Numerator-Denominator Structures of n-D MFDs

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***Abstract*— The approach to defining the zero/pole structure of a n-D rational matrix via the numerator/denominator zero structures of its MFDs is investigated. The possible coprimeness types a MFD can possess is resolved, and relationships between the zero structures which arise are derived.**

I. Introduction

In linear multivariable systems ([7], [1]) the poles and zeros of a rational matrix play a central role, governing response and transmission properties. These poles and zeros can be defined in a number of ways, for example via the Smith-McMillan form, or via matrix fraction descriptions (MFDs). Whichever approach is adopted a consistent notion of pole/zero structure is obtained and the theory is well expoused.

The counterpart of this analysis in linear multidimensional (n-D) theory is by contrast more recent and incomplete. [13] has defined a basic zero structure for n-D polynomial matrices and established connections with physical system properties. [5] has extended this definition to give a more detailed zero structure and has shown its invariance with respect to a matrix transformation of importance in n-D theory ([3], [4]). [8], [12] have considered the rational matrix case and developed pole/zero definitions which have physical interpretation.

This paper considers whether the pole/zero structure of a n-D rational matrix can be defined via the numerator and denominator matrices in a MFD. Coprimeness splits into three distinct notions in n-D and brings into question a number of tenets of 1-D theory, for example, the coprimeness type of an MFD and its constancy across the left and right handedness of the various MFDs. A number of such issues are addressed, and the detail of the numerator/denominator structures examined.

II. Matrix Zero Structure

Let $F[x] = F[x_1, \ldots, x_n]$ denote the polynomial ring over the field F in n indeterminates x_i. Invariably F is taken as $\mathbb{R}$ or $\mathbb{C}$. If $S = \{f_1, \ldots, f_s\}$ is a set of polynomials in $F[x]$, $I = \langle f_1, \ldots, f_s\rangle = \sum_{i=1}^{s} F[x]f_i \subseteq F[x]$ is called the *ideal generated by* $f_1, \ldots, f_s \in F[x]$, or *by* S. The *variety defined by* I is $V(I) \triangleq \{a = (a_1, \ldots, a_n) \in F^n; f_i(a) = 0, i = 1, \ldots, s\}$, and $a \in V(I)$ is a *zero of* I. S is said to be a *zero coprime set* in case $V(I) = \emptyset$ and a *factor coprime set* in case its elements have no nontrivial common factor.

Lemma 1: Let $g, h \in F[x_1, \ldots, x_n]$ and suppose that g divides $f_i h$ (written $g|f_i h$) for $i = 1, \ldots, s$, where $S = \{f_1, \ldots, f_s\}$ is a factor coprime set of polynomials, then $g|h$.

A matrix $P(x)$ over $F[x]$ is called a *n-D polynomial matrix*. Let r denote the rank of $P(x)$ then

Definition 1: The i^{th} DETERMINANTAL DIVISOR $d_i(x)$, $i = 1, \ldots, r$, of $P(x)$ is the g.c.d. of its i^{th} order minors, and the zeros of $d_i(x)$ are the i^{th} DETERMINANTAL ZEROS of $P(x)$.

The simple example $P(x) = (x_1 \ x_2)$ reveals $d_1(x) = 1$ and so there are no determinantal zeros. Nevertheless $P(x)$ loses rank for $x_1 = x_2 = 0$. A more encompassing definition is required.

Denote the ideal generated by the $i \times i$ minors of $P(x)$ by $I_i^{[P]}$. Write $I_i^{[P]} = d_i J_i^{[P]}$. Clearly $J_i^{[P]}$ is generated by a set of factor coprime polynomials, which may not be additionally zero coprime, a distinctive feature of n-D ($n > 1$), and which the example $P(x) = (x_1 \ x_2)$ illustrates. This leads to ([13], [5]).

Definition 2: The i^{th} ORDER INVARIANT ZEROS, $i = 1, \ldots, r$, of $P(x)$, are the elements of $V(I_i^{[P]})$. The ALGEBRAIC ORDER of $a \in V(I_i^{[P]})$ is the positive integer

$$n(a) \triangleq r - \text{rank } P(a)$$

and its i^{th} GEOMETRIC DEGREE, $\delta_i(a)$ is the number of times a occurs in $V(I_i^{[P]})$.

