ON THE COMPUTATION OF THE MINIMAL POLYNOMIAL OF A TWO-VARIABLE POLYNOMIAL MATRIX

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ABSTRACT

The main contribution of this work is to provide an algorithm for the computation of the minimal polynomial of a two variable polynomial matrix, based on the solution of linear matrix equations. The whole theory is implemented via an illustrative example.

Keywords : minimal polynomial, Cayley-Hamilton theorem, characteristic polynomial.

1. INTRODUCTION

It is well known form the Cayley Hamilton theorem that every matrix $A \in \mathbb{R}^{r\times r}$, satisfies its characteristic equation [3] i.e. if $p(s) := \det(sI_n - A) = s^n + p_1 s^{n-1} + \cdots + p_n$ then $p(A) = 0$. The Cayley Hamilton theorem is still valid for all cases of matrices over a commutative ring [6], and thus for multivariable polynomial matrices. Another form of the Cayley-Hamilton theorem also known as a *relative* Caley-Hamilton theorem is given in terms of the fundamental matrix sequence of the resolvent of the matrix i.e. if $(sI - A)^{-1} = \sum_{i=0}^{\infty} \Phi_i s^{-i}$ then $\Phi_k + p_1\Phi_{k-1} + \cdots + p_n\Phi_{k-n} = 0$. The Cayley-Hamilton theorem have been investigated for the matrix pencil case $A(s) = A_0 + A_1 s$ by [5] and the respective relative Cayley-Hamilton theorem by [4]. The Cayley-Hamilton theorem has been extended to matrix polynomials [2],[14], [15], to standard and singular two variable matrix pencils [7], [8],[10], [11], [18] and M-D matrix pencils by [9], [17]. The reason of interest in the Caley-Hamilton theorem is due to its applications in control systems i.e. calculation of the controllability and observability grammians and the state-transition matrix, electrical circuits, systems with delays, singular systems, 2-D linear systems, calculation of powers of matrices and inverses e.t.c. Except of the characteristic polynomial of a constant matrix let $p(s)$ with the nice property $p(A) = 0$ there is also another polynomial, known as minimal polynomial let $m(s)$, which is the least degree polynomial that satisfies the equation $m(A) = 0$ [3]. Since the minimal polynomial has less degree than the characteristic polynomial, it help us to solve quicker, problems such as the computation of the inverse or power of a matrix. A number of algorithms have been given for the computation of the minimal polynomial of a constant matrix [16] but there is no much interest as concerns the polynomial matrices of one or more variables. Therefore, the aim of this work is to propose an algorithm for the computation of the minimal polynomial of a two-variable polynomial matrix based on the solution of linear matrix equations. The proposed algorithm is illustrated via an example.

2. COMPUTATION OF THE MINIMAL POLYNOMIAL

Consider the polynomial matrix

$$A(s_1, s_2) = \sum_{i=0}^{q_1}\sum_{j=0}^{q_2} A_{i,j} s_1^i s_2^j \in \mathbb{R}[s_1, s_2]^{r\times r} \tag{1}$$

where q_1 (resp. q_2) is the greatest power of s_1 (resp. s_2) of $A(s_1, s_2)$. Then we can give the following definition.

Definition 1 *Every polynomial*

$$p(z) = z^p + p_1(s_1, s_2) z^{p-1} + \cdots + p_n(s_1, s_2)$$

for which

$$p(A(s_1, s_2)) = A(s_1, s_2)^p + p_1(s_1, s_2) A(s_1, s_2)^{p-1} + .. + p_n(s_1, s_2) I_n = 0 \tag{2}$$

is called an annihilating polynomial for the polynomial matrix $A(s_1, s_2) \in R^{r\times r}[s_1, s_2]$. The monic annihilating polynomial with the less degree is called minimal polynomial.

It is known that the characteristic polynomial $p(z) = \det(zI_n - A(s_1, s_2))$ is an annihilating polynomial but not necessary a minimal polynomial.

