\}x_{N-1-k,0} + \\
+T_{2(N-1)-p,q}Ax_{0,0} + \\
+\sum_{j=q+1-M}^{q-1}\sum_{k=0}^{N-2} T_{N-1-p+k,j}Bu_{N-k-1,M-1-j} + \\
+\sum_{j=q+1-M}^{q-1} T_{2(N-1)-p,j}Bu_{0,M-1-j} + T_{2(N-1)-p,q}Bu_{0,0} + \\
+\sum_{k=0}^{N-2}\{T_{N-1+p+k,q}B\}u_{N-1-k,0}
\end{gathered}$$

Using now the first and last block row equations of (30) we get the following extra boundary conditions for ($i = -1, N-1$ & $q = 0, 1, ..., M-2$), or ($q = -1, M-1$ & $i = -1, 0, ..., N-2, N-1$) (the boundary equations that we have described before in terms of block matrices)

$$\begin{gathered}
T_{-i+N-1,-q}E_{II}x_{0,M} + \sum_{k=0}^{N-2} T_{-i+k,-q}E_{II}x_{N-1-k,M} + \\
+\sum_{j=1-q}^{M-q}\{T_{-i,j-1}E_I x_{N,M-q+1-j} + \\
+\left(T_{-i+N-1,j}E_{II} - T_{-i+N-1,j-1}A\right)x_{0,M-q+1-j}\} - \\
-T_{-i,M-1-q}E_I x_{N,0} - T_{-i+N-1,M-1-q}Ax_{0,0} - \\
-\sum_{k=0}^{N-2}\left(\begin{array}{c} T_{-i+k,M-1-q}A + \\ -T_{-i+k+1,M-1-q}E_I \end{array}\right)x_{N-1-k,0} - \\
= \sum_{j=-q}^{M-2-q}\sum_{k=0}^{N-2} T_{-i+k,j}Bu_{N-k-1,M-1-j} + \\
+\sum_{j=-q}^{M-2-q} T_{-i+N-1,j}Bu_{0,M-1-j} + \\
+\left(T_{-i+N-1,M-1-q}B\right)u_{0,0} + \\
+\sum_{k=0}^{N-2} T_{-i+k,M-1-q}Bu_{N-1-k,0} \qquad (31)
\end{gathered}$$

Therefore, a necessary and sufficient condition so that (2) has a solution is that the initial conditions, final conditions and input sequences satisfy the relations (5), (28), (29) and (31).

IV. Conclusions

In the case of discrete time singular Roesser models, exact solutions where proposed in two different forms : a) backward solutions, and b) symmetric solutions. All the closed formula solutions were represented in terms of the forward and backward fundamental matrix of the singular Roesser model. It is easily seen that the proposed solutions : a) are extensions of the ones proposed in [3] for 1-D discrete time singular systems, and b) accomplish the work that have been done by [4] and [2] for the forward solution of the general singular model and the implicit Roesser model respectively. Certain controllability and observability criteria based on the proposed solutions are being studied and will be discussed in a future work.

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