That every $i \times i$ minor can be written as a linear combination of $(i-1)\times(i-1)$ minors has a number of consequences summarised in 1-D by

$$d_i(x)|d_{i+1}(x), \ i = 1, \ldots, r-1 \quad (1)$$

In n-D additional implications are

Lemma 2: If $P(x)$ is a $p \times q$ matrix then

$$I_r^{[P]} \subseteq \cdots \subseteq I_1^{[P]}$$

$$V(I_r^{[P]}) \supseteq \cdots \supseteq V(I_1^{[P]})$$

Clearly if $a \in \mathbb{C}$ is an i^{th} invariant zero of $P(x)$ then it is also an $(i+1)^{th}$ invariant zero and will always be an r^{th} invariant zero. Because of this we refer to the r^{th} invariant zeros simply as *the invariant zeros* of $P(x)$ in line with [13].

Corollary 1: If a is an invariant zero of $P(x)$ of algebraic order $n(a)$, and $\delta_i(a)$, $i = 1, \ldots, r$, are its geometric degrees then

$$0 = \delta_1(a) = \cdots = \delta_{r-n(a)}(a) < \delta_{r-n(a)+1}(a) \leq \cdots \leq \delta_r(a)$$

if $r > n(a)$, and $\delta_0(a) \triangleq 0$ if $r = n(a)$.

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Corollary 2: If $d_i(x)$ denotes the i^{th} determinantal divisor of $P(x)$ then $d_1|d_2 \cdots |d_r$ and so

$$\langle d_r \rangle \subseteq \cdots \subseteq \langle d_1 \rangle$$
$$V(\langle d_r \rangle) \supseteq \cdots \supseteq V(\langle d_1 \rangle)$$

Example 1: For

$$P(x,y,z) = \begin{bmatrix} x^2 & 0 & xy \\ 0 & z^2 & xz \end{bmatrix}$$

$$I_2^{[P]} = \langle x^3 z, x^2 z^2, -xyz^2 \rangle \tag{2}$$
$$I_1^{[P]} = \langle x^2, xy, xz, z^2 \rangle \tag{3}$$

Clearly $I_2 \subset I_1$ in line with Lemma 2. Note that

$$J_2^{[P]} = \langle x^2, xz, -yz \rangle \tag{4}$$
$$J_1^{[P]} = \langle x^2, xy, xz, z^2 \rangle \tag{5}$$

Now, for example $-yz \in J_2^{[P]}$, but clearly $-yz \notin J_1^{[P]}$. Therefore $J_2^{[P]} \nsubseteq J_1^{[P]}$. Also $xy \in J_1^{[P]}$ but $xy \notin J_2^{[P]}$, for the same reason. Thus $J_2^{[P]} \nsupseteq J_1^{[P]}$. This illustrates the non-existence of a definite inclusion between the ideals $J_i^{[P]}$ of $P(x)$.

III. n-D MFD TYPES

The basic form of coprimeness is that achieved by cancelling common factors. In n-D the more special kinds of coprimeness possess the nicer algebraic characterisations. Throughout analogous statements hold for right coprimeness.

Definition 3 ([10]): Two $p \times q, p \times l$ n-D polynomial matrices $T(x), U(x)$ with $p \leq q + l$, are said to be

(i) LEFT ZERO COPRIME (written lzc) if

$$\big(T(x) \quad U(x)\big) = p \ , \forall \ x \in \mathbb{C}^n. \tag{6}$$

(ii) LEFT MINOR COPRIME (lmc) in case the $p \times p$ minors of $\big((T(x) \quad U(x)\big)$ are factor coprime

(iii) LEFT FACTOR COPRIME (lfc) in case the existence of a polynomial factorisation

$$\big((T(x) \quad U(x)\big) = Q(x) \big((T'(x) \quad U'(x)\big) \tag{7}$$

implies $Q(x)$ is unimodular.

In the following we generally adhere to the convention of referring to coprimeness in its most specific form. Thus for example minor coprime refers to a matrix pair which are not additionally zero coprime.

Lemma 3 ([10], [5], [11]): The $p \times q, p \times l$ (with $p \leq q + l$) n-D polynomial matrices $T(x), U(x)$ are lzc iff one of the following equivalent conditions holds:

(i) $\exists$ polynomial matrices $X(x), Y(x)$ such that

$$T(x)X(x) + U(x)Y(x) = I_p \tag{8}$$

(ii) $\big(T(x) \quad U(x)\big)$ possesses no invariant zeros.