Example 1 *Let*

$$A(s_1, s_2) = \begin{bmatrix} s_1 & 1 & 0 \\ 0 & s_2 & 0 \\ 0 & 0 & s_1 \end{bmatrix}$$

Then $p(z) = \det(zI_n - A) = (z - s_1)^2 (z - s_2)$ *and* $p(A) = (A - s_1 I)^2 (A - s_2 I) = 0_{3,3}$.

The coefficients of the characteristic polynomial can be computed in a recursive way by an algorithm presented in [12] and [13]. As we shall see below, the characteristic polynomial of the above example is not the only polynomial of third order that satisfies (2), and does not coincide with the minimal polynomial. Let

$$B(s_1, s_2) = \sum_{i=0}^{p_1} \sum_{j=0}^{p_2} B_{i,j} s_1^i s_2^j \in \mathbb{R}[s_1, s_2]^{r \times r}$$

where p_1 (resp. p_2) is the greatest power of s_1 (resp. s_2) of $A(s_1, s_2)$. Then the product of $B(s_1, s_2)A(s_1, s_2)$ is given by :

$$B(s_1, s_2)A(s_1, s_2) = \\ = \sum_{l_1=0}^{p_1+q_1} \sum_{m_1=0}^{p_2+q_2} \left(\sum_{i=0}^{l_1} \sum_{j=0}^{m_1} (B_{i,j} A_{l_1-i,m_1-j}) \right) s_1^{l_1} s_2^{m_1}$$

If $B(s_1, s_2) = \Phi_{0,0,0} := I_r$ then

$$A(s_1, s_2) =: \sum_{i=0}^{q_1} \sum_{j=0}^{q_2} \Phi_{1,i,j} s_1^i s_2^j \equiv \Phi_{0,0,0} A(s_1, s_2) = \\ = \sum_{l_1=0}^{q_1} \sum_{m_1=0}^{q_2} \left(\sum_{i=0}^{l_1} \sum_{j=0}^{m_1} (\Phi_{0,i,j} A_{l_1-i,m_1-j}) \right) s_1^{l_1} s_2^{m_1}$$

where $\Phi_{0,i,j} = 0 \; \forall (i,j) \neq (0,0)$. Similarly, if we set $B(s_1, s_2) = \sum\limits_{i=0}^{q_1} \sum\limits_{j=0}^{q_2} \Phi_{1,i,j} s_1^i s_2^j := A(s_1, s_2)$, where $\Phi_{1,i,j} = A_{i,j}, i$ (resp. $j) = 0, 1, ..., q_1$ (resp. q_2) then

$$A^2(s_1, s_2) =: \sum_{i=0}^{q_1} \sum_{j=0}^{q_2} \Phi_{2,i,j} s_1^i s_2^j = \\ = \sum_{l_1=0}^{2q_1} \sum_{m_1=0}^{2q_2} \left(\sum_{i=0}^{l_1} \sum_{j=0}^{m_1} (\Phi_{1,i,j} A_{l_1-i,m_1-j}) \right) s_1^{l_1} s_2^{m_1}$$

In the general case, where $\Phi_{k,i,j}$ is the matrix coefficient of $s_1^i s_2^j$ in the matrix $A(s_1, s_2)^k$, then

$$A^k(s_1, s_2) = \begin{cases} I_r & k = 0 \\ \sum\limits_{l_1=0}^{kq_1} \sum\limits_{m_1=0}^{kq_2} \Phi_{k,l,m} s_1^{l_1} s_2^{m_1} & k \geq 1 \end{cases} \tag{3}$$

where

$$\Phi_{k,l_1,m_1} = \sum_{i=0}^{l_1} \sum_{j=0}^{m_1} (\Phi_{k-1,i,j} A_{l_1-i,m_1-j}) \tag{4}$$

with $l_1 = 0, 1, ..., kq_1, m_1 = 0, 1, ..., kq_2$
$\Phi_{1,l_1,m_1} = A_{l_1,m_1}$ and $\Phi_{0,0,0} = I_r$