(iii) $(T(x) \ U(x))$ can be extended to a unimodular matrix $\begin{pmatrix} T & U \\ X' & Y' \end{pmatrix}$

Lemma 4 ([10]): The $p \times q, p \times l$ (with $p \leq q + l$) n-D polynomial matrices $T(x), U(x)$ are lmc iff for each $i = 1, \ldots, n$ $\exists$ a polynomial $\psi_i(x)$ independent of the variable x_i and polynomial matrices $X_i(x), Y_i(x)$ such that

$$T(x)X_i(x) + U(x)Y_i(x) = \psi_i(x) I_p \tag{9}$$

Suppose the $p \times q$ rational matrix $G(x)$ is written

$$G(x) = D_1(x)^{-1} N_1(x) = N_2(x) D_2(x)^{-1} \ . \tag{10}$$

The coprimeness type of MFDs can only be guaranteed to be factor coprime, but may exceptionally be minor coprime or, more so, zero coprime. The question arises as whether the coprimeness type of, say, a left MFD is always the same.

Theorem 1 ([6], [9]): All left (resp. right) coprime MFDs of the same n-D rational matrix have the same type of coprimeness.

Proof: Assume that $G(x) = D_1^{-1} N_1 = D_2^{-1} N_2$, with D_1, N_1 left coprime and D_2, N_2 left minor coprime. From Lemma 4 for $i = 1, \ldots, n$ $\exists$ polynomial matrices X_i, Y_i and polynomials ψ_i such that

$$D_2 X_i + N_2 Y_i = \psi_i I_p$$

With $N_2 = D_2 D_1^{-1} N_1$ polynomial this gives

$$D_1 X_i + N_1 Y_i = \psi_i D_1 D_2^{-1}$$

Now $\psi_i D_1 D_2^{-1}$ is polynomial, and the factor coprimeness of the polynomials ψ_i determines that $D_1 D_2^{-1} = E$, say, is polynomial. Thus $D_1 = E D_2$ and $N_1 = E N_2$. Now D_1, N_1 are at least left factor coprime and so E is unimodular, thus

$$D_1 X E^{-1} + N_1 Y E^{-1} = \psi_i I_p$$

from which it follows that D_1, N_1 are left minor coprime. Hence if $G(x)$ has one left minor coprime MFD then all left coprime MFDs are of this type. By using the Bezout relation of Lemma 3, the same proof establishes that if $G(x)$ has one left zero coprime MFD then all left coprime MFDs are of this type. It thus follows that if $G(x)$ has one left factor coprime MFD then all left coprime MFDs will be of this type. ■

As regards the comparative coprimeness of left and right MFDs we have

Theorem 2: If a rational matrix has one zero coprime MFD then all of its coprime MFDs, both left and right, are zero coprime.

Proof: Let $G = D^{-1} N$ be left zero coprime. By Lemma 3 $\exists$ polynomial matrices such that $\begin{pmatrix} D & N \\ X & Y \end{pmatrix}$ is unimodular, and hence $\exists$ polynomial matrices such that

$$\begin{pmatrix} D & N \\ X & Y \end{pmatrix} \begin{pmatrix} X' & -N' \\ Y' & D' \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \tag{11}$$

The 2,2 block equation

$$-XN' + YD' = I \tag{12}$$

indicates that D', N' are right zero coprime. To show that D' is invertible, suppose $\exists$ a vector η such that $D'\eta = 0$. Fom (11)

$$-DN' + ND' = 0$$

and so $DN'\eta = 0$ which, since D is invertible, implies $N'\eta = 0$. Post-multiplying (12) by η thus gives $\eta = 0$. Hence D' is invertible and so $N'D'^{-1}$ is a right zero coprime MFD of G. Thus by Theorem 1, all right coprime MFDs of G are zero coprime. ∎

The above results can be summarised in the following table

$\frac{\text{Right}}{\text{Left}}$	Zero	Minor	Factor
Zero	$\exists$	$\nexists$	$\nexists$
Minor	$\nexists$	$\exists$	$\exists$
Factor	$\nexists$	$\exists$	$\exists$

The possibilities indicate that zero coprimeness of a MFD of an n-D rational matrix $G(x)$ is a property of $G(x)$ rather than a property of the specific MFD. Note that in 2-D, since minor and factor coprimeness coincide, the coprimeness type of an MFD is entirely a property of $G(x)$.

IV. The Internal Structure of n-D MFDs

With the establishment of the various combinations of coprimeness types for MFDs of $G(x)$, the concern is now as to the nature of the structural information which the MFDs carry through the ideals determined by the minors of their numerator and denominator matrices. An obvious initial question that arises is whether all left (resp. right) MFDs of $G(x)$ carry the same information, and the first result is discouraging.

Lemma 5 ([2]): If $G(x)$ has a left (resp. right) factor coprime MFD then it possesses left (resp. right) coprime MFDs which are not related by a unimodular factor.

Clearly it follows from this that if $G(x)$ has say a lfc MFD then any other left MFD will not necessarily carry the same structure as determined by the ideals of its numerator and denominator. As regards other factorisation types the following is a crucial result.

Lemma 6: $G(x)$ has a left (resp. right) minor coprime MFD iff all left (resp. right) coprime MFDs of $G(x)$ differ only by a unimodular factor.

Proof: If $G(x)$ has a lmc MFD, then all left MFDs of $G(x)$ are minor coprime by Theorem 1. It also follows from the proof of this Theorem that if $D_1^{-1}N_1 = D_2^{-1}N_2$ are two such left MFDs then $D_2D_1^{-1} = U$ is a unimodular matrix. Hence $D_2 = UD_1$. Also $N_2 = D_2D_1^{-1}N_1 = UN_1$, as required.

Conversely suppose that all left MFDs of $G(x)$ are related by a unimodular factor. If $G(x)$ has a left factor coprime MFD then Lemma 5 gives an immediate contradiction. ∎

Theorem 3: If the n-D rational matrix $G(x)$ has a left minor (resp. right) coprime MFD, then all left (resp. right) MFDs of $G(x)$ give rise to identical numerator and denominator ideal structure.

Proof: Suppose $G(x)$ has a left minor coprime MFD, then all left MFDs of $G(x)$ are minor coprime by Theorem 1. If $D_1^{-1}N_1 = D_2^{-1}N_2$ are two such left MFDs then by Lemma 6 $(D_2,\ N_2) = U(D_1,\ N_1)$ for some unimodular matrix U.

For any matrix Q let $Q^{i_1,\dots,i_k}_{j_1,\dots,j_k}$ denote the $k\times k$ submatrix formed from rows $i_1,\dots,i_k$ and columns $j_1,\dots,j_k$. The Cauchy-Binet Theorem and the relation $D_2 = UD_1$ then gives

$$|{D_2}^{i_1,\dots,i_k}_{j_1,\dots,j_k}| = \sum_{1\le l_1\le\cdots\le l_k\le p} |U^{i_1,\dots,i_k}_{l_1,\dots,l_k}||{D_1}^{l_1,\dots,l_k}_{j_1,\dots,j_k}|$$

from which it is clear that $I_k^{[D_2]} \subseteq I_k^{[D_1]}$. In a similar way the relation $D_1 = U^{-1}D_2$ yields $I_k^{[D_2]} \supseteq I_k^{[D_1]}$ since U is unimodular, and so

$$I_k^{[D_2]} = I_k^{[D_1]}$$

The relation $N_2 = UN_1$ yields $I_k^{[N_2]} = I_k^{[N_1]}$. ∎

It is thus seen that left (resp. right) MFDs of the rational matrix $G(x)$ have identical numerator/denominator structures provided they are at least lmc (rmc). It remains to determine how these structures correspond between the different left and right MFDs. The following result is required.

Lemma 7: If $G(x)$ is a $p\times q$ rational matrix with $p\ge q$ and coprime factorisations

$$D_1(x)^{-1}N_1(x) = N_2(x)D_2(x)^{-1} \tag{13}$$

then

$$D_1'(x)^{-1}N_1'(x) = N_2'(x)D_2'(x)^{-1} \tag{14}$$

are factorisations of the square rational matrix $G'(x) = (G(x), 0_{p,p-q})$ of the same coprimenesstypes as the corresponding factorisations in (13) where

$$\begin{aligned} D_1' &= D_1 & N_1' &= (N_1, 0_{p,p-q}) \\ D_2' &= \begin{pmatrix} D_2 & 0 \\ 0 & I_{p-q}\end{pmatrix} & N_2' &= (N_2, 0_{p,p-q}) \end{aligned} \tag{15}$$

Proof: It is clear that the non-zero high order minors of, for example, $\begin{pmatrix} {D_2'}^T & {N_2'}^T\end{pmatrix}^T$ are identical to those of $\begin{pmatrix} {D_2}^T & {N_2}^T\end{pmatrix}^T$. It thus follows that any factorisation in (14) is zero (resp. minor) coprime iff the corresponding factorisation in (13) is zero (resp. minor) coprime. For factor coprimeness it is clear that any common right factor of $D_2,\ N_2$ will give rise to a common right factor of $D_2',\ N_2'$. For the converse suppose that $D_2',\ N_2'$ have a common right factor Q. Then

$$\begin{pmatrix} D_2 & 0 \\ 0 & I \\ N_2 & 0 \end{pmatrix} = \begin{pmatrix} D''_{11} & D''_{12} \\ D''_{21} & D''_{22} \\ N''_{11} & N''_{12} \end{pmatrix}\begin{pmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22}\end{pmatrix}$$