Let now the minimal polynomial of $A(s_1, s_2)$ be of the form

$$p(z) = z^m + p_{r-1}(s_1, s_2) z^{m-1} + ... + \\ + p_1(s_1, s_2) z + p_0(s_1, s_2)$$

where $m \leq r$, with

$$p_i(s_1, s_2) = \sum_{k=0}^{(m-i)q_1} \sum_{l=0}^{(m-i)q_2} p_{i,k,l} s_1^k s_2^l$$

for $i = 1, 2, ..m$ and $p_{i,k,l} \in \mathbb{R}$. Then (2) can be rewritten as

$$p(A(s_1, s_2)) = A(s_1, s_2)^m + \\ p_{m-1}(s_1, s_2) A(s_1, s_2)^{m-1} + ... + p_0(s_1, s_2) I_r = 0_{r,r} \tag{5}$$

or equivalently

$$p_{m-1}(s_1, s_2) A(s_1, s_2)^{m-1} + ... + \\ + p_0(s_1, s_2) I_r = -A(s_1, s_2)^m \tag{6}$$

Equation (6), may be rewritten as

$$\sum_{i=0}^{mq_1} \sum_{j=0}^{mq_2} f_{i,j} s_1^i s_2^j = - \sum_{i=0}^{mq_1} \sum_{j=0}^{mq_2} \Phi_{m,i,j} s_1^i s_2^j \tag{7}$$

By using the equations (3),(4) and (6), in (7) we get the following formula

$$F = \Phi P = \overline{\Phi} \tag{8}$$

where Φ, P and $\overline{\Phi}$ are defined as follows,

$$F = \begin{bmatrix} f_0 \\ f_1 \\ \vdots \\ f_{mq_1} \end{bmatrix} \; ; \; f_i = \begin{bmatrix} f_{i,0} \\ f_{i,1} \\ \vdots \\ f_{i,mq_2} \end{bmatrix}$$

$$P = \begin{bmatrix} p_{m-1,0}I_r \\ p_{m-1,1}I_r \\ \vdots \\ p_{m-1,q_1}I_r \\ \hline p_{m-2,0}I_r \\ \vdots \\ p_{m-2,2q_1}I_r \\ \hline \vdots \\ \hline p_{0,0}I_r \\ \vdots \\ p_{0,mq_1}I_r \end{bmatrix} \; ; \; p_{i,n} = \begin{bmatrix} p_{i,n,0}I_r \\ p_{i,n,1}I_r \\ \vdots \\ p_{i,n,q_2}I_r \end{bmatrix} \qquad (9)$$

$$\overline{\Phi} = \begin{bmatrix} -\Phi_{m,0} \\ -\Phi_{m,1} \\ \vdots \\ -\Phi_{m,mq_1} \end{bmatrix}$$

$$\Phi = \left[\underbrace{\begin{matrix} \Phi_{m-1,0} & \cdots & 0 \\ \Phi_{m-1,1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ \Phi_{m-1,q_1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ \Phi_{m-1,(m-1)q_1} & \cdots & \Phi_{m-1,(m-2)q_1} \\ 0 & \cdots & \Phi_{m-1,(m-2)q_1+1} \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \Phi_{m-1,(m-1)q_1} \end{matrix}}_{q_1+1} \right.$$

$$\cdots \underbrace{\begin{matrix} \Phi_{1,0} & 0 & \cdots & 0 \\ \Phi_{1,1} & \Phi_{1,0} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \Phi_{1,q_1} & \Phi_{1,q_1-1} & \cdots & 0 \\ 0 & \Phi_{1,q_1} & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & \Phi_{1,0} \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & \Phi_{1,q_1} \end{matrix}}_{(m-1)q_1+1}$$

$$\left. \underbrace{\begin{matrix} \Phi_{0,0} & 0 & 0 & \cdots & 0 & 0 \\ 0 & \Phi_{0,0} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \Phi_{0,0} & 0 \\ 0 & 0 & 0 & \cdots & 0 & \Phi_{0,0} \end{matrix}}_{mq_1+1} \right] \qquad (10)$$