The 2,2 block equation is

$$I = D''_{21}Q_{12} + D''_{22}Q_{22}$$

which indicates that $D''_{21},\ D''_{22}$ are zlc. Thus $\exists$ polynomial matrices X, Y such that

$$Z = \begin{pmatrix} X & Y \\ D''_{21} & D''_{22}\end{pmatrix}$$

is unimodular. Now

$$ZQ = \begin{pmatrix} Q'_{11} & Q'_{12} \\ 0 & I \end{pmatrix}$$

and so

$$Z_1 = \begin{pmatrix} I & -Q'_{12} \\ 0 & I\end{pmatrix} Z$$

is unimodular and

$$Z_1 Q = \begin{pmatrix} Q'_{11} & 0 \\ 0 & I \end{pmatrix}$$

is a right factor of $\begin{pmatrix} D'_2 \\ N'_2 \end{pmatrix}$. Thus $\exists$ polynomial matrices such that

$$\begin{pmatrix} D_2 & 0 \\ 0 & I \\ N_2 & 0 \end{pmatrix} = \begin{pmatrix} D'_{11} & D'_{12} \\ D'_{21} & D'_{22} \\ N'_{11} & N'_{12} \end{pmatrix} \begin{pmatrix} Q'_{11} & 0 \\ 0 & I \end{pmatrix}$$

The 1,1 and 3,1 block equations then give

$$\begin{pmatrix} D_2 \\ N_2 \end{pmatrix} = \begin{pmatrix} D'_{11} \\ N'_{11} \end{pmatrix} Q'_{11}$$

Thus $\begin{pmatrix} {D'_2}^T & {N'_2}^T \end{pmatrix}^T$ is factor right coprime iff $\begin{pmatrix} {D_2}^T & {N_2}^T \end{pmatrix}^T$ is factor right coprime. A simpler proof establishes that $\begin{pmatrix} D'_1 & N'_1 \end{pmatrix}$ is factor left coprime iff $\begin{pmatrix} D_1 & N_1 \end{pmatrix}$ is factor left coprime ■

It is clear from the above result that the ideals generated by the $i \times i$ minors of N_1, N'_1 coincide for $i = 1, \ldots, r$ where r is the rank of $G(x)$, as do those of N_2, N'_2. Those of D_1, D'_1 coincide for each $i = 1, \ldots, p$. The ideals of D_2, D'_2 correspond as follows

$$I_{p-i}^{[D'_2]} = I_{q-i}^{[D_2]}, \ i = 0, \ldots, q-1 \tag{16}$$

and for any $q \leq i \leq p-1$, $I_{p-i}^{[D'_2]} = \langle 1 \rangle$. A similar result, and correspondence of ideals, holds for $q > p$.

As seen above there are various combinations of coprimeness types for the left and right MFDs of $G(x)$. The lzc/rzc combination gives

Theorem 4: If the $p \times q$ rational matrix $G(x)$, of rank r, possesses the zlc and zrc factorisations

$$D_1(x)^{-1} N_1(x) = N_2(x) D_2(x)^{-1} \ . \tag{17}$$

Then

$$I_{r-i}^{[N_1]} = I_{r-i}^{[N_2]}, \ i = 0, \ldots, r-1 \tag{18}$$

and

$$I_{p-i}^{[D_1]} = I_{q-i}^{[D_2]}, \ i = 0, \ldots, h-1 \tag{19}$$

where $h = \min(p, q)$. For any $i \geq h$, $I_{p-i}^{[D_1]} = \langle 1 \rangle$ in case $p - i \geq 0$ or $I_{q-i}^{[D_2]} = \langle 1 \rangle$ in case $q - i \geq 0$.