with

$$\Phi_{i,n} = \begin{bmatrix} \Phi_{i,n,0} & 0 & \cdots & 0 \\ \Phi_{i,n,1} & \Phi_{i,n,0} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \Phi_{i,n,mq_2} & \Phi_{i,n,mq_2-1} & \cdots & \Phi_{i,n,0} \\ \vdots & \Phi_{i,n,mq_2} & \cdots & \Phi_{i,n,1} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \Phi_{i,n,mq_2} \end{bmatrix}$$

for $i = 2, 3, .., r$ and

$$\Phi_{1,n} = \begin{bmatrix} A_{n,0} & 0 & \cdots & 0 \\ A_{n,1} & A_{n,0} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_{n,q_2} & A_{n,q_2-1} & \cdots & A_{n,0} \\ \vdots & A_{n,q_2} & \cdots & A_{n,1} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_{n,q_2} \end{bmatrix}$$

$$\Phi_{0,0} = \begin{bmatrix} I_r & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & I_r & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & I_r & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & I_r & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & I_r & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & I_r \end{bmatrix}$$

Let Φ_i be the matrices that contains the $i \bmod r$ columns of the matrix Φ and K_i be the matrices that contain the i columns of the matrix $\overline{\Phi}$. Then (8) rewritten as

$$
\underbrace{\begin{bmatrix} \Phi_1 \\ \Phi_2 \\ \vdots \\ \Phi_r \end{bmatrix}}_{\mathsf{F}m}
\underbrace{\begin{bmatrix} p_{m-1,0} \\ p_{m-1,1} \\ \vdots \\ p_{m-1,q} \\ \hline p_{m-2,0} \\ \vdots \\ p_{m-2,2q} \\ \hline \vdots \\ \hline p_{0,0} \\ \vdots \\ p_{0,mq} \end{bmatrix}}_{\mathsf{P}_m}
=
\underbrace{\begin{bmatrix} K_1 \\ K_2 \\ \vdots \\ K_r \end{bmatrix}}_{\mathsf{K}_m}
\qquad (11)
$$

where $p_{i,n}$ have been defined in (9) and $\mathsf{F}_m \in \mathbb{R}^{n_1 \times m_1}$, where

$$
\begin{aligned}
n_1 &= (q_1 m + 1)(q_2 m + 1) r^2 \\
m_1 &= \sum_{i=1}^{m} (iq_1 + 1)(iq_2 + 1)
\end{aligned}
$$

(with $n_1 \geq m_1$ (with equality hold only for $r = m = 1$)). We can use numerical methods, such as Gauss elimination method or QR factorization, for the solution of (8) or (11) in terms of $p_{i,k,l}$ and thus to find the two–variable polynomials $p_i(s_1, s_2)$, $i = 0, 1, ..., m-1$ where $m \leq r$. An algorithm for the computation of the minimal polynomial or otherwise for the coefficients $p_{i,k,l}$ for $i = 0, 1, ..., m-1$ and ($k = 0, 1, .., mq_1$ and $l = 0, 1, ..., mq_2$) is given below.

Algorithm 1. Computation of the minimal polynomial matrix of $A(s_1, s_2) \in \mathbb{R}[s_1, s_2]^{r \times r}$.

Step 1. Determine the matrices $\Phi_{0,0}, \Phi_{1,i}$ for $i = 0, 1, .., q$ directly in terms of the coefficients of the polynomial matrix $A(s_1, s_2)$ and then the matrices Φ_1, Φ_2 defined below

$$
\Phi_1 = \begin{bmatrix}
\Phi_{0,0} & 0 & 0 & \cdots & 0 & 0 \\
0 & \Phi_{0,0} & 0 & \cdots & 0 & 0 \\
0 & 0 & \Phi_{0,0} & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & \Phi_{0,0} & 0 \\
0 & 0 & 0 & \cdots & 0 & \Phi_{0,0}
\end{bmatrix}
$$