Proof: By Lemma 7 expand $G(x)$ to ensure it is $l \times l$ where $l = max(p, q)$. The associated correspondence of ideals is detailed in (16). Note that (17) implies

$$N_1 D_2 = D_1 N_2 \tag{20}$$

From the zero coprimeness requirements $\exists$ polynomial matrices X, Y, W, Z such that

$$\begin{aligned} N_1 X + D_1 Y &= I_l \\ W D_2 + Z N_2 &= I_l \end{aligned} \tag{21}$$

From (20) and (21) it follows that

$$\begin{pmatrix} W & -Z \\ N_1 & D_1 \end{pmatrix} \begin{pmatrix} D_2 & X \\ -N_2 & Y \end{pmatrix} = \begin{pmatrix} I_l & J \\ 0 & I_l \end{pmatrix} \tag{22}$$

where $J = WX - ZY$.

Consider the following equation formed from (22)

$$\underbrace{\begin{pmatrix} 0 & E^{i_1, \ldots, i_k} \\ N_1 & D_1 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} D_{2_{j_1, \ldots, j_k}} & X \\ -N_{2_{j_1, \ldots, j_k}} & Y \end{pmatrix}}_{B} = \begin{pmatrix} {N_2}_{j_1, \ldots, j_k}^{i_1, \ldots, i_k} & Y^{i_1, \ldots, i_k} \\ 0 & I_l \end{pmatrix} \tag{23}$$

where $1 \leq k \leq l$, and $E^{i_1, \ldots, i_k}$ is that matrix whose t, s^{th} element is 1 if $s = i_t$, and zero otherwise.

Take determinants in (23), and use the Cauchy-Binet theorem to expand the left hand side to give

$$\sum_m |A_{m_1, \ldots, m_{l+k}}^{1, \ldots, l+k}| \, |B_{1, \ldots, l+k}^{m_1, \ldots, m_{l+k}}| = |{N_2}_{j_1, \ldots, j_k}^{i_1, \ldots, i_k}| \tag{24}$$

The form of A indicates that any minor of A of the type occurring in the left hand side of (24) for which $\{i_1, \ldots, i_k\}$ is not a subset of $\{m_1, \ldots, m_{l+k}\}$ is zero. Thus all minors of A which occur in the left hand side of (24) contain the columns $\{i_1, \ldots, i_k\}$. Such a factor is then expressible via Laplace expansion in terms of products of minors of N_1 and D_1. The smallest minor of N_1 occuring in this Laplace expansion is of order k. Thus $|{N_2}_{j_1, \ldots, j_k}^{i_1, \ldots, i_k}|$ is expressible as a linear combination of minors of N_1 of order k and greater. Since any minor can be expanded in terms of lower order minors, it follows that $|{N_2}_{j_1, \ldots, j_k}^{i_1, \ldots, i_k}|$ can be written as a linear combination of the k order minors of N_1. Thus

$$I_k^{[N_2]} \subset I_k^{[N_1]}, \ k = 1, \ldots, l \tag{25}$$

The above argument may be repeated, with

$$\underbrace{\begin{pmatrix} W & -Z \\ N_1^{i_1, \ldots, i_k} & D_1^{i_1, \ldots, i_k} \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} D_2 & E_{j_1, \ldots, j_k} \\ -N_2 & 0 \end{pmatrix}}_{B} = \begin{pmatrix} I & W_{j_1, \ldots, k_k} \\ 0 & {N_1}_{j_1, \ldots, j_k}^{i_1, \ldots, i_k} \end{pmatrix} \tag{26}$$

replacing (23), to establishes the reverse inclusions. Hence

$$I_k^{[N_1]} = I_k^{[N_2]}$$

for $k = 1, \ldots, l$.

An entirely analogous argument, with the roles of N_1, N_2 and D_1, D_2 interchanged, completes the proof. ■

Corollary 3: Under the conditions of Theorem 4

$$V(I_{r-i}^{[N_1]}) = V(I_{r-i}^{[N_2]}) \text{ for } i = 0, 1, \ldots, r-1 \tag{27}$$

and

$$V(I_{p-i}^{[D_1]}) = V(I_{q-i}^{[D_2]}) \text{ for } i = 0, 1, \ldots, h-1 \tag{28}$$

while for any $i \geq h$, $V(I_{p-i}^{[D_1]}) = \emptyset$ in case $p - i \geq 0$, and $V(I_{q-i}^{[D_2]}) = \emptyset$ in case $q - i \geq 0$.

Suppose the ideal I_i generated by the $i \times i$ minors of the n-D matrix P is written as $d_i J_i^{[P]}$. Although we have seen that no particular relation of inclusion holds between the ideals $J_i^{[P]}$

there is something that can be said about the corresponding ideals of the numerator and denominator matrices of the zero coprime MFDs of a rational matrix.

Corollary 4: Under the conditions of Theorem 4

$$J^{[N_1]}_{r-i} = J^{[N_2]}_{r-i}$$
$$V(J^{[N_1]}_{r-i}) = V(J^{[N_2]}_{r-i})$$

where $i = 0, \ldots, r-1$, and

$$J^{[D_1]}_{p-i} = J^{[D_2]}_{q-i}$$
$$V(J^{[D_1]}_{p-i}) = V(J^{[D_2]}_{q-i})$$

where $i = 0, \ldots, h-1$. For any $i \geq h$, $J^{[D_1]}_{p-i} = \langle 1 \rangle$ (and so $V(J^{[D_1]}_{p-i}) = \emptyset$) in case $p-i \geq 0$, while $J^{[D_2]}_{q-i} = \langle 1 \rangle$ (and so $V(J^{[D_2]}_{q-i}) = \emptyset$) in case $q-i \geq 0$.

The above results reveal the commonality of numerator/denominator zero structures of any rational matrix which possesses a zero coprime MFD. The other cases are considered next.

Theorem 5: If the $p \times q$ rational matrix $G(x)$ possesses the lmc and rmc factorisations

$$D_1(x)^{-1}N_1(x) = N_2(x)D_2(x)^{-1} \ . \qquad (29)$$

then

$$d^{[N_1]}_{r-i} = d^{[N_2]}_{r-i}, \ i = 0, \ldots, r-1 \qquad (30)$$

and

$$d^{[D_1]}_{p-i} = d^{[D_2]}_{q-i}, \ i = 0, \ldots, h-1 \qquad (31)$$

where $h = \min(p, q)$. For any $i \geq h$, $d^{[D_1]}_{p-i} = 1$ in case $p-i \geq 0$ or $d^{[P_2]}_{q-i} = 1$ in case $q-i \geq 0$.

Proof: By Lemma 7 expand $G(x)$ to ensure it is $l \times l$ where $l = max(p, q)$. The associated correspondence of ideals is detailed in (16). (29) implies that

$$N_1(x)D_2(x) = D_1(x)N_2(x) \qquad (32)$$

From the minor coprimeness requirements $\exists$ polynomial matrices X_m, Y_m, W_m, Z_m such that

$$N_1X_m + D_1Y_m = \phi_m(x)I_p \qquad (33)$$
$$W_mD_2 + Z_mN_2 = \psi_m(x)I_q$$

where for $m = 1, \ldots, n$ the polynomials $\phi_m(x), \psi_m(x)$ do not contain the variable x_m. Thus each set $\{\phi_1, \ldots, \phi_n\}$, $\{\psi_1, \ldots, \psi_n\}$ is factor coprime.

From (32) and (33) it follows that

$$\begin{pmatrix} W_m & -Z_m \\ N_1 & D_1 \end{pmatrix} \begin{pmatrix} D_2 & X_m \\ -N_2 & Y_m \end{pmatrix} = \begin{pmatrix} \psi_m(x)I_q & J_m \\ 0 & \phi_m(x)I_p \end{pmatrix} \qquad (34)$$

Consider then the following equation formed from (34), which is analogous to (23),

$$\underbrace{\begin{pmatrix} 0 & E^{i_1,\ldots,i_k} \\ N_1 & D_1 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} D_{2_{j_1,\ldots,j_k}} & X_m \\ -N_{2_{j_1,\ldots,j_k}} & Y_m \end{pmatrix}}_{B} \qquad (35)$$
$$= \begin{pmatrix} N_2{}^{i_1,\ldots,i_k}_{j_1,\ldots,j_k} & Y_m^{i_1,\ldots,i_k} \\ 0 & \phi_m(x)I_p \end{pmatrix}$$

where $1 \leq k \leq l$, and $E^{i_1,\ldots,i_k}$ is that matrix whose t, s^{th} element is 1 if $s = i_t$, and zero otherwise.

Repeating the argument in the proof of Theorem 4 reveals that $\phi_m(x)^p | N_2{}^{i_1,\ldots,i_k}_{j_1,\ldots,j_k}|$ can be written as a linear combination of the k order minors of N_1 from which it follows that $d^{[N_1]}_k | \phi^p_m d^{[N_2]}_k$. Since $\{\phi_1, \ldots, \phi_n\}$ is factor coprime Lemma 1 shows that $d^{[N_1]}_k | d^{[N_2]}_k$. Continuing as in the proof of Theorem 4 it is concluded that $d^{[N_1]}_k = d^{[N_2]}_k$ modulo a multiplicative constant, with a corresponding statement holding for D_1, D_2. ■

Corollary 5: Under the conditions of Theorem 5

$$\langle d^{[N_1]}_{r-i} \rangle = \langle d^{[N_2]}_{r-i} \rangle$$
$$V(\langle d^{[N_1]}_{r-i} \rangle) = V(\langle d^{[N_2]}_{r-i} \rangle)$$

where $i = 0, \ldots, r-1$. A similar statement holds in relation to D_1, D_2..