$$
\Phi_2 = \left[\begin{matrix}
\Phi_{1,0} & 0 & \cdots & 0 \\
\Phi_{1,1} & \Phi_{1,0} & \cdots & 0 \\
\vdots & \vdots & \ddots & 0 \\
\Phi_{1,q_1} & \Phi_{1,q_1-1} & \cdots & 0 \\
0 & \Phi_{1,q_1} & \cdots & 0 \\
\vdots & \vdots & \cdots & \vdots \\
0 & 0 & \cdots & \Phi_{1,0} \\
\vdots & \vdots & \cdots & \vdots \\
0 & 0 & \cdots & \Phi_{1,q_1}
\end{matrix} \right|
\left. \begin{matrix}
\Phi_{0,0} & 0 & \cdots & 0 \\
0 & \Phi_{0,0} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \Phi_{0,0}
\end{matrix} \right]
$$

where

$$
\Phi_1 \in \mathbb{R}^{r(q_1+1)^2 \times r(q_1+1)^2}
$$

$$
\Phi_2 \in \mathbb{R}^{r(3q_1+2)(3q_2+2) \times r(2q_1+1)(2q_2+1)}
$$

Step 2. x=1

Rewrite the equations $\Phi_i P_i = \overline{\Phi}_i$ as $\mathsf{F}_i \mathsf{P}_i = \mathsf{K}_i$ for $i = 1, 2$ (see (11)).

While $\left(rank_{\mathbb{R}} \mathsf{F}_{x+1} = \sum_{i=1}^{x+1} (iq_1 + 1)(iq_2 + 1) \right)$

x=x+1

Define the matrix

$$
\Phi_{x+1} = \underbrace{\left[\begin{matrix}
\Phi_{x,0} & 0 & \cdots & 0 \\
\Phi_{x,1} & \Phi_{x,0} & \cdots & 0 \\
\Phi_{x,2} & \Phi_{x,1} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
\Phi_{x,q_1} & \Phi_{x,q_1-1} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
\Phi_{x,xq_1} & \Phi_{x,xq_1-1} & \cdots & \Phi_{x,(x-1)q_1} \\
0 & \Phi_{x,xq_1} & \cdots & \Phi_{x,(x-1)q_1+1} \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \Phi_{x,xq_1}
\end{matrix} \right.}_{(q_1+1)(q_2+1)}
$$

$$
\cdots \underbrace{\left. \begin{matrix}
\Phi_{1,0} & 0 & \cdots & 0 & 0 \\
\Phi_{1,1} & \Phi_{1,0} & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & 0 & 0 \\
\Phi_{1,q_1} & \Phi_{1,q_1-1} & \cdots & \cdots & 0 \\
0 & \Phi_{1,q_1} & \cdots & \cdots & 0 \\
\vdots & \vdots & \cdots & \ddots & \vdots \\
0 & 0 & \cdots & \cdots & \Phi_{1,0} \\
\vdots & \vdots & \cdots & \cdots & \vdots \\
0 & 0 & \cdots & \cdots & \Phi_{1,q_1}
\end{matrix} \right|}_{((x-2)q_1+1)((x-2)q_2+1)}
$$

$$\underbrace{\begin{bmatrix} \Phi_{0,0} & 0 & \cdots & 0 \\ 0 & \Phi_{0,0} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \Phi_{0,0} \end{bmatrix}}_{((x+1)q_1+1)((x+1)q_2+1)}$$

where $\Phi_{x+1} \in \mathbb{R}^{n_1 \times n_2}$ with

$$n_1 = ((x+1) q_1 + 1) ((x+1) q_2 + 1)$$
$$n_2 = \sum_{i=1}^{x+1} (iq_1 + 1) (iq_2 + 1)$$

End While

Step 3. The coefficients of the minimal polynomial are given by the solution of the system (11) i.e. $\mathsf{F}_x \mathsf{P}_x = \mathsf{K}_x$. ∎

Similarly with the previous section, the above algorithm can be easily modified in order to find the characteristic polynomial of a two-variable polynomial matrix as follows.