Corollary 6: Suppose $G(x)$ possesses the lmc and rfc factorisations

$$D_1(x)^{-1}N_1(x) = N_2(x)D_2(x)^{-1} \ . \qquad (36)$$

Let $d^{[\cdot]}_i$ denote the i^{th} determinantal divisor of the indicated matrix. Then

$$d^{[N_1]}_{r-i} | d^{[N_2]}_{r-i}, \ i = 0, \ldots, r-1 \qquad (37)$$

and

$$d^{[D_1]}_{p-i} | d^{[D_2]}_{q-i}, \ i = 0, \ldots, h-1 \qquad (38)$$

where $h = \min(p, q)$. For any $i \geq h$, $d^{[D_1]}_{p-i} = 1$ in case $p-i \geq 0$ or $d^{[P_2]}_{q-i} = 1$ in case $q-i \geq 0$.

Proof: The lmc of D_1, N_1 may be exploited as in Theorem 5 to give the result. The lack of a Bezout identity for factor coprimeness precludes the reverse divisibility statements, and so equality cannot be concluded. ■

Example 2:

$$G(x) = \begin{pmatrix} \frac{(z_3+0.5)^2}{(z_2+2)(z_3+2.5)} & \frac{1}{(z_2+2)(z_3+4.5)} \\ \frac{(z_3+0.5)}{(z_1+3)(z_3+2.5)} & \frac{(1}{(z_1+3)(z_3+4.5)} \end{pmatrix}$$

has the rfc MFD [2] $N_2D_2^{-1}$ with

$$N_2 = \begin{pmatrix} (z_3-0.5)(z_3+0.5) & 0 \\ 0 & (z_3-0.5)(z_3+0.5) \end{pmatrix}$$
$$D_2 = \begin{pmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{pmatrix}$$
$$d_{11} = (z_2+2)(z_3+2.5), \ d_{12} = -(z_1+3)(z_3+2.5)$$
$$d_{21} = -(z_2+2)(z_3+0.5)(z_3+4.5)$$
$$d_{22} = (z_1+3)(z_3+0.5)^2(z_3+4.5)$$

It also has the left minor coprime MFD $D_1^{-1}N_1$ with

$$D_1 = \begin{pmatrix} (z_2+2)(z_3+2.5) & -(z_1+3)(z_3+2.5) \\ 2(z_2+2) & (z_1+3)(3z_3+11.5) \end{pmatrix}$$
$$N_1 = \begin{pmatrix} (z_3-0.5)(z_3+0.5) & 0 \\ 5(z_3+0.5) & 3 \end{pmatrix}$$

It can then be seen for example that

$$I_1^{[N_1]} = \langle 1 \rangle;\ d_1^{[N_1]} = 1$$
$$I_2^{[N_1]} = \langle (z_3 - 0.5)(z_3 + 0.5) \rangle = I_1^{[N_2]}$$
$$d_2^{[N_1]} = (z_3 - 0.5)(z_3 + 0.5) = d_1^{[N_2]}$$
$$I_2^{[N_2]} = \langle (z_3 - 0.5)^2 (z_3 + 0.5)^2 \rangle$$
$$d_2^{[N_2]} = (z_3 - 0.5)^2 (z_3 + 0.5)^2$$

from which the relationships determined in Corollary 6 may be verified. The relationship between the determinantal divisors of the denominator matrices may be similarly verified.

V. CONCLUSIONS

This paper has adopted the definition of the zero structure of a polynomial matrix ([13], [5]) and considered its possible role in the definition of the zero/pole stucture of a n-D rational matrix $G(x)$ via its MFDs. The coprimeness type of the MFD has been seen to be unique to its handedness, while this coprimeness type is only seen to be unique to $G(x)$ when it possesses a zero coprime MFD. In this case there is unified agreement of the numerator/denominator structure, whichever MFD is used. In the other cases, of minor or factor coprime MFDs, there appears only partial agreement between these structures as has been described. The results provide interesting comparison with the 1-D case where there is consistent agreement across all such structures, and highlights the complexities inherent in n-D.

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