Algorithm 2. Computation of the characteristic polynomial of $A(s_1, s_2) \in \mathbb{R}[s_1, s_2]^{r \times r}$.

Step 1. Construct the matrices $\Phi_{i,0}, \Phi_{i,1}, ..., \Phi_{i,iq_1}$ for $i = 0, 1, ..., r-1$.

Step 2. Define the matrix Φ_r.

Step 3. Construct the matrices F_r and K_r that contains the $i \bmod r$ columns of Φ_r and Φ respectively.

Step 4. The coefficients of the characteristic polynomial are given by the solution of the system (11) with $m = r$ i.e. $\mathsf{F}_r \mathsf{P}_r = \mathsf{K}_r$. ∎

The upper bound for the complexity of the algorithm for the computation of the minimal polynomial, which is the same for the characteristic polynomial, is $O\left(q_1^3 q_2^3 r^{12}\right)$ (q_1 (q_2) =the greatest power of s_1 (s_2) in $A(s_1, s_2)$, r=the dimension of the matrix $A(s_1, s_2)$), if we use the Gauss elimination method for checking the rank of the matrix F_x and solving the equation $\mathsf{F}_x \mathsf{P}_x = \mathsf{K}_x$.

Example 2 *Let*

$$A(s_1, s_2) = \begin{bmatrix} s_1 & 1 & 0 \\ 0 & s_2 & 0 \\ 0 & 0 & s_1 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{A_{00}} s_1^0 s_2^0 +$$
$$+ \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{A_{10}} s_1^1 s_2^0 + \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{A_{01}} s_1^0 s_2^1$$

Define the matrices

$$\Phi = \begin{bmatrix} \Phi_{20} & 0 & \Phi_{10} & 0 & 0 & I & 0 & 0 & 0 \\ \Phi_{21} & \Phi_{20} & \Phi_{11} & \Phi_{10} & 0 & 0 & I & 0 & 0 \\ \Phi_{22} & \Phi_{21} & 0 & \Phi_{11} & \Phi_{10} & 0 & 0 & I & 0 \\ 0 & \Phi_{22} & 0 & 0 & \Phi_{11} & 0 & 0 & 0 & I \end{bmatrix}$$

where

$$\Phi_{11} = \begin{bmatrix} \Phi_{110} & 0 & 0 \\ \Phi_{111} & \Phi_{110} & 0 \\ 0 & \Phi_{111} & \Phi_{110} \\ 0 & 0 & \Phi_{111} \end{bmatrix}$$

$$\Phi_{20} = \begin{bmatrix} \Phi_{200} & 0 \\ \Phi_{201} & \Phi_{200} \\ \Phi_{202} & \Phi_{201} \\ 0 & \Phi_{202} \end{bmatrix}, \Phi_{21} = \begin{bmatrix} \Phi_{210} & 0 \\ \Phi_{211} & \Phi_{210} \\ \Phi_{212} & \Phi_{211} \\ 0 & \Phi_{212} \end{bmatrix}$$

$$\Phi_{22} = \begin{bmatrix} \Phi_{220} & 0 \\ \Phi_{221} & \Phi_{220} \\ \Phi_{222} & \Phi_{221} \\ 0 & \Phi_{222} \end{bmatrix}$$

$$\Phi_{10} = \begin{bmatrix} \Phi_{100} & 0 & 0 \\ \Phi_{101} & \Phi_{100} & 0 \\ 0 & \Phi_{101} & \Phi_{100} \\ 0 & 0 & \Phi_{101} \end{bmatrix}$$

The matrices $\Phi_{i,j,k}$ can be computed by using (4). Write the relation (11) with

$$\underbrace{\begin{bmatrix} \Phi_1 \\ \Phi_2 \\ \Phi_3 \end{bmatrix}}_{\mathsf{F3}} \underbrace{\begin{bmatrix} p_{20} \\ p_{21} \\ p_{10} \\ p_{11} \\ p_{12} \\ p_{00} \\ p_{01} \\ p_{02} \\ p_{03} \end{bmatrix}}_{\mathsf{P}_3} = \underbrace{\begin{bmatrix} K_1 \\ K_2 \\ K_3 \end{bmatrix}}_{\mathsf{K}_3} \tag{12}$$

where

$$p_{20} = \begin{bmatrix} p_{200} \\ p_{201} \end{bmatrix}, p_{21} = \begin{bmatrix} p_{210} \\ p_{211} \end{bmatrix}, p_{10} = \begin{bmatrix} p_{100} \\ p_{101} \\ p_{102} \end{bmatrix}$$

$$p_{11} = \begin{bmatrix} p_{110} \\ p_{111} \\ p_{112} \end{bmatrix}, p_{12} = \begin{bmatrix} p_{120} \\ p_{121} \\ p_{122} \end{bmatrix}, p_{00} = \begin{bmatrix} p_{000} \\ p_{001} \\ p_{002} \\ p_{003} \end{bmatrix}$$

$$p_{01} = \begin{bmatrix} p_{010} \\ p_{011} \\ p_{012} \\ p_{013} \end{bmatrix}, p_{0,2} = \begin{bmatrix} p_{020} \\ p_{021} \\ p_{022} \\ p_{023} \end{bmatrix}, p_{0,3} = \begin{bmatrix} p_{030} \\ p_{031} \\ p_{032} \\ p_{033} \end{bmatrix}$$

and $\Phi_i, i = 1, 2, 3$ *be the matrices that contains the* $i \bmod r$ *columns of the matrix* Φ *and* K_i *be the matrices that contain the* i *columns of the matrix* $\overline{\Phi}$. *(12) has the following solution*

$$p_{200}, p_{201}, p_{210}, p_{200} \in \mathbb{R}, p_{211} = 0, p_{100} = 0$$
$$p_{101} = -p_{200}, p_{102} = -1 - p_{201}$$
$$p_{110} = -p_{200}, p_{111} = -1 - p_{210} - p_{201}, p_{112} = 0$$
$$p_{120} = -1 - p_{210}, p_{121} = 0, p_{122} = 0$$
$$p_{000} = 0, p_{001} = 0, p_{002} = 0, p_{003} = 0, p_{010} = 0$$
$$p_{011} = p_{200}, p_{012} = 1 + p_{201}, p_{013} = 0$$
$$p_{020} = 0, p_{021} = 1 + p_{210}, p_{022} = 0$$
$$p_{023} = 0, p_{030} = 0$$
$$p_{031} = 0, p_{032} = 0, p_{033} = 0$$

and therefore an annihilating polynomial of third order is given by

$$p(z) = z^3 + \sum_{i=0}^{2}\sum_{j=0}^{3}\sum_{k=0}^{3} p_{i,j,k} z^i s_1^j s_2^k =$$
$$= (s_1 - z)(s_2 - z) \times$$
$$\times (z + p_{200} + s_1 (p_{210} + 1) + s_2 (p_{201} + 1))$$

Note that $p(A(s_1, s_2)) = 0$. *The characteristic polynomial of* $A(s_1, s_2)$ *is* $(z - s_1)^2 (z - s_2)$ *and coincides with the above annihilating polynomial iff* $p_{200} = 0, p_{201} = -1, p_{210} = -2$. *Since the matrix* Φ_3 *has not full rank, we can repeat the above procedure with the matrix* Φ_2 *in place of* Φ_3 *and then we get the minimal polynomial to be the following*

$$p(z) = z^2 + z(-s_1 - s_2) + s_1 s_2 = (s_1 - z)(s_2 - z)$$

3. CONCLUSIONS

Algorithms for the computation of the minimal polynomial and the characteristic polynomial of a two-variable polynomial matrix have been developed. The proposed algorithms are easily implemented in a digital computer and are very useful in many problems, such as the computation of the power and inverse of polynomial matrices. An extension of these algorithms to the n-variable case can easily be done. However, an obvious disadvantage of the above algorithms is their complexity. Therefore, further research is undertaken in order to overcome this problem.

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